

GENERALIZED HERMITE POLYNOMIAL AND IT'S SUPERGAUSSIAN FORM

A

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College of Basic Science and Humanities,
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In partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE IN
MATHEMATICS

BY

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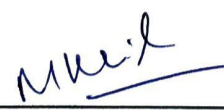
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I **Subham Kumar Rout**, MSc Student, Department of Mathematics, College of Basic Science and Humanities, OUAT, Bhubaneswar, Odisha, do hereby declare that the thesis entitled “**generalized hermite polynomial and it's supergaussian form**” submitted to the College of Basic Science and Humanities, Odisha University of Agriculture and Technology, Bhubaneswar- 751003, Odisha for the degree of Master of Science in the subject Mathematics is an original piece of research work and no part of this thesis had been submitted for any Degree or Diploma to any other University / Institution.

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ABSTRACT

The Generalized Hermite polynomial and its relationship with Bell polynomial and its Super-Gaussian form are presented in this thesis. chapter 2 contains three theorems of Generalized Hermite polynomial which proofs are based upon the relationship of Bell polynomial and Generating Hermite Polynomial . We describe some examples as well as some properties of Bell Polynomial in chapter 3. A brief information of Super -Gaussian form of Hermite Polynomial was discussed in chapter – 4. This thesis concluded that Hermite Polynomial plays an important role in both mathematics and physics.

CHAPTER-1

Introduction: -

The Hermite Polynomials was presented over a year prior. Hermite Polynomials were characterized by Pierre-Simon Laplace in 1810. However, it was unmistakable structure and in 1859 studied by Pafnuty Chebyshev in details. Chebyshev's work was neglected and finally in 1864 Charles Hermite composed on the Polynomials describing them as new. Since that day it was perceived as a remarkable strategy in pure and applied mathematics both.

Hermite recommended a set of summed up polynomials. Right now, the importance of Hermite polynomial ($H_m(y) = \sum_{m=0}^{\frac{m}{2}} \frac{(-1)^u m!}{u!(m-2u)!} (2y)^{m-2u}$) has been seen and use to address the quantum mechanical and optical beam transport issues. Inside this rear system, summed up types of harmonic oscillator eigen functions is being proposed. The Hermite polynomials evolution theory has been sluggish but steady. Recently, the generalised special functions theory has undergone a considerable transformation.

The motivations for being interest dare varied. Limiting ourselves to the practical, we observe that the application of new classes of special functions enabled answers to several physical issues that were difficult to achieve using traditional analytical and numerical methods. Just the generalised Bessel functions have recently supplied arguments for reconsidering the generalised Hermite polynomials theory (G.H.P.) in a broader framework. Further research has mainly focused on the orthogonal qualities of as well as the potential of establishing an infinite variable Hermite polynomial. This study contributes the theory of Generalised Hermite Polynomials ($e^{2xv-v^2} = \sum_{m=0}^{\infty} \frac{v^m}{m!} H_m(y)$) in a new way.

In this thesis we study about Bell Polynomials and investigate the relationship between Generalised Hermite Polynomial and Bell polynomials and also a new type of polynomial's properties. We will explain the super-Hermite for its significance to the super-Gaussian Hermite functions. Which is discussed in chapter 4 This is going to play an essential role in the theory of super Gaussian beam. There has yet to be devised a general mode development theory. We shall introduce super Gaussian functions in this work.

In mathematics a class of Orthogonal Polynomial was formed by Hermite Polynomial. In physics then use as Eigen functions of the harmonic oscillator. In numerical analysis they appear as Gauss-Hermite quadrature. In this chapter we proof some results by using the relationship between Generating Hermite Polynomial and Bell Polynomial. And we also give a brief idea of Supergaussian form of Hermite Polynomial.

CHAPTER-2

PRELIMINARY OF GENERATING HERMITE POLYNOMIALS:

For Hermite polynomial we consider differential equation as follows

$$\frac{d^2x}{dy^2} - 2y\frac{dx}{dy} + 2yx = 0 \quad \dots\dots\dots (1)$$

Which is known as Hermite differential equation.

The solution of equation (1) is

$$\sum_{m=0}^{\frac{m}{2}} \frac{(-1)^u m!}{r! (m - 2u)!} (2y)^{y-2u}$$

Where, $[m/2] = \{m/2, \text{ if } m \text{ is even}\}$

$\{(m-1)/ 2, \text{ if } m \text{ is odd}\}$

$$H_m(y) = \sum_{m=0}^{\frac{m}{2}} \frac{(-1)^u m!}{u!(m-2u)!} (2y)^{m-2u} \dots\dots\dots (2)$$

This is Hermite polynomial.

Putting $m = 0,1,2,3 \dots\dots\dots$ in (2) we have

$$H_0 (y) = 1$$

$$H_1 (y) = 2y$$

$$H_2(y) = 4y^2 - 2$$

$$H_3(y) = 8xy^3 - 12y$$

$$H_4(y) = 16y^4 - 48y^2 + 12$$

Generating Hermite Polynomial

$$e^{2xy-v^2} = \sum_{m=0}^{\infty} \frac{v^m}{m!} H_m(y)$$

Proof: -

$$e^{2yv-v^2}.$$

$$= e^{2yv} \cdot e^{-v^2}$$

$$\{ e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots \dots \dots \}$$

$$= e^{-v^2} \cdot e^{2yv}$$

$$\{ e^y = \sum_{u=0}^{\infty} \frac{y^u}{u!} \}$$

$$e^{-v^2} = \sum_{u=0}^{\infty} \frac{(-v)^{2u}}{u!}$$

$$e^{2yv} = \sum_{t=0}^{\infty} \frac{(2yv)^t}{t!}$$

$$= \sum_{u=0}^{\infty} \frac{(-v)^{2u}}{u!} \sum_{t=0}^{\infty} \frac{(2yv)^t}{t!}$$

$$= \sum_{u=0}^{\infty} \frac{(-1)^{u+2u}}{u!} \sum_{t=0}^{\infty} \frac{(-1)^{t+2u}}{u!}$$

$$= \sum_{u=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^u (2y)^t v^{t+2u}}{u! t!}$$

Put $t+2u = m$

$$t = m - 2u$$

$$= \sum_{m=0}^{\infty} \sum_{u=0}^{m/2} \frac{(-1)^u (2y)^{m-2u} v^m}{u! (m-2u)!}$$

$$= \sum_{m=0}^{\infty} \sum_{u=0}^{\infty} \frac{(-1)^u (2y)^{m-2u} m! v^m}{u! (m-2u)! m!}$$

$$= \sum_{m=0}^{\infty} \frac{v^m}{m!} H_m(y) \quad \cdot \text{(Proved)}$$

Orthogonal Property: -

Prove that $\int_{-\infty}^{\infty} e^{-y^2} H_m(x)H_n(x) dx$
 $= \begin{cases} 0 & m \neq n \\ \sqrt{\pi} 2^m m! & m=n \end{cases}$

Recurrence relation: -

- 1) $H'_m(y) = 2m H_{m-1}(y) \quad m \geq 1$
- 2) $H_{m+1}(y) = 2y H_m(y) - 2m H_{m-1}(y) \quad m \geq 1$
- 3) $H'_m(y) = 2y H_m(y) - H_{m+1}(y).$
- 4) $H''_m(y) - 2y H'_m(y) + 2m H_m(y) = 0$

Rodrigue's formula: -

i.e- $H_m(y) = (-1)^m e^{y^2} \frac{d^m}{dy^m} e^{-y^2}$

Squares of Hermite Polynomial

The squares of Hermite polynomial is defined as

$$\frac{1}{\sqrt{1-v^2}} \exp \frac{2y^2 v}{1+v} = \sum_{m=0}^{\infty} \frac{H_m^2(y)}{2^m} \frac{v^m}{m!} \dots\dots\dots(3)$$

Reformulating equation (3) we get

$$\frac{1}{\sqrt{1-v^2}} \exp \frac{yv}{1+v} = \sum_{m=0}^{\infty} H_m^2(\sqrt{y}) \frac{v^m}{m!}$$

$$\frac{1}{\sqrt{1-v^2}} \exp \frac{yv}{1+v} = \sum_{m=0}^{\infty} \frac{H_m^2(\frac{\sqrt{y}}{2}) v^m}{2^m} \frac{v^m}{m!} \dots\dots\dots(4)$$

Let's define $F(v) = F(v,y) = \frac{1}{\sqrt{1-v^2}} \exp \frac{yv}{1+v} \dots\dots\dots(5)$

In this paper, we are going to review the generating functions e^{2vy-v^2} and $F(v) = F(v,y)$ defines in (5), represent the clear-cut formulas for the nth order derivatives of the functions $F(t)$ and e^{2vy-v^2} . This can be gazed as ordinary differential equations or derivative polynomials [8], find further differential equations for Hermite polynomials $H_n(x)$ and their squares $H_n^2(x)$.

The main outcomes of this thesis can be expressed as the succeeding theorems.

Theorems:

Theorem 2.1.

For $n \geq 0$, the nth order derivative of the function $F(v) = F(v, y)$ define in (4) can be evaluated by

$$F(v) = F(v, y) = \frac{1}{\sqrt{1-v^2}} \exp \frac{yv}{1+v},$$

$$\frac{d^n F(v)}{dv^n} = \left\{ \frac{(-1)^n n!}{(1+v)^n} \sum_{m=0}^n \frac{(-1)^m}{m!} \frac{1}{(1+t)^m} \left(\sum_{k=0}^{n-m} \frac{(-1)^k (1+t)^k}{2^k} (n-k-m-1) \right) \times \left[\frac{1}{t^k} \sum_{l=0}^k \frac{(2l-1)! 2^l}{l!} (lk-l) \frac{t^{2l}}{1-t^2} \right] y^m \right\} F(v)$$

.....(6)

where $(0 \ 0) = 1$ and $(p \ q) = 0$ for $q > p \geq 0$.

Theorem 2.2

The Hermite polynomials $H_m(y)$ can be evaluated by

$$H_m(y) = (-1)^m \frac{m!}{2^m} \sum_{k=0}^m (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} x^{2k-n}$$

For $m \geq 0$

And the nth derivative of their generating function e^{2yv-v^2} can be computed by

$$\frac{d^n e^{2yv-v^2}}{dv^n} = e^v \frac{m!}{2^m} \sum_{k=0}^m (-1)^k \frac{2^{2k}}{k!} \binom{k}{m-k} (v-y)^{2k-m} \dots\dots(7)$$

Theorem 2.3

The Hermite polynomials $H_m(y)$ and their derivatives $H'_m(y)$ fulfil the condition: for $m \geq 0$

$$H'_n(y) = 0,$$

$$\text{and } H'_m(y) = 2m H_{m-1}(y) \dots\dots\dots(8)$$

for $m \in \mathbb{N}$. Consequently,

$$H_m(y) = 2y H_{m-1}(y) - H'_{m-1}(y) \dots\dots\dots(9)$$

for $m \geq 2$.

$$H_m(y) = 2y H_{m-1}(y) - 2(m-1) H_{m-2}(y) \dots\dots\dots(10)$$

we proof these theorems using some properties of Bell Polynomials.

CHAPTER-3

Bell Polynomials:

The class of Bell Polynomials includes extensions of Generalised Hermite polynomials.

We will give a brief overview of bell polynomials in this chapter, highlighting their connection to the exponential generating function in combinatorics as well as the formal power series composition & inversion.

Numerous attributes are going to start on the fly, with quick proofs. The majority of the key findings can be discovered in LovisComtet's advanced combinatorics book. Combinatorial theory by Martin Aigner is a good foundation topolynomials' Binomial sequences and how they are often employed in polynomials operators' theory.

The relationship between Generating Hermite polynomials and Bell Polynomials is

$$\exp\left(\sum_{s=1}^m y_s t^s\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(\{s\}_m)}(\{y_s\}_m)$$

where $\{s\}_m = 1, 2, 3, \dots, m$.

$$\{y_s\}_m = x_1, x_2, \dots, x_m,$$

One can identify $H_n^{(\{s\}_m)}(\{y_s\}_m)$ as a

G.H.P with m variables.

Let's consider a sequence of real numbers

$$\{f_l\}_{k=0}^{\infty} \dots \dots \dots (1)$$

Since problems about convergence are never raised, the power series is considered formal only at $t=0$ do we want to assess it and its derivatives

$$f_k = \frac{d^k}{dt^k} f(t) \Big|_{t=0} \dots\dots\dots(2)$$

Here (ii) is defined as the coefficient of a Taylor series

- (i) Known as a formal Taylor series and it is k^{th} Taylor coefficient of $f(k)$

The variables in those polynomials, which have an unlimited number of them, are actually the coefficient f_1, f_2 and so on of an unknown exponential generating functions

$$f(t) = \sum_{k=1}^{\infty} f_k \frac{t^k}{k!} \dots\dots\dots(3)$$

Where $f_0=f(0)=0$ is our assumption.

The bell polynomials $B_{n,k}=B_{n,k}(f_1, f_2, \dots)$ are defined by ,

$$e^{yf(t)} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_{n,k} z^k \frac{t^n}{n!} \dots\dots\dots(4)$$

for example :

$$B_0 = 1,$$

$$B_1(y_1) = y,$$

$$B_2(y_1, y_2) = y_1^2 + y_2,$$

$$B_3(y_1, y_2, y_3) = y + 3y_1y_2 + y_3,$$

$$B_4(y_1, y_2, y_3, y_4) = y_1^4 + 6y_1^2y_2 + 4y_1y_3 + 3y_2^2 + y_4$$

$$B_5(y_1, y_2, y_3, y_4, y_5) = y_1^5 + 10y_2y_1^3 + 15y_2^2y_1 + 10y_3y_1^2 + 10y_3y_2 + 5y_4y_1 + y_5$$

By differentiating their quire amount of times and setting $x=t=0$, we can isolate $B_{n,k}$ in eq-iv as

$$B_{n,k} = \frac{\partial^n}{\partial t^n} \frac{1}{k!} \frac{\partial^k}{\partial z^k} e^{yf(t)} \Big|_{z=0, t=0}$$

$$B_{n,k} = \frac{1}{k!} \frac{d^n}{dy^n} f^k(t) \Big|_{t=0} \dots\dots\dots(5)$$

For all $n, k \geq 0$.

Specifically, this states that $B_{n,k}$ is the n th Taylor coefficient of $\frac{1}{k!} f^k(t)$

$$\frac{1}{k!} f^k(t) = \sum_{n=0}^{\infty} B_{n,k} \frac{t^n}{n!} \dots\dots\dots(6)$$

As soon as we set $k=0$ we get $B_{0,0} = 1$ and $B_{n,0} = 0$ for $n \geq 1$

By setting $k=1$ we get $B_{n,1} = f_n$ for $n \geq 0$.

As $f_0 = 0$ we have.

$$\frac{1}{k!} f^k(t) = \frac{1}{k!} (f_1 t + f_2 \frac{t^2}{2!} + \dots\dots\dots)^k = f_1^k \frac{t^k}{k!} + \dots\dots\dots(7)$$

$$B_{n,k} = 0$$

Where $k > n, B_{n,n} = f_1^n$.

We obtain the following explicit expression for $B_{n,k}$ by carefully expanding out eq- (7) and looking at the coefficient of $\frac{t^n}{n!}$

$$B_{n,k} = \sum_{j_1, j_2, \dots \geq 0, j_1 + j_2 + \dots = k, j_1 + 2j_2 + 3j_3 + \dots = n} \frac{n!}{j_1! j_2! \dots (1!)^{j_1} (2!)^{j_2}} f_1^{j_1} f_2^{j_2} \dots\dots\dots(8)$$

We can infer a number of things from this formula, including first, the constraints of summation implies for some $j_1 \geq 0, j_2 \geq, \dots, j_5 \geq 0, \dots$ we have if the variable f_s occurs in $B_{n,k}$,

$$s-1 \leq j_2 + 2j_3 + \dots + (s-1)j_s + \dots = n-k$$

Giving us $s \leq n - k + 1$.

Therefore, $B_{n,k} = B_{n,k}(f_1, \dots, f_{n-k+1})$ which means that B_n depends only on the variables $f_1, f_2, \dots, f_{n-k+1}$ and no other value.

In addition, it is simple to observe that $B_{n,k}$ is homogeneous of degree k and a combinatorial argument may be used to prove that all of the coefficients are in fact integers.

Sometimes it is more convenient to write $B_{n,k}(f(t))$ instead of $B_{n,k}(f_1, f_2, \dots)$ whereas always $f(0)$ defined by (iii). When doing this, it's crucial to keep in mind that $B_{n,k}(f(t))$ is a function of f_1, f_2 not t .

As previously mentioned, for constant $f(t)$, the following polynomials can be defined in z

$$B_n(y) = B_n(y; f(t)) = \sum_{k=0}^n B_{n,k}(f(t)) y^k$$

These polynomials can be seen as being produced by the expansion of

$$e^{yf(t)} = \sum_{n=0}^{\infty} B_n(y) \frac{t^n}{n!}.$$

Note that since $B_{n,k} = f_1^n$

We have that $\deg B_n(y) = n$, for all n

If $f_1 \neq 0$ and $\deg B_n(y) < n$ for all $n > 0$ if $f_1 = 0$

It is inevitably true that $\deg B_0(y) = 0$ since $B_0(y) = 1$ for every $f(t)$.

Let $B = B(f(t))$ denote the infinite lower triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & f_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & f_2 & f_1^2 & 0 & 0 & 0 & \dots \\ 0 & f_3 & B_{3,2} & f_1^3 & 0 & 0 & \dots \\ 0 & f_4 & B_{4,2} & B_{4,3} & f_1^4 & \dots & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Example:

The following are a few of the almost familiar examples of bell polynomials $B(f(t))$ and the corresponding polynomials $B(y;f(0))$.

1. The standard polynomials $y^n = B_n(y; t)$

2. Bell Polynomial of 1st kind

The following factorials $[y]_n = B_n(y; \log(1+t))$.

$$[y]_n = y(y-1) \dots (y-n+1) = \sum_{k=0}^n S_{n,k} y^k$$

Where the Stirling numbers of the first kind are the integers $s, k = B, K(\log(1+t))$.

3. The rising factorials

$$[y]^n = y(y+1) \dots (y+n-1) = \sum_{k=0}^n |S_{n,k}| y^k$$

So that

$$|S_{n,k}| = B_{n,k}(-\log(1-t)).$$

4. Bell Polynomial of 2nd kind

The exponential polynomials $\phi_n(y) = B_n(y; e^t - 1)$

$$\phi_n(y) = \sum_{k=0}^n S_{n,k} y^k$$

where the numbers $S_{n,k} = B_{n,k}(e^t - 1)$ are known as the string numbers of the second kind.

5. Laguerre Polynomial

The Laguerre Polynomials $L_n(y) = B_n(y, t / (t-1))$

$$L_n(y) = \sum_{k=0}^n L_{n,k} y^k$$

Where the numbers

$$L_{n,k} = (t / (t - 1))$$

are called Lah numbers given by

$$L_{n,k} = (-1)^n \frac{n!}{k!} [n - 1 \ k - 1]$$

In order to proof our main theorems that discuss in chapter 2 we desire handful of lemmas below.

Lemma 3.1

For $n \geq k \geq 0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $B_{n,k}(z_1, z_2, \dots, z_{n-k+1})$, are defined by

$$B_{n,k}(z_1, z_2, \dots, z_{n-k+1}) = \sum_{1 \leq j \leq n, l_i \in \{0\} \cup \mathbb{N}} \frac{n!}{\pi_{j=k}^{n-k+1} l_i! \sum_{j=1}^n j l_j = n \sum_{j=1}^n l_j = k} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{j!}\right)^{l_i}$$

The Fa' a di Bruno formula can be explained by using the second-class Bell polynomial $B_{n,k}(y_1, y_2, \dots, y_{n-k+1})$.

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)) \dots \dots \dots (i)$$

Lemma 3.2

For complex number a and b, we have

$$B_{n,k}(aby_1, ab^2y_2, \dots, ab^{n-k+1}y_{n-k+1}) = a^k B_{n,k}(y_1, y_2, \dots, y_{n-k+1}) \dots \dots \dots (ii)$$

Lemma 3.3

The Bell polynomials of the second kind $B_{n,k}$ satisfy.

$$B_{n,k}(y, 1, 0, \dots, 0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} y^{2k-n} \dots \dots \dots (iii)$$

For $0 \leq k \leq n$

Lemma 3.4

for $n \geq k \geq 0$,

We have

$$B_{n,k}(1! 2! \dots (n-k+1)!) = \binom{n-k}{k-1} \frac{n!}{k!} \dots \dots \dots \text{(iv)}$$

Some more properties of Hermite polynomials

- $\sum_{r+s=k} \sum_{l+m=n} \binom{n}{l} \binom{l-1}{r-1} \frac{l!}{r!} (-1)^m B_{n,k}(1!, 2!, \dots, (m-s+1))$
- $\sum_{r=0}^k \sum_{l=0}^n (-1)^{n-l} \binom{n}{l} \binom{l-1}{r-1} \binom{n-l-1}{k-r-1} \frac{l!(n-l)!}{r!(k-r)!}$
- $\frac{n!}{k!} \sum_{r=0}^k \sum_{l=0}^n (-1)^{n-1} \binom{k}{r} \binom{l-1}{r-1} \binom{n-k-1}{k-r-1}$

Which is not easier than the delicate appearance;

PROOFS OF MAIN RESULTS THAT DISCUSS IN CHAPTER 2

Proof of theorem 2.1

Using the lemma 3.1 , 3.2 and 3.3

we derive

$$\begin{aligned} \frac{d^k}{dt^k} \left(\frac{1}{\sqrt{1-t^2}} \right) &= \sum_{t=0}^k \frac{d^l}{du^l} \frac{1}{\sqrt{u}} B_{k,l}(-2t, 0 \dots 0) \\ &= \sum_{t=0}^k \left\langle -\frac{1}{2} \right\rangle \frac{1}{e^{u^{l+\frac{1}{2}}}} (-2)^l B_{k,l}(t, 1, \dots, 0) \\ &= \sum_{t=0}^k \left\langle -\frac{1}{2} \right\rangle \frac{1}{e^{(1-t^2)^{l+1/2}}} 2^l \frac{1}{2^{k-l}} \frac{k!}{l!} \binom{l}{k-l} t^{2t-k} \\ &= \sum_{t=0}^k \frac{(2l-1)!}{2^l} \frac{1}{(1-t^2)^{l+1/2}} 2^l \frac{1}{2^{k-1}} \frac{k!}{l!} \binom{l}{k-l} t^{2t-k} \frac{(2l-1)!!}{2^l} \\ &= \frac{1}{\sqrt{1-t^2}} \frac{k!}{(2t^k)} \sum_{t=0}^k \frac{(2l-1)! 2^l}{l!} \binom{l}{k-l} \frac{t^{2l}}{(1-t^2)^t} \end{aligned}$$

Where $u=u(t)=1-t^2$

Likewise , by the equation (i),(ii) we acquire

$$\begin{aligned} & \frac{d^k}{dt^k} \left(\exp \frac{yt}{1+t} \right) \\ &= \sum_{t=0}^k y^l e^z B_{k,l} \left(\frac{1!}{(1+t)^2}, \frac{-2!}{(1+t)^3}, \dots, \frac{(-1)^{k-l}(k-l+1)!}{(1+t)^{k=l+2}} \right) \\ &= \sum_{t=0}^k y^l e^{\frac{yt}{1+t}} \frac{(-1)^{k+l}}{(1+t)^k} B_{k,l}(1!, 2!, \dots, (k-l+1)! y^l \\ & \sum_{t=0}^k e^{\frac{xt}{1+t}} \frac{(-1)^{k+l}}{(1+t)^k} \sum_{l=0}^k \frac{(-1)^l (k-1)}{l! (l-1)} \frac{y^l}{(1+t)^l} z^l, \end{aligned}$$

Where $v = v(t) = \frac{t}{1+t}$. Using the above two results and applying the Leibniz rule give

$$\begin{aligned} \frac{d^n F(t)}{dt^n} &= \sum_{k=0}^n \binom{n}{k} \frac{d^k}{dt^k} \left(\frac{1}{\sqrt{1-t^2}} \right) \frac{d^{n-k}}{dt^{n-k}} \left(\exp \frac{xt}{1+t} \right) \\ &= \sum_{k=0}^n \binom{n}{k} \frac{1}{\sqrt{1-t^2}} \frac{k!}{(2t)^k} \sum_{l=0}^k \frac{(2l-1)! 2^l}{l!} \binom{l}{k-l} \frac{t^{2l}}{(1-t^2)^l} \\ & \times e^{\frac{yt}{1+t}} \frac{(-1)^{n-k}(n-k)!}{(1+t)^{n-k}} \sum_{l=0}^{n-k} \frac{(-1)^l (n-k-1)}{l! (l-1)} \frac{y^l}{(1+t)^l} \\ &= \frac{e^{\frac{yt}{1+t}}}{\sqrt{1-t^2}} \frac{(-1)^n n!}{(1+t)^n} \sum_{k=0}^n \frac{(-1)^k (1+t)^k}{(2t)^k} \sum_{l=0}^k \frac{(2l-1)! 2^l}{l!} \times \\ & \binom{l}{k-l} \frac{t^{2l}}{(1-t^2)^l} \sum_{m=0}^{n-k} \frac{(-1)^m (n-k-1)}{m! (m-1)} \frac{y^m}{(1+t)^m} \\ &= F(t) \frac{(-1)^n n!}{(1+t)^n} \sum_{k=0}^n \frac{(-1)^k (1+t)^k}{(2t)^k} \sum_{l=0}^k \frac{(2l-1)! 2^l}{l!} \binom{l}{k-l} \times \\ & \frac{t^{2l}}{(1-t^2)^l} \sum_{m=0}^{n-k} \frac{(-1)^m (n-k-1)}{m! (m-1)} \frac{y^m}{(1+t)^m} \\ &= F(t) \frac{(-1)^n n!}{(1+t)^n} \sum_{m=0}^{n-k} \frac{(-1)^m}{m!} \frac{1}{(1+t)^m} \left(\sum_{k=0}^{n-m} \frac{(-1)^k (1+t)^k}{2^k} \times \right. \\ & \left. \binom{n-k-1}{m-1} \left[\frac{1}{t^k} \sum_{t=0}^k \frac{(2l-1)! 2^l}{l!} \binom{l}{k-l} \frac{t^{2l}}{(1-t^2)^l} \right] \right) y^m \end{aligned}$$

Hence proved.

Proof of theorem 2.2

By the equations (i),(ii),(iii)

$$\begin{aligned} \frac{d^n e^{2yt-t^2}}{dt^n} &= \sum_{k=0}^n (e^u)^{(k)} B_{n,k}(2t-2y, -2, 0, \dots, 0) \\ &= \sum_{k=0}^n e^{2yt-t^2} (-2)^k B_{n,k}(t-y, 1, 0, \dots, 0) \\ &= e^{2yt-t^2} \sum_{k=0}^n (-2)^k \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} (t-y)^{2k-n} \\ &= \frac{n!}{2^n} \frac{e^{2yt-t^2}}{(t-y)^n} \sum_{k=0}^n -1^k \frac{2^{2k}}{k!} \binom{k}{n-k} ((t-y)^{2k}) \end{aligned}$$

Where $u=u(t)=2yt-t^2$.

Thus, we obtain

$$\begin{aligned} H_n(y) &= \frac{d^n e^{2yt-t^2}}{dt^n} \\ &= \frac{n!}{2^n} \frac{e^{2yt-t^2}}{(t-y)^n} \sum_{k=0}^n (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} (t-y)^{2k} \\ &= \frac{n!}{2^n} \frac{1}{(-y)^n} \sum_{k=0}^n (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} (-y)^{2k} \\ &= (-1)^n \frac{n!}{2^n} \sum_{k=0}^n (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} y^{2k-n} \end{aligned}$$

Formula (8) follows that

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

$$\text{Or } H_n(y) = e^{\frac{y^2}{2}} (y - d/dy)^n e^{-\frac{y^2}{2}}$$

This completes the proof.

Proof of the theorem 2.3

Differentiating the Hermite Polynomial equation with respect to y on both sides we get

$$2te^{2yt-t^2} = \sum_{n=0}^{\infty} H'_n(y) \frac{t^n}{n!}$$

$$2t \sum_{n=0}^{\infty} H_n(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H'_n(y) \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} 2H_n(y) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H'_n(y) \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} H_{n-1}(y) \frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} H'_n(y) \frac{t^n}{n!}$$

It follows that $H'_0(y)=0$ and (9) valid.

Differentiating the Generating Hermite Polynomial with respect to y we get

$$\frac{4yt}{1+t\sqrt{1-t^2}} \exp \frac{2y^2t}{1+t} = \sum_{n=0}^{\infty} \frac{2H_n(y)H'_n(y)}{2^n} \frac{t^n}{n!}$$

$$\frac{2yt}{1+t} \sum_{n=0}^{\infty} \frac{H_n^2(y)}{2^n} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{2H_n(y)H'_n(y)}{2^n} \frac{t^n}{n!}$$

$$2yt \sum_{n=0}^{\infty} \frac{H_n^2(y)}{2^n} \frac{t^n}{n!} = (1+t) \sum_{n=0}^{\infty} \frac{H_n(y)H'_n(y)}{2^n} \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{2yH_n^2(y)}{2^n} \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{H_n(y)H'_n(y)}{2^n} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{2H_n(y)H'_n(y)}{2^n} \frac{t^{n+1}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{2xH_{n-1}^2(y)}{2^{n-1}} \frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{H_n(y)H'_n(y)}{2^n} \frac{t^n}{n!} +$$

$$\sum_{n=0}^{\infty} \frac{2H_{n-1}(y)H'_{n-1}(y)}{2^{n-1}} \frac{t^n}{(n-1)!}$$

This can be simplified as

$$H_n(y)H'_n(y) = 2n H_{n-1}(y)[2yH_{n-1}(y) - H'_{n-1}(y)]$$

Proof is complete.

CHAPTER-4

Super-Gaussians form of Hermite Polynomials.

From the relationship between Bell Polynomial and Generating Hermite polynomial we get the idea that

$$H_n^{\left(\{s\}_{m \rightarrow \infty}\right)}\left(\left\{\frac{y_s}{s!}\right\}_{m \rightarrow \infty}\right) = y_n(x_1, \dots, \dots, x_n) \quad \dots\dots\dots(1)$$

Where, $H_n(x)$ is Hermite Polynomial and

Bell Polynomial is denoted as $y_n(x_1, \dots, \dots, x_n)$

We notice from Bell Polynomials is

$$\left(\frac{d}{dx}\right)^m e^{g(x)} = (e^{g(x)} B_m(g^{(1)}(x), \dots, \dots, g^{(m)}(x))) \quad \dots\dots\dots(2)$$

Assuming $g(x) = -x^m$ equation 2 for, m is an integer

$$\left(\frac{d}{dx}\right)^n \exp(-x^m) = \exp(-x^m) y_{n,m} \times ((-mx^{m-1}), -m(m-1)x^{m-2}, \dots, \dots, \frac{m!}{(m-1)!} x^{m-n}) \quad \dots(3)$$

For $n > m$ equation 3 vanishes.

When $m = 2$ we can identify that

$$y_{n,2}(-2\{x^{2-s}\}_n) = (-1)^n H_n(x) \quad \dots\dots\dots(4)$$

According to equation to equation (3) we also find

$$\left(\frac{d}{dx}^{n(m-1)}\right) y_{n,m} \left[\left\{\frac{m!}{(m-s)!} x^{m-s}\right\}_n\right] = (-1)^n m^n (nm - n)! \quad \dots\dots\dots(5)$$

In order to represent the beams comprised in optical cavities with mirrors showing Supergaussian reflectivity tapering the following equation from Bell Polynomial, supergaussians have been used in optics.

$$\left\{y \frac{d}{dx} + y^2 - \frac{1}{m^{m-1}} \left(\frac{d}{dy} + y\right)^m\right\} \phi_{n,m}(y) = n\phi_{n,m}(y) \quad \dots(6)$$

Where $\phi_{n,m}(x)$ was introduced as the Hermite oscillator like functions

$$\phi_{n,m}(y) = \frac{1}{\sqrt{m^n n!}} H_{n,m}(y) \exp\left(-\frac{y^2}{2}\right)$$

We introduce the super- Gaussian Hermite functions with the assumption that is even and larger than 2

$$G_{n,m}(y) = -1^n Y_{n,m} \left(-\left\{\frac{m!}{(m-s)!} x^{m-s}\right\}_n\right) \exp\left(-\frac{y^m}{2}\right) \quad \dots\dots(7)$$

$$n \leq m.$$

For $m = 2$ equation (1) is reduced to the usual Gauss- Hermite functions.

Here $G_{n,m}(y)$ are not orthogonal but it can show the overlapping integral

$$P_{n,n',m} = \int_{-\infty}^{+\infty} G_{n,m}(x) G_{n',m}(x) dx \quad \dots\dots(8)$$

For $n > n'(m-1)$ equation is vanishing.

When $n = n'(m-1)$

The equation becomes

$$\begin{aligned} \text{i.e. } -P_{(n,n',m)} &= (-1)^{n+n'} \int_{-\infty}^{+\infty} \exp(-xm) Y_{n,m}(x) Y_{n',m}(x) dx \\ &= \int_{-\infty}^{+\infty} \left[\left(\frac{d}{dx}\right)^n \exp(-x)^m \right] y_{n',m}(x) dx \\ &= (-1)^{n'(m+1)+n} (n'm - n')! \int_{-\infty}^{+\infty} \left[\left(\frac{d}{dx}\right)^{n-n'(m-1)} \exp(-x)^m \right] \\ &= \frac{(-1)^{n'(m+1)+n}}{m} m^{n'} (n'm - n')! \gamma\left(\frac{1}{m}\right) \quad \dots\dots(9) \end{aligned}$$

The integral vanishes when $n > n'(m-1)$

If $n' < n < n'(m-1)$, $p_{n,n',m}$ is non zero bt we get a more complected expression as compare to equation (9)

Clearly when $m = 2$ equation (9) diminishes to the usual orthogonally condition.

Conclusion

The paper illustrates the relationship between generating Hermite polynomial and bell polynomial. A basic information about Super – Gaussian form of Hermite polynomial is also described using this relationship. The use of Hermite polynomial is invariant. We use Hermite polynomial as a set of orthogonal function in quantum mechanics. The properties of Hermite polynomial and bell polynomial are used as a natural tool for super- Gaussian form. In optics, this form plays an important role. Generally, the role of Hermite polynomial is take a important place in mathematics as well as physics.

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