A POPULATION GROWTH MODEL WITH VARYING GROWTH RATES

By

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CERTIFICATE - I

This is to certify that this thesis entitled **"A population growth model with varying growth rates"** submitted for the degree of **Master of Science** in the subject of **Statistics** to Chaudhary Charan Singh Haryana Agricultural University, Hisar, is a bonafide research work carried out by **Mr. Jatinder Kumar** under my supervision and guidance and that no part of this dissertation has been submitted for any other degree.

The assistance and help received during the course of investigation have been fully acknowledged.

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CERTIFICATE - II

This is to certify that this thesis entitled **"A Population growth model with varying growth rates"** submitted by **Mr. Jatinder Kumar** to Chaudhary Charan Singh Haryana Agricultural University, Hisar, in partial fulfillment of the requirements for the degree of **Master of Science** in the subject of **Statistics,** has been approved by the student's Advisory committee after an oral examination on the same.

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Hisar *Jatinder Kumar*

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LIST OF SYMBOLS

Symbol

$X_1(t)$	Population size of group-I at time '*t*'
$X_2(t)$	Population size of group-II at time '*t*'
$X_3(t)$	Population size of group-III at time '*t*'
$\underline{\mathbf{X}}(t)$	Population vector at time '*t*'
b_2	Birth rate of group-II
b_3	Birth rate of group-III
d_1	Death rate of group-I
d_2	Death rate of group-II
d_3	Death rate of group-III
h_1	Harvesting rate of group-I
h_2	Harvesting rate of group-II
h_3	Harvesting rate of group-III
m_1	Migration rate from group-I to group-II
m_2	Migration rate from group-II to group-III
λ	Eigen value
ξ	Eigen vector corresponding to eigen value λ

INTRODUCTION

Growth, an essential and particular characteristic of all living matters has created interest of investigators in the field of biological sciences. The term "population growth" refers to how the number of individuals in a population increases (or decreases) with time. The growth is controlled by the rate at which new individuals are added to the population - the birth rate and the rate at which individuals leave the population - the death and harvesting rate. Since growth is an essential and important phenomenon in biological sciences so investigators are much interested in the study of growth process of all living matters particularly in animal population.

Population growth attracts wide notice because people often feel that it is related to their national survival. Hence whether we are concerned with animal or plant population, bacterial or human population a useful theory of growth process is extremely desirable. Growth is a physiological activity of great practical importance in all living beings including animals. Populations grow according to the number of individuals that are capable of reproduction. At the same time, their growth is limited according to scarcity of land, food or the presence of external forces such as predators. In this module we examine simple differential equations that model populations.

Brody (1945) defined growth as ‘relatively irreversible time change in magnitude in the measured dimension’. More simply growth is an increase in size during a specific period of a part or the whole of whatever is being measured. However, ‘true’ growth may be described as a composite of many diverse physiological and chemical processes. Measurement of growth depends upon the selection of a unit which best describes the type of physiological change being evaluated. Growth can be represented and expressed mathematically in several ways:

(i) As an average rate of percentage increase in the measured dimension per unit of time.

(ii) As a percentage increase of an initial dimension per unit of time.

(iii) As a cumulative increase throughout a prescribed period of time.

Any biological, physical or social phenomenon in terms of growth can best be studied through mathematical modeling. For the formulation of models, some hypothesis is needed which explains the underlying phenomenon. For example, in case of human or animal population the hypothesis may be that increase in population in a unit time is equal to the excess of births over the number of deaths in that unit of time and the number of births and deaths are proportional to the size of the population. The hypothesis gives a mathematical model, which, in turn, shows that the population size is a function of time. The predictions from the model are compared with the size of the actual population in the past. If there is a perfect agreement between the values predicted through the model and the actual data, then the model may be used for future predictions. If on the other hand, the agreement is not good, the hypothesis can be modified in the light of the discrepancies and go on modifying the hypothesis till one finds agreement between observed and predicted values.

An equation or set of equations, which represents the behaviour of a system, is called model. The model helps to include only the important variables to represent a real situation. Thus models simplify the given situation for convenience of analysis and are capable of providing population projections. Their usefulness arises out of the fact that actual counts of the population by census method are available only at infrequent intervals. Whereas for statistical purpose, it is necessary to have an accurate estimate of the population as far as possible in intercensal years and these estimates may be used by vital statisticians. Kendall and Buckland (1960) defined a model as “a formalized expression of a theory or the casual situation which is regarded as having generated observed data. In statistical analysis the model is generally expressed in a mathematical form, but diagrammatic models are also found”.

Population growth is of two types viz

(1) Exponential growth and

(2) Logistic growth

Exponential Growth:

If a population has a constant birth rate throughout time and is never limited by food or disease, it has what is known as exponential growth. With exponential growth the birth rate alone controls how fast (or slow) the population grows.

Logistic Growth:

In a population showing exponential growth the individuals are not limited by food or disease. However, in most real situations both food and disease become important as population increases beyond limits. There is an upper limit to the number of individuals the environment can support. Ecologists refer to this as the "carrying capacity" of the environment. Populations in this kind of environment show what is known as logistic growth.

(2).5 Population Growth Models:

Population growth models are widely used in various disciplines to study the growth and structure of the population at different times. For example, actuaries and demographers are interested in models of growth for human population for predicting expected duration of life at various ages and for estimating future population trends (Pollard, 1973). Bio-economists are interested in models of growth for populations of sheep, fish, forests and other renewable resources for the sake of their optimal exploitation (Clark, 1976). Medical scientists and biologists are interested in models of growth of bacterial population for the sake of controlling diseases and epidemics and for genetic studies. Chemical engineers are interested in the growth models for the study of micro organisms.

Growth of a population depends mainly on three factors birth, death and migration and can not be determined by any one factor alone. It can be measured in three ways:

(a) as an observed change in the total number of people,

(b) as a current process of replacement, or

(c) as a change in the size of some sections of the population, such as age groups or livelihood classes.

Population growth models are of three types, as given below:

(i) Discrete-time discrete-age-scale models

(ii) Continuous-time discrete-age-scale models

(iii) Continuous-time continuous-age-scale models

In the present study, continuous-time discrete-age-scale model has been considered as cattle growth can best be described by this model.

(iii).6 Effect of Age-Factor:

The actual population structure may be more complex in the sense that its vital rates do not remain fixed, and therefore, give a less definite pattern to its structure. Its composition is related to past history of growth and helps to determine the future capacity of growth.

Growth is most evident in the reproductive stage in proportion to the total population. If the population in reproductive age group is more than the other age groups then the birth rate will be higher and death rate will be lower which results in the increase in the population size very speedily and vice-versa. In a stable population, growth depends on a set of age specific birth-rates and age specific death rates. Here an age-dependent model is developed where the growth depends upon the parameters of pre-reproductive, more-reproductive and less-reproductive age groups

(iii).7 Effect of Harvesting:

Harvesting may be defined as the removal of some of the individuals from the reproduction process at different times. A population of living beings always exhibits some growth pattern and tends to grow beyond limits. A stage may come that it becomes unmanageable. Owing to the limited availability of resources, there is a need to keep constant or some manageable population size. This could be done by harvesting a proportion of the population as and when needed. Thus a balance may be maintained between birth on one side and death and harvesting on the other.

Harvesting rates in pre-reproductive, more-reproductive and less-reproductive groups of animal population may be determined to ensure that the population of any age group does not die out completely. The growth models after incorporating the effect of harvesting may provide answer to questions such as:

(1) What would be the effect of given harvesting rate on the growth and structure of a population or how would a given harvesting rate affect the growth and structure?

(2) How much harvesting rate is required to maintain a stable or desired population size?

The process of harvesting results in stable populations of deer, fish and other game animals and can be used to reduce the number of animals who needlessly die from starvation or other natural causes. On the other hand, unregulated harvesting can lead a population to the brink of extinction, as is evidenced by well-known examples such as the North American Bison (*Bison bison*) and several populations of whales.

(2).8 Effect of Migration:

Migration may be of two types – internal and external migrations. Internal migration does not affect the total size of population but affect the population in various age groups. Therefore, in addition to birth, death and harvesting the component of migration has significant impact in changing the composition of various age groups in the population. Thus the study on the effect of migration in population growth models is necessary.

In present study the word migration is used in its narrow sense, that is only those animals are called migrants which come from the previous age-groups.

(2).9 Outline of the Present Study:

Most of the population models have been developed by considering single birth rate either in one reproductive group or in the population as a whole. Since the birth and death rates of different age groups of animal populations are varying, so the study of population growth considering different reproductive age groups becomes necessary. It, therefore, becomes more important to study the age distribution of the population after a lapse of time given the initial population distribution. The model, therefore, must ensure that birth and death rates are the functions of age distribution of the population. Moreover a realistic model must account for in addition to birth and death, the other growth parameters such as age distribution, harvesting and migration etc. as these are responsible for changing the structure of the population.

To account for the above phenomena the attention in this work has been focused to develop a continuous-time discrete-age-scale population model taking into account the different birth, death and migration parameters of different age groups with different harvesting rates. The behaviour of the model has been studied and the model has been illustrated using the data on cattle population. More specifically, the objectives of study are:

(1) To develop a population growth model with different reproductive groups and illustrate with suitable example.

(2) To obtain the conditions of stability, growth and extinction of the population

Chapter-2 gives a review of literature of work related to the objectives. Chapter-3 deals with the formulation of continuous-time discrete-age-scale model with varying growth parameters. The conditions of growth, extinction and stability have been found. Some special cases under certain assumptions have also been discussed.

Chapter-4 is devoted to the application of the model to the study of cattle growth. The cattle projections under different conditions of growth have been obtained.

REVIEW OF LITERATURE

(2).1 Population growth models:

The first fundamental model of population growth was given by Malthus (1798), who found that population growth is an explosive process and represented it by a continuous function.

$$\underline{N}(t) = \underline{N}(0)\, e^{at}$$

Thus the population size $\underline{N}(t)$ depends on 'a', the difference between the intrinsic birth and death rates. The model did not attract much attention for a long time and the formulation of population growth models have come into prominence in the mid of twentieth century. Sharp and Lotka (1911) were the first to formulate continuous-time and continuous-age-scale models. These workers obtained the population projections in terms of an integral equation. Feller (1941) continued this work and employed the Laplace transformation to solve the integral equation. Later the model was solved by finding the largest root of the integral equation.

Pearl and Reed (1920) described that an empirical equation can determine the normal rate of population growth. The empirical equation for population growth has distinct and considerable usefulness. They described that population can be estimated in intercensal years by fitting an appropriate curve to the available data. Rhodes (1940) studied the growth of female population and found that the fertility is zero when the age of female is either less than the beginning of reproductive period or greater than the end of this period.

The systematic formulation of discrete-time and discrete-age-scale population models was done initially by Bernardelli (1941), Lewis (1942) and Leslie (1945). Later the model was modified by Leslie (1959) to include the effect of overcrowding or density dependence. These workers introduced matrix theory in population mathematics and formulated the model in terms of a system of difference equations by dividing the population into different age groups of equal duration. Let initial population vector be denoted by

$$\underline{\boldsymbol{X}}(0) = [X_1(0), X_2(0), X_3(0), \ldots, X_n(0)]'$$

As time progresses, the size of each group changes because of three biological processes namely birth, deaths and aging. By describing these processes Leslie projected the population vector for the future.

Let $f_1, f_2, \ldots, f_n$ be the expected number of offspring per individual in different age groups and p_i (i = 1, 2, ..., n-1) be the fraction of individuals that survive in the i^{th} group and migrate to $(i+1)^{st}$ age group. Then the population projection at time $(t+1)$ is given in the matrix notation as follows:

$$\begin{pmatrix} X_1(t+1) \\ X_2(t+1) \\ X_3(t+1) \\ \vdots \\ \vdots \\ X_n(t+1) \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_2(t) \\ \vdots \\ \vdots \\ X_n(t) \end{pmatrix}$$

i.e. $\underline{\mathbf{X}}(t+1) = \underset{\sim}{\mathbf{M}}\, \underline{\mathbf{X}}(t)$,

By recurrence relation it implies that

$$\underline{\mathbf{X}}(t) = \underset{\sim}{\mathbf{M}}^{t}\, \underline{\mathbf{X}}(0)$$

where matrix $\underset{\sim}{\mathbf{M}}$ is known as Leslie matrix.

Thus by knowing $\underline{X}(O)$ and matrix $\underset{\sim}{M}$, population structure at any time can be obtained. This model may be used to study the growth of human or animal populations.

Kapur (1978) developed a continuous-time discrete-age-scale population model showing that, in general, a stable population structure exists for which the population of female groups depend on parameters of female groups only while the population of male group depends on parameters of both male and female groups taking into account the difference in birth, death and migration parameters of the two sexes.

Kapur (1979a) further obtained conditions for the invariance of the proportion of pre-reproductive children, reproductive adults and post-reproductive persons in a growing population. He presented a unique reproductive structure that does not change with time. He also studied the effect of changes in births, deaths and internal migration parameters on the reproductive structure of the population. Kapur and Khan (1979, 1980) studied various population models and suggested that logistic model is a realistic one for explaining the growth of a population. Kapur (1985a) also examined the stability of a population model in which the growth rate at any time depends both on the current population size as well as on the population size at an earlier instant of time. Prajneshu (1998) proposed a non-linear deterministic model for aphid abundance and used it to approximate the peak aphid count and final cumulative aphid count. The per-capita death in his model was proportion to the cumulative population size and the solution was the symmetric analytic function. El Ghordaf (2004) proposed a mathematical model to study the urban dynamics, adjusting to a ecological model given by Lotka and Voltara (1939). Two first order non-linear ordinary differential equations were used to predict the population size.

There may be some uncertainty associated with the size of the population estimated at any time and the result is not unique and tends to fluctuate in a random manner about some mean size and variance. Therefore, some stochastic population growth models have been developed by the researchers. Renshaw (1986) presented a survey of the deterministic and stochastic models in the area of population dynamics. Vlad (1989) proposed a non-linear age-dependent model for age-structured sexual population based on the assumption that birth function depends on ages of two parents and the death function in additive terms depending on age, sex and time of evolution of population densities, respectively. Chiu (1990) applied some age dependent population growth models to compute the growth of a human population.

The literature on two sex stochastic models of population growth starts with article by Kendall (1949). He outlined some of the analytic difficulties in the study of the stochastic aspects of two-sex population growth models. Goodman (1953) considered a stochastic process which is a probabilistic analogue of the deterministic model describing a population where females are marriage dominant i.e. where the birth rate depends on the female population size. Bartlett, Grower and Leslie (1960) developed a stochastic logistic model and obtained the equilibrium distribution of the population size numerically and developed elegant approximations for the mean, variance and skewness of the equilibrium distribution.

Pollard (1966) developed a stochastic model based on the assumptions similar to those made by Leslie in his deterministic analogue. Goodman (1968) developed stochastic models for describing the changes with time in the number of females and males in the various age groups of the population. He gave the method for calculating the expected values, variances and co-variances of the number of males and females in each age groups at any time. Goodman (1969) further gave the method for analyzing the growth of a two-sex population in which the birth and death rates may depend upon several factors such as sex, age, parity etc.

Sykes (1969) showed that in discrete stable population theory the matrix of birth and survival rates has a positive maximal eigen value associated with which is a positive eigen vector that leads to the population structure. Bosch (1971) modeled the growth and survival of redwoods using matrix method and drew a meaningful conclusion in the growth of redwoods for maximizing the profit.

Matis *et. al.* (1994) developed a model for the spread of insect populations using stochastic birth-death-migration process and gave statistical methodology for predicting population size. Matis (2005) further developed a stochastic population growth model for the black-margined pecan aphid.

(2).2 Models with Harvesting:

The natural populations whether of plants or animals have a tendency to grow beyond limits. It becomes unmanageable in view of limited resources and an appropriate action would be to consider harvesting. Several researchers proposed models to study various applied problems on harvesting. Douglas (1955) estimated the size of an animal population using binomial model in the study of wild life on the basis of the changes in sex ratio following harvesting in one sex only. Watt (1955) had investigated the effect of harvesting in a population of flour beetle *Tribolium confusum* removing adults, pupae and larvae. He observed that harvesting increases the numerical productivity of the population.

Harvesting has been considered in the Leslie model by many workers for different purposes. Lefkovitch (1965, 1967) studied the effect of harvesting in different stages of a pest population. Beddington and Taylor (1973) developed a model for maximizing the sustainable yield from a population of constant size using simple extension of the Leslie model. He assumed the complete removal of one age group and partial removal of second age group and demonstrated its application in demography. Doubleday (1975) optimized the harvesting strategy for animal population.

Rorres and Fair (1975) considered the objective of maximizing the sustainable yield subject to the population vector satisfying a general linear constraint. Beddington and May (1977) obtained the relative variance of the yield as a point of maximum sustained yield by harvesting an animal population in a randomly fluctuating manner. Kapur (1978, 1979a, 1979b, 1980), in a series of papers proposed various harvesting models. He proposed discrete and continuous time models and also studied the condition for the stability of population. Reed (1980) proposed an age-specific harvesting policy for maximizing equilibrium yield in discrete time, self-regenerating, age-structured population model. Kapur (1985b) developed an optimal harvesting strategy for animal populations. He considered harvesting in one-sex population and gave a policy for optimizing the total biomass. Rawal *et. al.* (1994) studied the growth of *Shaiwal* cows from thirty nine years data where adult cows are being harvested and heifer calvings are joining the milking herd. They found that such information is required for the formulation and operation of the breeding plan for genetic improvement.

Goyal *et. al.* (1995) developed a two-sex model to study the cattle growth (*Hariana* breed) with harvesting, where the birth in small interval (t, $t + dt$) is proportional to the size of the female segment at time 't'. Further Goyal and Gupta (1997) studied cattle growth through a two-sex harvesting model under the assumption that the growth in cattle takes place in proportion to the geometric average of the male and female cattle population present in the herd.

Giorgio and Serena (1997) gave an optimal harvesting policy for age dependent female population with two-life stages termed as eggs and adults. They considered that certain fraction of individuals of fixed ages is being harvested per unit of time in both stages to bring the population to an equilibrium level using a continuous-time linear model. Jenson (2000) proposed a two-sex density dependent matrix model to study the effect of harvesting on a white tailed deer population. He observed that if only males were harvested, yield increased asymptotically with increased harvest effort and there was no maximum sustainable yield and if only females were harvested, the harvest increased to a maximum and then decreased. Finally he concluded that the best harvesting policy for maximum yield is when both males and females are harvested. Suman (2005) studied the cattle growth of *Shaiwal* breed through a two-sex age-dependent population growth model with and without harvesting.

For detailed discussion and comprehensive bibliography in this field, the references may be made to the books by Keyfitz (1968), Usher (1972), Pollard (1973), Clark (1976), Pielou (1977) and Sivamurthy (1982). The book by Kapur (1985) gives various models and their applications in population growth problems.

AN AGE DEPENDENT POPULATION GROWTH MODEL WITH HARVESTING

(2).1 Introduction:

Population growth models play an important role in a variety of growth phenomena in various fields. After the famous work of Leslie (1945), Lewis (1942) Bernardelli (1941), and Lotka (1939), these models have been modified several times to explain appropriate growth situations. Such modifications to construct models with appropriate variation in harvesting rate at various age groups have been given by Kapur (1978), Doubleday (1975), Lefkovitch (1966) and Williamson (1967).

In this chapter an age-dependent model has been proposed where birth, death and harvesting rates are the function of the parameter of three population age groups viz. group-I (pre-reproductive), group-II (more-reproductive) and group-III (less-reproductive). The conditions for stability, growth and extinction have been obtained. The model has also been derived for the following special cases:

- Population structure considering birth only in IInd age-group
- Harvesting for stable population structure
- Population structure without harvesting

(2)2 Formulation of the model

Let $\underline{\boldsymbol{X}}(t) = [X_1(t), X_2(t), X_3(t)]'$ be an age-dependent population vector at any time '*t*', where the components representing the populations of group-I, group-II and group-III, respectively. The members of all the age-groups may grow (by birth and migration from previous age-group) and decay (by death, harvesting and migration to next age-group) and are independent of each other. Let birth, death, and harvesting rates be $(0, b_2, b_3)$, (d_1, d_2, d_3), (h_1, h_2, h_3) in Ist, IInd and IIIrd age groups respectively. m_1 is the rate at which the members of Ist age-group migrate to IInd age-group and m_2 is the rate at which the members of IInd age-group migrate to IIIrd age-group on maturity and survival respectively. It is assumed that

(i) Birth in small interval $(t, t + dt)$ occurs in proportional to size of groups II and III.

(ii) Death in any group occurs in proportional to size of the group.

(iii) Harvesting in any group is allowed in proportional to its size.

We now prove the following theorem which leads to the population structure:

Theorem 3.1:

For an age-dependent population growth model with the assumptions (i – iii), the population structure at any time '*t*' is given by:

$$\underline{X}(t) = \underset{\sim}{Y}\, e^{\underset{\sim}{\Lambda} t}\, \underset{\sim}{Y}^{-1}\, \underline{X}(0)$$

$\underset{\sim}{\Lambda}$ = diag $(\lambda_1, \lambda_2, \lambda_3)$, where λ_i (i = 1, 2, 3) denote eigen values of matrix $\boldsymbol{K}$.

and $\underset{\sim}{Y}$ denotes the matrix of right eigen vectors corresponding to the eigen values of matrix $\underset{\sim}{K}$.

$$\underset{\sim}{\boldsymbol{K}} = \begin{matrix} -(d_1+m_1+h_1) & b_2 & b_3 \\ m_1 & -(d_2+m_2+h_2) & 0 \\ 0 & m_2 & -(d_3+h_3) \end{matrix}$$

Proof:

The rate of change in the size of population of all three age-groups is given by following linear differential equations

$$\frac{dX_1}{dt} = (b_2X_2 + b_3X_3) - (d_1+m_1+h_1)\,X_1$$

$$\frac{dX_2}{dt} = m_1X_1 - (d_2+m_2+h_2)\,X_2$$

$$\frac{dX_3}{dt} = m_2X_2 - (d_3+h_3)\,X_3 \qquad \ldots (3.2.1)$$

The first equation of (3.2.1), for example, says that the change in the size of the I^{st} age-group is equal to,

(i) The number of births produced by the individuals of I^{st} and II^{nd} groups

(ii) Minus the number of deaths, number harvested from I^{st} age-group and migrations to II^{nd} age-group.

The set of equations (3.2.1) can be written in matrix form as:

$$\frac{d\underline{\boldsymbol{X}}}{dt} = \underset{\sim}{\boldsymbol{K}}\,\underline{\boldsymbol{X}} \qquad \ldots (3.2.2)$$

For the sake of simplification, put

$a = (d_1+m_1+h_1)$

$b = (d_2+m_2+h_2)$

$c = (d_3+h_3)$

and we get,

$$\underset{\sim}{\boldsymbol{K}} = \begin{vmatrix} -a & b_2 & b_3 \\ m_1 & -b & 0 \\ 0 & m_2 & -c \end{vmatrix}$$

The eigen values of matrix $\underset{\sim}{\boldsymbol{K}}$ are given by

$$| \underset{\sim}{\boldsymbol{K}} - \lambda \underset{\sim}{\boldsymbol{I}} | = 0 \qquad \dots (3.2.3)$$

Thus we get,

$$\begin{vmatrix} -(a+\lambda) & b_2 & b_3 \\ m_1 & -(b+\lambda) & 0 \\ 0 & m_2 & -(c+\lambda) \end{vmatrix} = 0 \qquad \dots (3.2.4)$$

Which on simplification gives,

$$\lambda^3 + \lambda^2 (a+b+c) + \lambda (ab+bc+ca-m_1 b_2)$$

$$+ (abc - m_1 b_2 c - m_1 m_2 b_3) = 0$$

Putting the values of a, b and c, we get

$$\lambda^3 + \lambda^2(d_1+m_1+h_1 + d_2+m_2+h_2 + d_3+h_3) + \lambda[(d_1+m_1+h_1)(d_2+m_2+h_2)$$
$$+ (d_2+m_2+h_2)(d_3+h_3) + (d_3+h_3)(d_1+m_1+h_1) - (m_1 b_2)]$$
$$+ [(d_1+m_1+h_1)(d_2+m_2+h_2)(d_3+h_3) - m_1 b_2(d_3+h_3) - m_1 m_2 b_3] = 0 \quad \ldots (3.2.5)$$

On solving above cubic equation, we can obtain three eigen values viz. λ_1, λ_2 and λ_3. Since it is difficult to obtain the solution for equation (3.2.5), theoretically so eigen values can be obtained empirically as illustrated in the Appendix.

The right eigen vectors corresponding to eigen values of matrix $\underset{\sim}{\boldsymbol{K}}$ are given by $(\underset{\sim}{\boldsymbol{K}} - \lambda \underset{\sim}{\boldsymbol{I}})\, \boldsymbol{\xi} = 0$, where $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]'$ is an eigen vector.

or
$$\begin{bmatrix} -(a+\lambda) & b_2 & b_3 \\ m_1 & -(b+\lambda) & 0 \\ 0 & m_2 & -(c+\lambda) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = 0 \quad \ldots (3.2.6)$$

Which gives:

$$-(a+\lambda)\,\xi_1 + b_2\,\xi_2 + b_3\,\xi_3 = 0$$

$$m1\,\xi_1 - (b+\lambda)\,\xi_2 = 0$$

$$m2\,\xi_2 - (c+\lambda)\,\xi_3 = 0 \qquad \ldots(3.2.7)$$

From above equations we get following eigen vector, corresponding to eigen value λ

$$\begin{matrix} 1 \\ \dfrac{m_1}{(b+\lambda)} \\ \dfrac{m_1 m_2}{(b+\lambda)(c+\lambda)} \end{matrix} \qquad \ldots(3.2.8)$$

Hence, the matrix $\underset{\sim}{\boldsymbol{Y}}$ whose columns are the right eigen vectors corresponding to the eigen values λ_1, λ_2 and λ_3 can be written as:

$$\underset{\sim}{\boldsymbol{Y}} = \begin{matrix} 1 & 1 & 1 \\ \dfrac{m_1}{(b+\lambda_1)} & \dfrac{m_1}{(b+\lambda_2)} & \dfrac{m_1}{(b+\lambda_3)} \\ \dfrac{m_1 m_2}{(b+\lambda_1)(c+\lambda_1)} & \dfrac{m_1 m_2}{(b+\lambda_2)(c+\lambda_2)} & \dfrac{m_1 m_2}{(b+\lambda_3)(c+\lambda_3)} \end{matrix} \qquad \ldots(3.2.9)$$

Now by diagonalization property of matrix (Searle and Hausman, 1970) the matrix $\underset{\sim}{\boldsymbol{K}}$ can be written as:

$$\underset{\sim}{\boldsymbol{K}} = \underset{\sim}{\boldsymbol{Y}}\, \underset{\sim}{\boldsymbol{\Lambda}}\, \underset{\sim}{\boldsymbol{Y}}^{-1}$$

…(3.2.10)

Hence from (3.2.2), we get

$$\frac{d\underline{\boldsymbol{X}}}{dt} = \underset{\sim}{\boldsymbol{Y}}\, \boldsymbol{\Lambda}\, \underset{\sim}{\boldsymbol{Y}}^{-1}\, \underline{\boldsymbol{X}}(t)$$

On solving, we get

$$\underline{\boldsymbol{X}}(t) = \underset{\sim}{\boldsymbol{Y}}\, e^{\boldsymbol{\Lambda} t}\, \underset{\sim}{\boldsymbol{Y}}^{-1}\, \underline{\boldsymbol{X}}(0) \qquad \text{…(3.2.11)}$$

Which gives population structure at any time '*t*'.

Hence Proved.

Let $u_{1i} = 1$

$$u_{2i} \quad \frac{m_1}{(b + \lambda_i)}$$

$$u_{3i} = \frac{m_1 m_2}{(b + \lambda_i)(c + \lambda_i)} \qquad \{\text{for } i = 1, 2, 3\}$$

Hence (3.2.9) becomes,

$$Y = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}, \text{ and let } \quad Y^{-1} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$$

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$$

Putting in (3.2.11) we get,

$$\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} X_1(0) \\ X_2(0) \\ X_3(0) \end{pmatrix}$$

...(3.2.12)

Solving (3.2.12) we get following system of equations:

$$X_1(t) = [\, u_{11}v_{11}X_1(0) + u_{11}\, v_{12}\, X_2(0) + u_{11}v_{13}X_3(0)\,]\, e^{\lambda_1 t}$$

$$+ [\, u_{12}v_{21}X_1(0) + u_{12}\, v_{22}\, X_2(0) + u_{12}v_{23}X_3(0)\,]\, e^{\lambda_2 t}$$

$$+ [\, u_{13}v_{31}X_1(0) + u_{13}\, v_{32}\, X_2(0) + u_{13}v_{33}X_3(0) \,]\, e^{\lambda 3t}$$

$$X_2(t) = [\, u_{21}v_{11}X_1(0) + u_{21}\, v_{12}\, X_2(0) + u_{21}v_{13}X_3(0) \,]\, e^{\lambda 1t}$$

$$+ [\, u_{22}v_{21}X_1(0) + u_{22}\, v_{22}\, X_2(0) + u_{22}v_{23}X_3(0) \,]\, e^{\lambda 2t}$$

$$+ [\, u_{23}v_{31}X_1(0) + u_{23}\, v_{32}\, X_2(0) + u_{23}v_{33}X_3(0) \,]\, e^{\lambda 3t}$$

$$X_3(t) = [\, u_{31}v_{11}X_1(0) + u_{31}\, v_{12}\, X_2(0) + u_{31}v_{13}X_3(0) \,]\, e^{\lambda 1t}$$

$$+ [\, u_{32}v_{21}X_1(0) + u_{32}\, v_{22}\, X_2(0) + u_{32}v_{23}X_3(0) \,]\, e^{\lambda 2t}$$

$$+ [\, u_{33}v_{31}X_1(0) + u_{33}\, v_{32}\, X_2(0) + u_{33}v_{33}X_3(0) \,]\, e^{\lambda 3t}$$

…(3.2.13)

Which takes the form,

$$X_1(t) = c_{11}\, e^{\lambda 1t} + c_{12}\, e^{\lambda 2t} + c_{13}\, e^{\lambda 3t}$$

$$X_2(t) = c_{21}\, e^{\lambda 1t} + c_{22}\, e^{\lambda 2t} + c_{23}\, e^{\lambda 3t}$$

$$X_3(t) = c_{31}\, e^{\lambda 1t} + c_{32}\, e^{\lambda 2t} + c_{33}\, e^{\lambda 3t}$$

…(3.2.14)

Hence the predicted population of group-I, group-II and group-III at any time '*t*' is given by equation

$$X_i(t) = c_{i1}\, e^{\lambda 1t} + c_{i2}\, e^{\lambda 2t} + c_{i3}\, e^{\lambda 3t}$$

...(3.2.15)

where c_{ij} $\{i,j = 1, 2, 3\}$ are finite quantities.

After a long time i.e. as $t \to \infty$, the behaviour of $X_i(t)$; ($i = 1, 2, 3$) will be determined by the largest eigen value (λ_1). If λ_1 is greater than zero, the population of all age-groups will grow; if it is less than zero, then all population groups will tend to extinction; and if it is equal to zero, the population of all age-groups will approach to a steady state value.

(ii)21 Conditions for growth, extinction and stability:

The eigen values of matrix $\underset{\sim}{\boldsymbol{K}}$ are given by (3.2.5). In general, λ_2 and λ_3 are negative and λ_1, which may be positive, zero or negative according as

$$m_1 b_2(d_3+h_3) + m_1 m_2 b_3 - (d_1+m_1+h_1)\,(d_2+m_2+h_2)\,(d_3+h_3) \geq 0$$

...(3.2.16)

We now derive the conditions for growth, extinction and stability of the population as follows:

(i) Growth

If $m_1 b_2 (d_3+h_3) + m_1 m_2 b_3 > (d_1+m_1+h_1)(d_2+m_2+h_2)(d_3+h_3)$

...(3.2.17)

Then λ_1 will be positive and population of all age-groups ultimately grows exponentially.

(ii) Extinction:

If $m_1 b_2 (d_3+h_3) + m_1 m_2 b_3 < (d_1+m_1+h_1)(d_2+m_2+h_2)(d_3+h_3)$

...(3.2.18)

Then λ_1 will be less than zero and thus whole of the population will die out in due course of time.

(iii) Stability:

If $m_1 b_2 (d_3+h_3) + m_1 m_2 b_3 = (d_1+m_1+h_1)(d_2+m_2+h_2)(d_3+h_3)$

...(3.2.19)

Then λ_1 will be zero and population will approach to its steady state value as $t \quad \infty$

(ii).3 Special Cases:

3.3.1 Population structure considering birth only in IInd age-group

The model has been proposed with single reproductive group only i.e. with pre-reproductive, reproductive and post-reproductive groups. So b_3 will be equal to zero.

Hence the matrix $\underset{\sim}{\boldsymbol{K}}$ reduces to,

$$\underset{\sim}{\boldsymbol{K}} = \begin{matrix} -(d_1+m_1+h_1) & b_2 & 0 \\ m_1 & -(d_2+m_2+h_2) & 0 \\ 0 & m_2 & -(d_3+h_3) \end{matrix}$$

...(3.3.1)

The above matrix is same as obtained by Kapur (1979a) for a population with only one reproductive group. Thus the model of Kapur is a special case of (3.2.11)

(ii).3.2 Harvesting for stable population structure

Sometimes, researcher requires to find the condition under which a stable population structure exists. This condition will be in terms of decision about the harvesting rate. Here a uniform harvesting rate is determined to maintain a stable population structure

For the existence of stable population structure the uniform harvesting rate $\bar{h}$ is the solution of the equation (3.2.19)

Let $h_1 = h_2 = h_3 = \bar{h}$

For stable population vector, we have

$$(d_1+m_1+\bar{h})\,(d_2+m_2+\bar{h})\,(d_3+\bar{h}) - m_1b_2(d_3+\bar{h}) - m_1m_2b_3 = 0$$

...(3.3.2)

For simplification, let

$= d^1+m^1$

$\beta = d_2+m_2$

$= d_3$

- $\bar{h}^3 + \bar{h}^2 (\; + \beta + \;) + \bar{h} (\; \beta + \beta \; + \; - m1_{b}2)$

$$+ (\; \beta \; - m_1b2 \; - m^1{}_{m}2b3) = 0 \quad ...(3.3.3)$$

On solving the above equation, we get one positive and two negative values for $\bar{h}$. Harvesting can never be negative so we select the positive value (largest root of 3.3.3) as uniform harvesting rate in all the three age-groups, and we obtain the characteristic equation as follows

Considering $h_1 = h_2 = h_3 = \bar{h}$ in (3.2.3), we get

$$\lambda^3 + \lambda^2 (a + b + c) + \lambda (ab + bc + ca + m_1 b_2) = 0$$

Where, $a = (d_1 + m_1 + \bar{h})$

$b = (d_2 + m_2 + \bar{h})$

$c = (d_3 + \bar{h})$

or $\lambda [\lambda^2 + \lambda (a + b + c) + (ab + bc + ca + m_1 b_2)] = 0$ ...(3.3.4)

which on solving give eigen values as,

$\lambda_1 = 0$

$\lambda_2 = -\frac{1}{2} (a + b + c) - \frac{1}{2} \{ (a + b + c)^2 - 4 (ab + bc + ca - m_1 b_2)\}^{1/2}$

$\lambda_3 = -\frac{1}{2} (a + b + c) + \frac{1}{2} \{ (a + b + c)^2 - 4 (ab + bc + ca - m_1 b_2)\}^{1/2}$

...(3.3.5)

Since $\lambda_1 = 0$, therefore, the population will approach to a stable steady state.

3.3.3 Population structure without harvesting

If $h_1 = h_2 = h_3 = 0$ then in the absence of harvesting the population will go on increasing and values of a, b and c in the characteristic equation (3.2.5) reduces to,

$a = d_1 + m_1$

$b = d_2 + m_2$

$c = d_3$

We can solve the characteristic equation for the largest eigen value and population structure can be predicted at any time ‘*t*’.

APPLICATION TO
A CATTLE POPULATION

Applications of population growth models are available in literature. The examples may be found in the field of animal ecology, poultry, entomology, genetics demography etc. Several researchers in particular Jensen (2000), Reed (1980), Kapur (1978), Rorres (1975), Pollard (1973), Usher (1972), Lefkovitch (1965) etc. have applied these models in various growth problems. Apart from this certain important problems related to animal growth can be studied by harvesting a growing population at various stages as per need of researcher. For example, an animal scientist may wish to investigate the structure of an animal population where in addition to birth and death, a certain proportion of animals have to be harvested from different age-groups at different times. The harvesting may be done to maintain some pre-specified population size due to limited resource availability or for more profitability.

The model developed in Chapter-3 provides a ground to study population growth problems with and without harvesting. In the present Chapter, an application of the model has been illustrated on the growth of a cattle population. An attempt has been made to study the age-dependent growth structure of cattle. The population projections have been obtained under different assumptions with respect to parameters of different age-groups and harvesting.

The data on the growth of a cattle population were collected from Animal Farm of CCS Haryana Agricultural University, Hisar. Natural birth and deaths occurred in the population from time to time for three age-groups, viz, group-I (upto 3 years), group-II (above 3 years but upto 8 years) and group-III (above 8 years). The yearly data for 12 years (1994 to 2005) in respect of birth, death and harvesting of cattle in all the age-groups are given in Table-4.1. Number of cattle migrated from group-I to group-II and from group-II to group-III has also been shown in the Table. Further the sale, disposal and emigration of cattle were considered as harvesting to explain the model developed in Chapter-3.

The population projections of cattle, up to year 2010 have been obtained in each case as mentioned above. The projections have been represented in both tabular and graphical form.

The year 1994 was considered as the initial period (i.e. $t = 0$). Initially, the cattle population consisted of 142, 219 and 71 cattle in group–I, II and III respectively. At the end of one year it grows to 160, 205 and 75 in three groups respectively.

(ii)1 Application of the model:

The application of the model has been discussed under the following cases:

(1) Growth of cattle population with two reproductive groups.

(2) Harvesting for stable population structure.

(3) Growth without harvesting.

By making use of Table–4.1 the average annual growth rates (Keyfitz, 1968) of cattle population were obtained as follows;

Birth rate for group-II (b_2) = 885/(225x12) = 0.329

Birth rate for group-III (b_3) = 235/(97x12) = 0.202

Death rate for group-I (d_1) = 166/(199x12) = 0.069

Death rate for group-II (d_2) = 31/(225x12) = 0.012

Death rate for group-III (d_3) = 20/(97x12) = 0.017

Table – 4.1 Actual number of births, deaths, harvesting and migrations in cattle in three age-groups

Year	Number of cattle			Number of births		Number of deaths			Number harvested			Number of migrations	
	group-I	group-II	group-III	group-II	group-III	group-I	group-II	group-III	group-I	group-II	group-III	From group-I to II	From group-II to III
1994	142	219	71	67	15	23	5	4	17	13	12	24	20
1995	160	205	75	57	12	18	3	1	31	9	11	23	17
1996	157	199	80	73	14	20	0	1	40	7	15	18	10
1997	166	200	74	87	28	22	3	2	14	10	9	41	13
1998	204	215	76	76	21	19	2	0	43	5	3	42	18
1999	197	232	91	94	22	20	4	2	21	33	35	71	49
2000	201	217	103	68	16	14	5	1	30	9	28	52	34
2001	189	221	108	57	18	12	2	4	38	23	16	11	12
2002	203	195	100	69	23	7	0	1	26	18	24	68	26
2003	194	219	101	77	24	3	1	1	26	23	9	59	21
2004	207	233	112	79	20	5	2	1	54	35	26	52	31
2005	195	217	116	81	22	3	4	2	38	12	16	40	7
Total	2215	2572	1107	885	235	116	31	20	378	197	204	501	258

Harvesting rate for group-I (h_1) = 378/(199x12) = 0.158

Harvesting rate for group-II (h_2) = 197/(225x12) = 0.073

Harvesting rate for group-III (h_3) = 204/(97x12) = 0.175

Migration rate from group-I to II (m_1) = 501/(199x12) = 0.210

Migration rate from group-II to III (m_2) = 258/(225x12)= 0.096

On substituting these values in equation (3.2.5) and solving for λ, we get:

λ_1 = 0.015

λ_2 = -0.583

λ_3 = -0.241

Using (3.2.9), we get

$$\underset{\sim}{\boldsymbol{Y}} = \begin{matrix} 1.000 & 1.000 & 1.000 \\ 1.050 & -0.519 & -3.500 \\ 0.500 & 0.123 & 6.333 \end{matrix}$$

and by equation (3.2.14) we get the projected number of cattle in each age group upto year 2010. The observed and projected cattle population in each group along with relative error are shown in Table – 4.2. It clearly shows that the population of group-I and group-III increases with time as expected (since λ_1 >

0) and the population of group-II is almost constant from 1994 to 2003 and increases from 2004 to 2010 at a constant rate.

Table – 4.2 Observed and projected cattle population with harvesting (h_1 = 0.158, h_2 = 0.073, h_3 = 0.175)

Year	Observed Population			Projected Population			% Relative Error		
	group-I	group-II	group-III	group-I	group-II	group-III	group-I	group-II	group-III
1994	142	219	71	142	219	71	0.0	0.0	0.0
1995	160	205	75	161	212	77	-0.6	-3.3	-2.6
1996	157	199	80	174	208	82	-9.8	-4.3	-2.4
1997	166	200	74	182	207	86	-8.8	-3.4	-14.0
1998	204	215	76	188	208	89	8.5	3.4	-14.6
1999	197	232	91	193	209	97	2.1	11.0	-6.2
2000	201	217	103	197	212	95	2.0	2.4	8.4
2001	189	221	108	201	214	97	-6.0	3.3	11.3
2002	203	195	100	204	216	100	-0.5	-9.7	0.0
2003	194	219	101	207	219	102	-6.3	0.0	-1.0
2004	207	233	112	211	222	104	-1.9	5.0	7.7
2005	195	217	116	214	225	106	-8.9	-3.6	9.4
2006				217	228	108			
2007				220	232	109			
2008				224	235	111			
2009				227	239	113			
2010				231	243	115			

OBSERVED AND PROJECTED POPULATION OF GROUP-I WITH HARVESTING

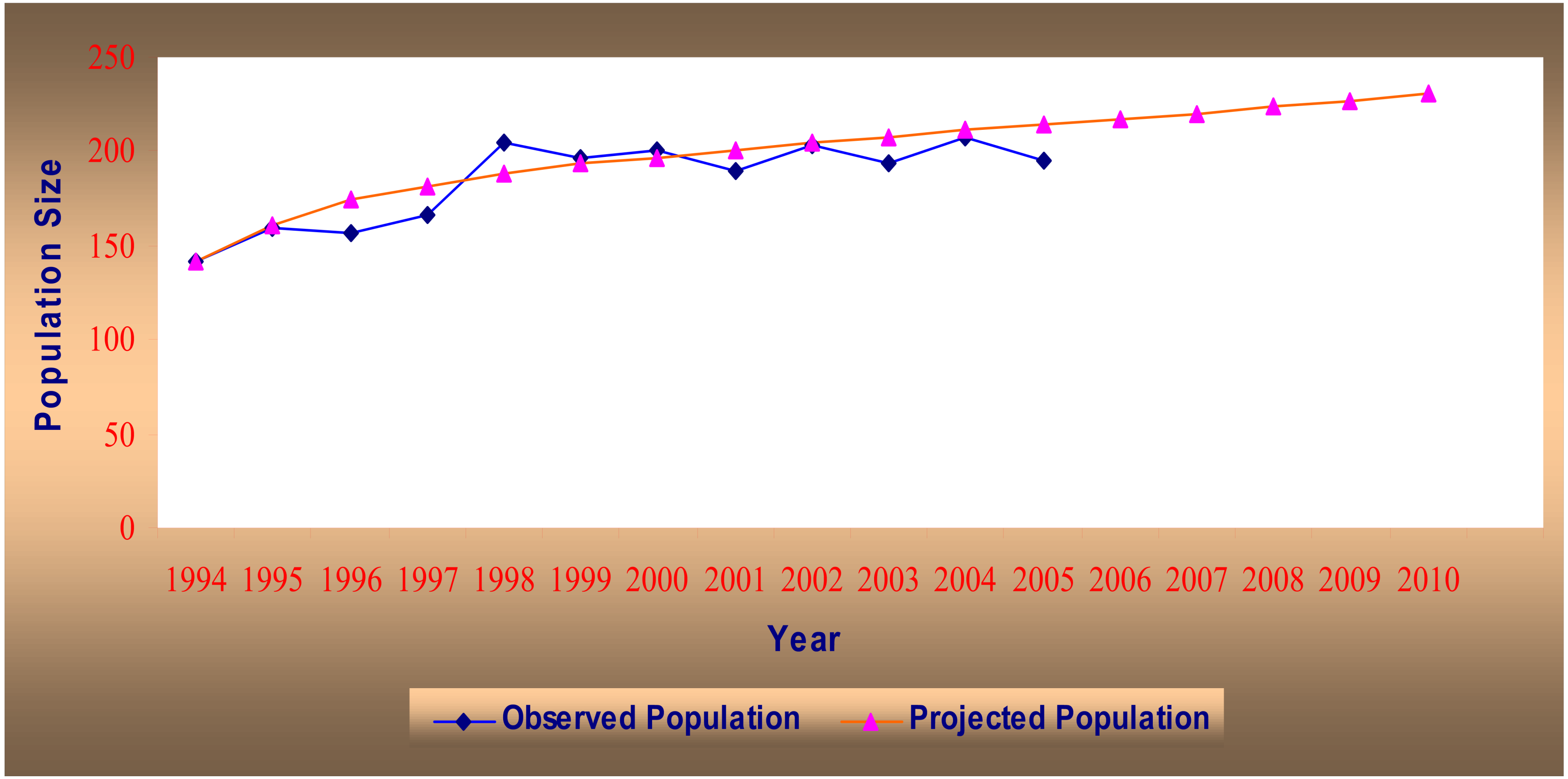

Figure No. 1(a)

OBSERVED AND PROJECTED POPULATION OF GROUP-II WITH HARVESTING

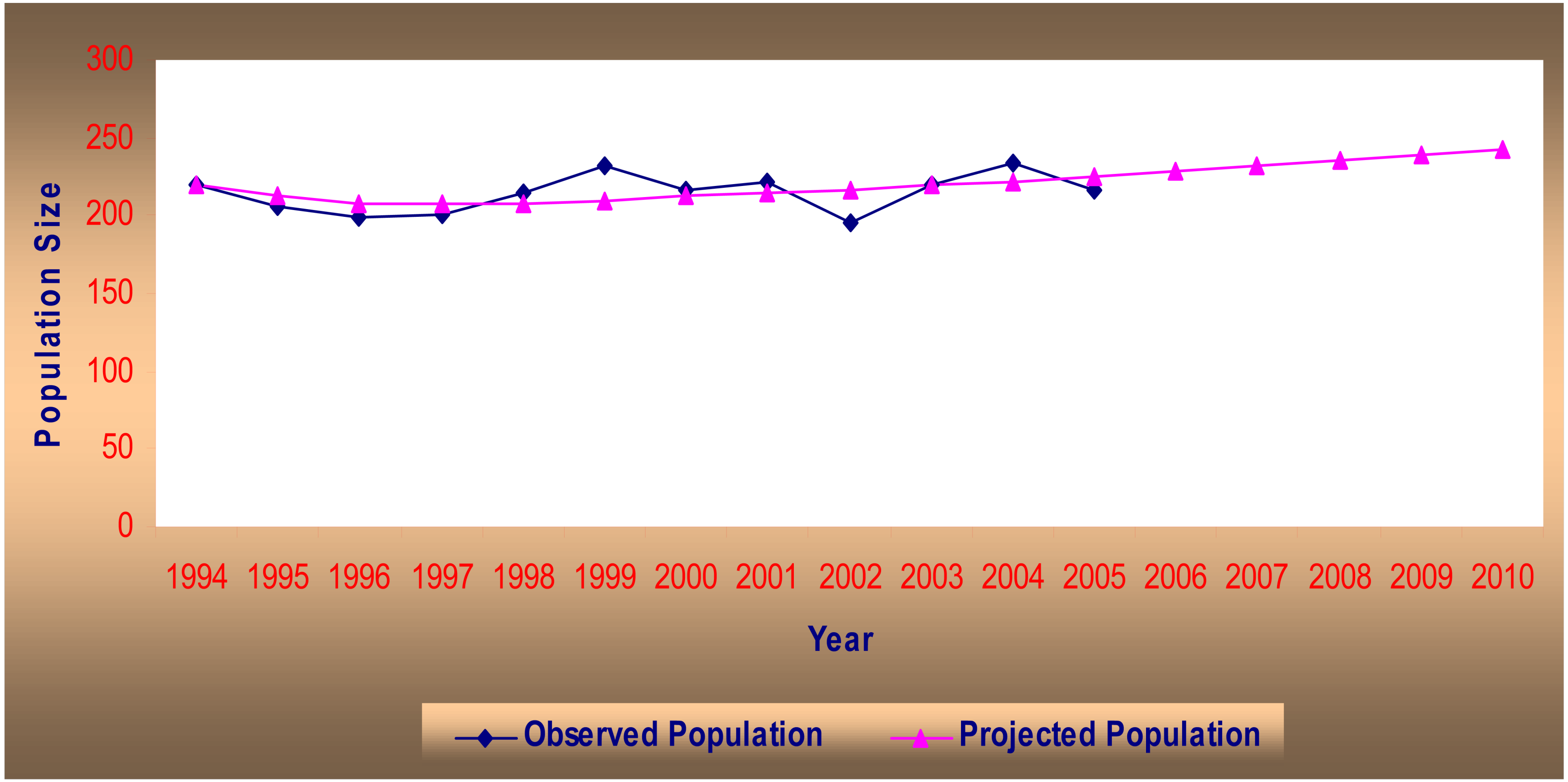

Figure No. 1(b)

OBSERVED AND PROJECTED POPULATION OF GROUP-III WITH HARVESTING

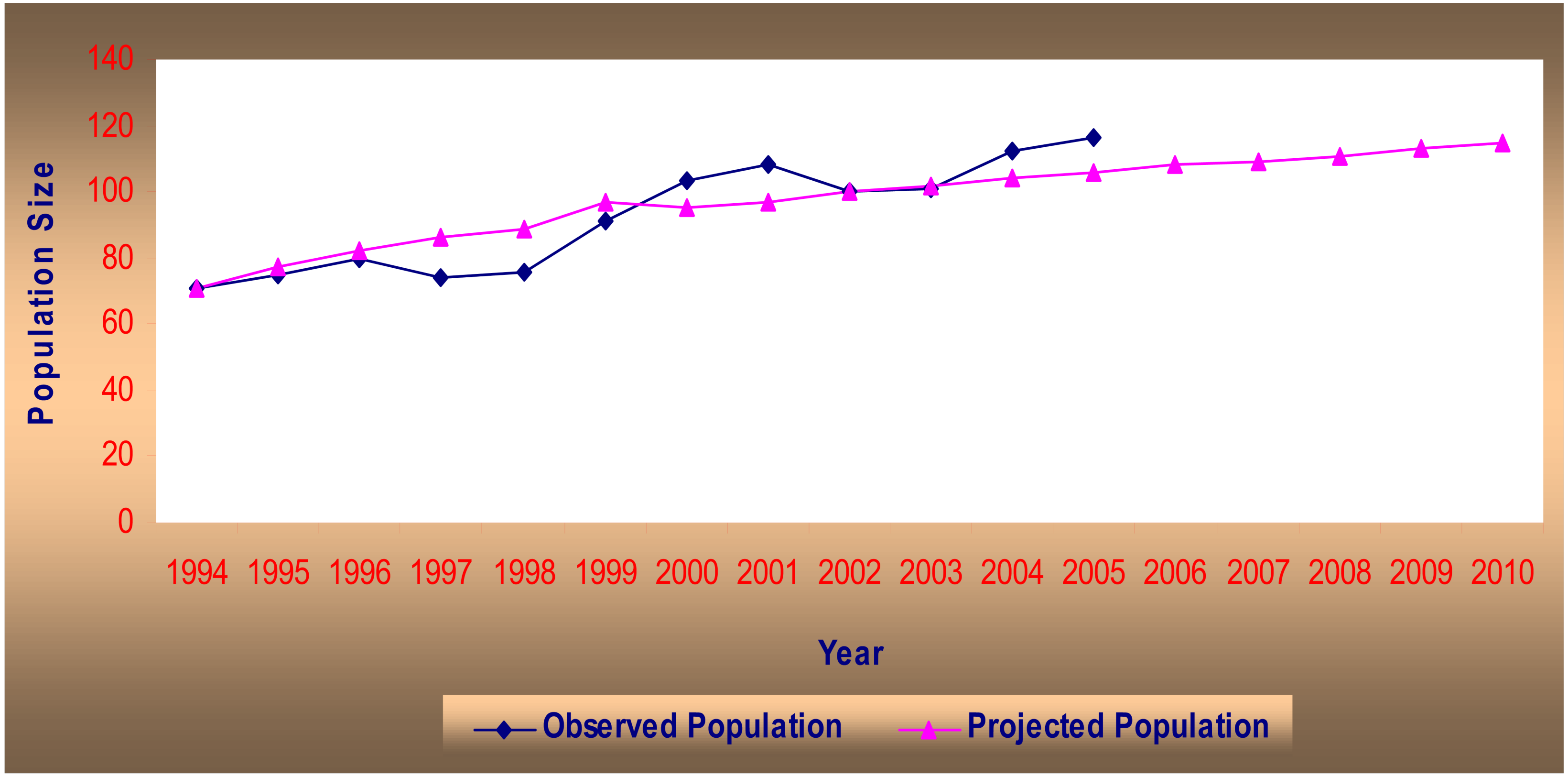

Figure No. 1(c)

Figs: 1(a) to 1(c) gives the graphical representation of the projected and observed cattle population for all the age groups.

(3)11 Chi-Square test:

For testing the validity of the model, chi-square test of goodness of fit is applied. Calculated values of Chi-square for different age groups are as follows:

Age group	^{2}calculated
group-I	7.899
group-II	6.842
group-III	7.534

These values are less than tabulated $^{2}{}_{11,\ 0.05}$ = 19.675. Therefore, the results are non-significant and we conclude that the model has been fitted well in all the age groups. For the total population (without considering the age groups) the value of Chi-square is;

$^{2}{}_{calculated}$ = 22.275

and tabulated value of ^{2} is

$^{2}{}_{35,\ 0.05}$ = 49.766

which show the validity of the model to the total cattle population without considering the age groups.

(3).2 Harvesting for stable population Structure:

The roots of cubic equation (3.3.3) are

$\bar{h}$ = 0.131, -0.450 and -0.085

Here the largest root i.e. $\bar{h}$ = 0.131 was selected as possible uniform harvesting rate for the stable population structure and after substituting it in (3.3.4) we get the following equation,

$$\lambda\,(\lambda^2 + 0.797\lambda + 0.125) = 0$$

and after solving it, we get three eigen values as given below,

$\lambda_1 = 0$

$\lambda_2 = -0.582$

$\lambda_3 = -0.215$

By adopting the same procedure as in the last section, the cattle projections in each group were computed upto year 2010, and are given in Table–4.3.

Table–4.3 shows the cattle projections for the stable cattle structure of three age-groups and we can observe the variation in cattle projections which is maximum upto year 1999 in each group.

From year 2000 to 2010 cattle projections approach to stability as expected (since $\lambda_1 = 0$).

Figs. 2(a) to 2(c) give the graphical representation of the cattle projections for the three age groups.

Table – 4.3 Cattle projection for the stable population structure

$(h_1 = h_2 = h_3 = \bar{h} = 0.131)$

Year	Projected Population		
	group-I	group-II	group-III
1994	142	219	71
1995	163	201	80
1996	175	190	86
1997	183	182	91
1998	186	178	94
1999	188	175	97
2000	189	173	99
2001	189	172	101
2002	190	171	102
2003	190	170	104
2004	189	169	104
2005	190	169	105
2006	190	168	105
2007	190	168	106
2008	190	168	106
2009	190	168	107
2010	190	168	107

PROJECTED POPULATION OF GROUP-I WITH UNIFORM HARVESTING

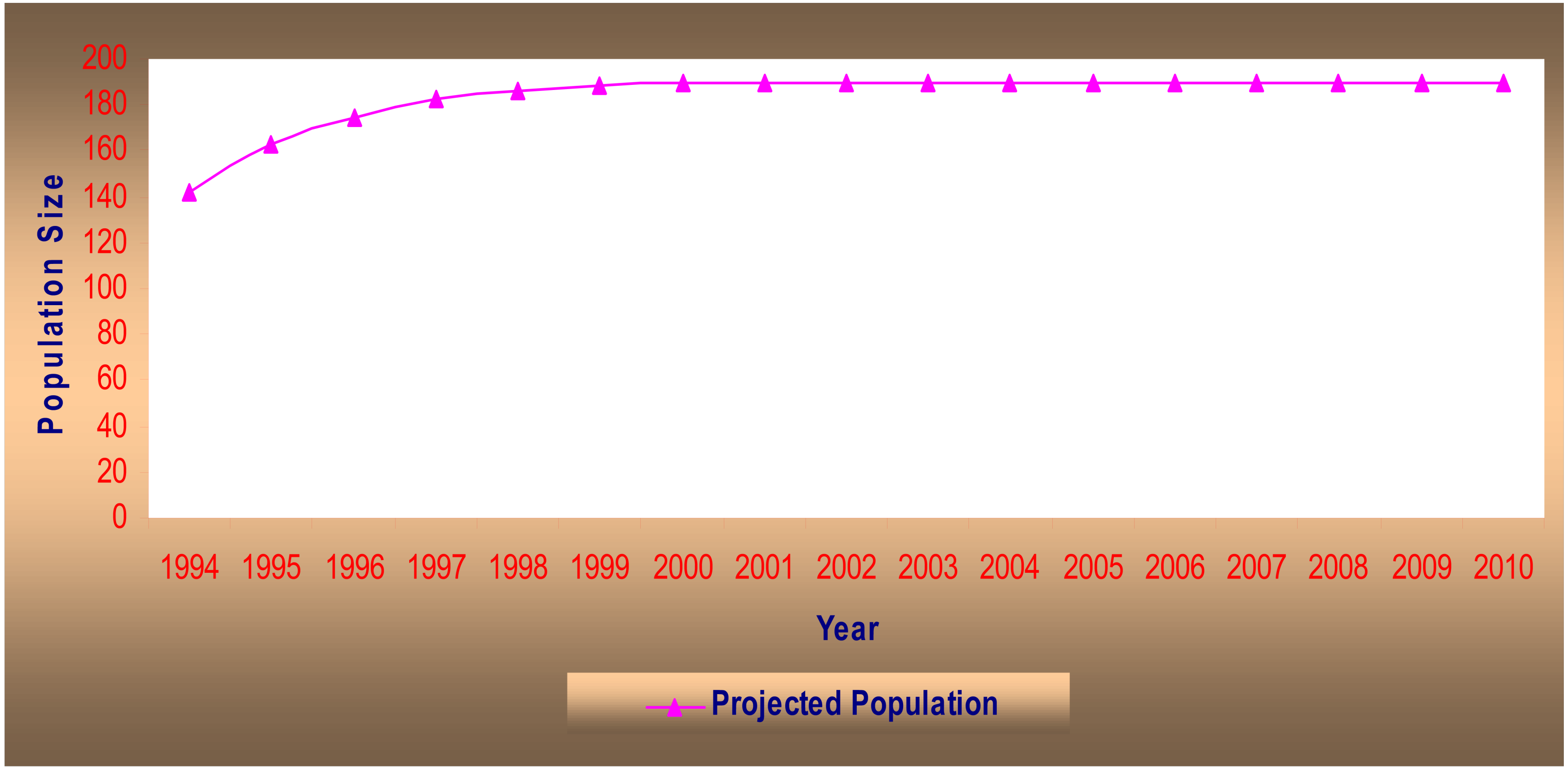

Figure No. 2(a)

PROJECTED POPULATION OF GROUP-II WITH UNIFORM HARVESTING

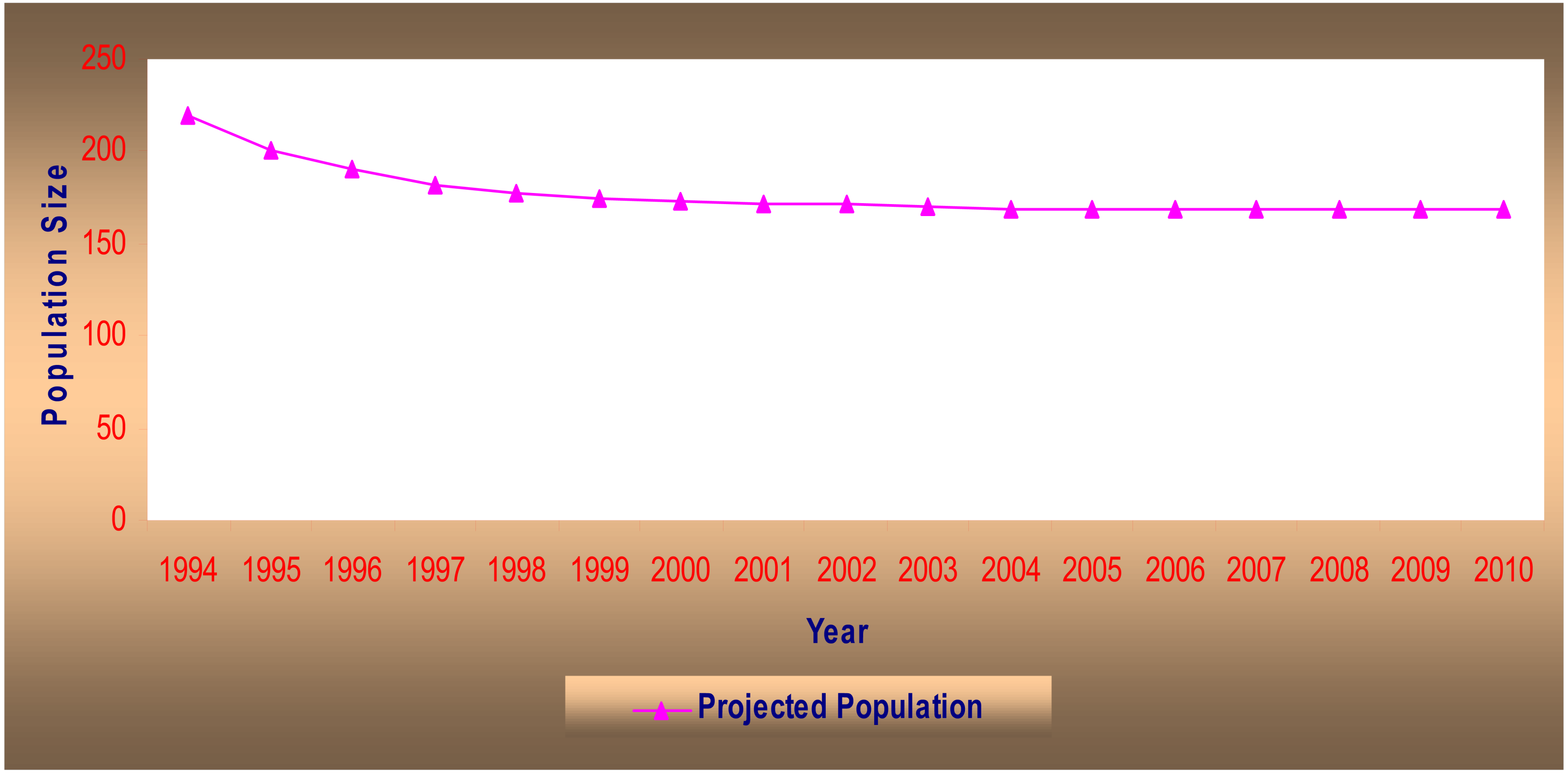

Figure No. 2(b)

PROJECTED POPULATION OF GROUP-III WITH UNIFORM HARVESTING

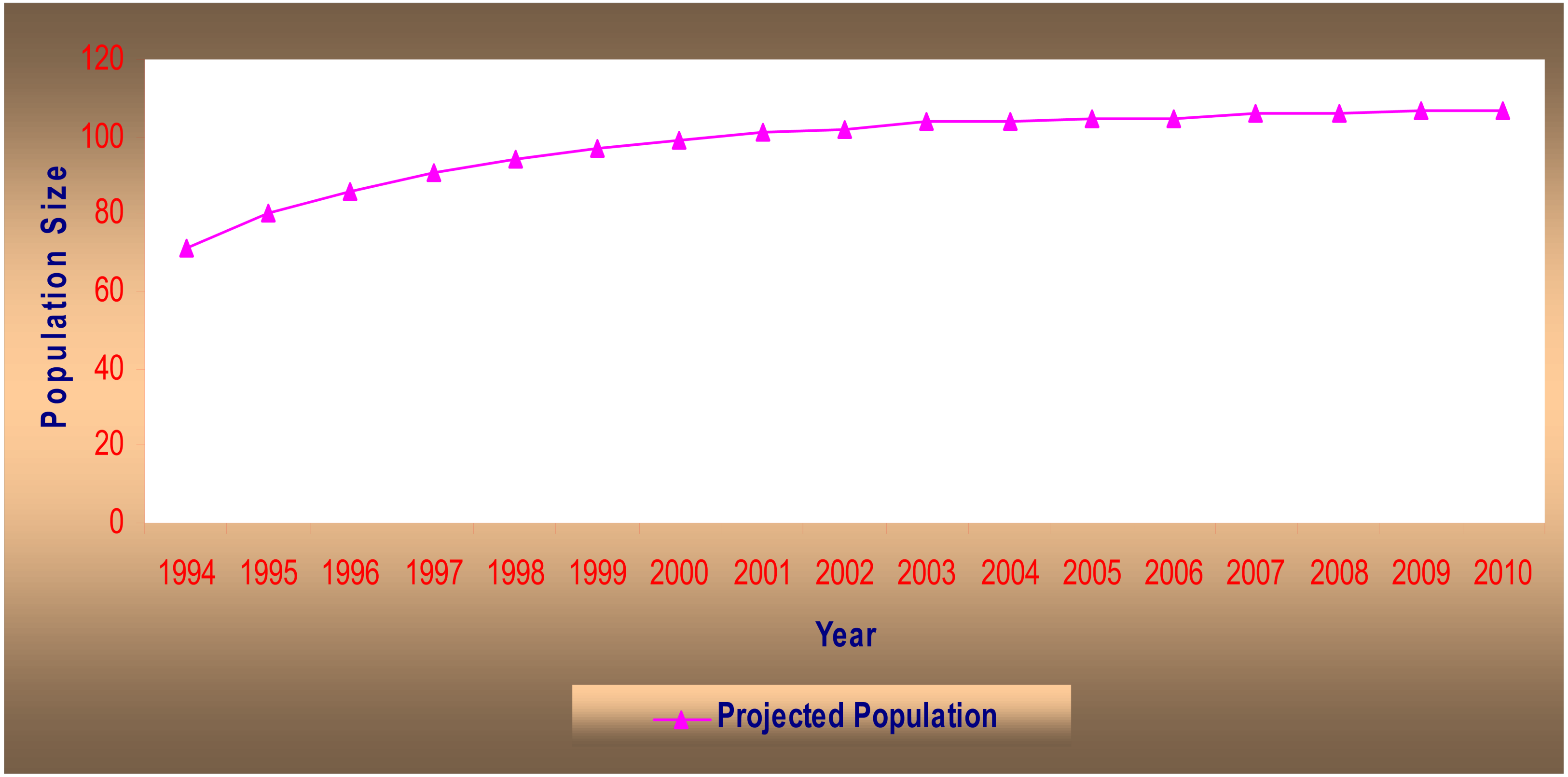

Figure No. 2(c)

(3).3 Population structure without harvesting

If the harvesting is not allowed then the population will grow exponentially and a stage will come that it grows beyond limits. The growth rates are considered same as that of the original population, because if we do not harvest any cattle from all the three groups then these cattle will also reproduce with the same rate while death and migration rate will remain same.

$b_2 = 0.329$; $b_3 = 0.202$

$d_1 = 0.069$; $d_2 = 0.012$; $d_3 = 0.017$

$m_1 = 0.210$; $m_2 = 0.096$

and harvesting rates will be zero i.e. $h_1 = h_2 = h_3 = 0$

On substituting the above growth rates in (3.3.8), we get

$\lambda_1 = 0.131$

$\lambda_2 = -0.450$

$\lambda_3 = -0.085$

The cattle projections for each group are shown in Table–4.4. The table indicates that in the absence of harvesting the population will grow exponentially as expected since $\lambda_1 > 0$.

Fig. 3(a) to 3(c) gives the graphical representation of the cattle projections of all the age groups in the absence of harvesting.

Table – 4.4 Cattle projections without harvesting ($h_1 = h_2 = h_3 = 0$)

Year	Projected Population		
	group-I	group-II	group-III
1994	142	219	71
1995	185	249	92
1996	226	265	112
1997	267	287	134
1998	310	315	158
1999	357	349	170
2000	409	390	202
2001	465	424	250
2002	531	480	289
2003	605	544	332
2004	690	617	382
2005	787	701	438
2006	897	797	502
2007	1020	898	574
2008	1165	1032	657
2009	1329	1175	751
2010	1515	1337	856

PROJECTED POPULATION OF GROUP-I WITHOUT HARVESTING

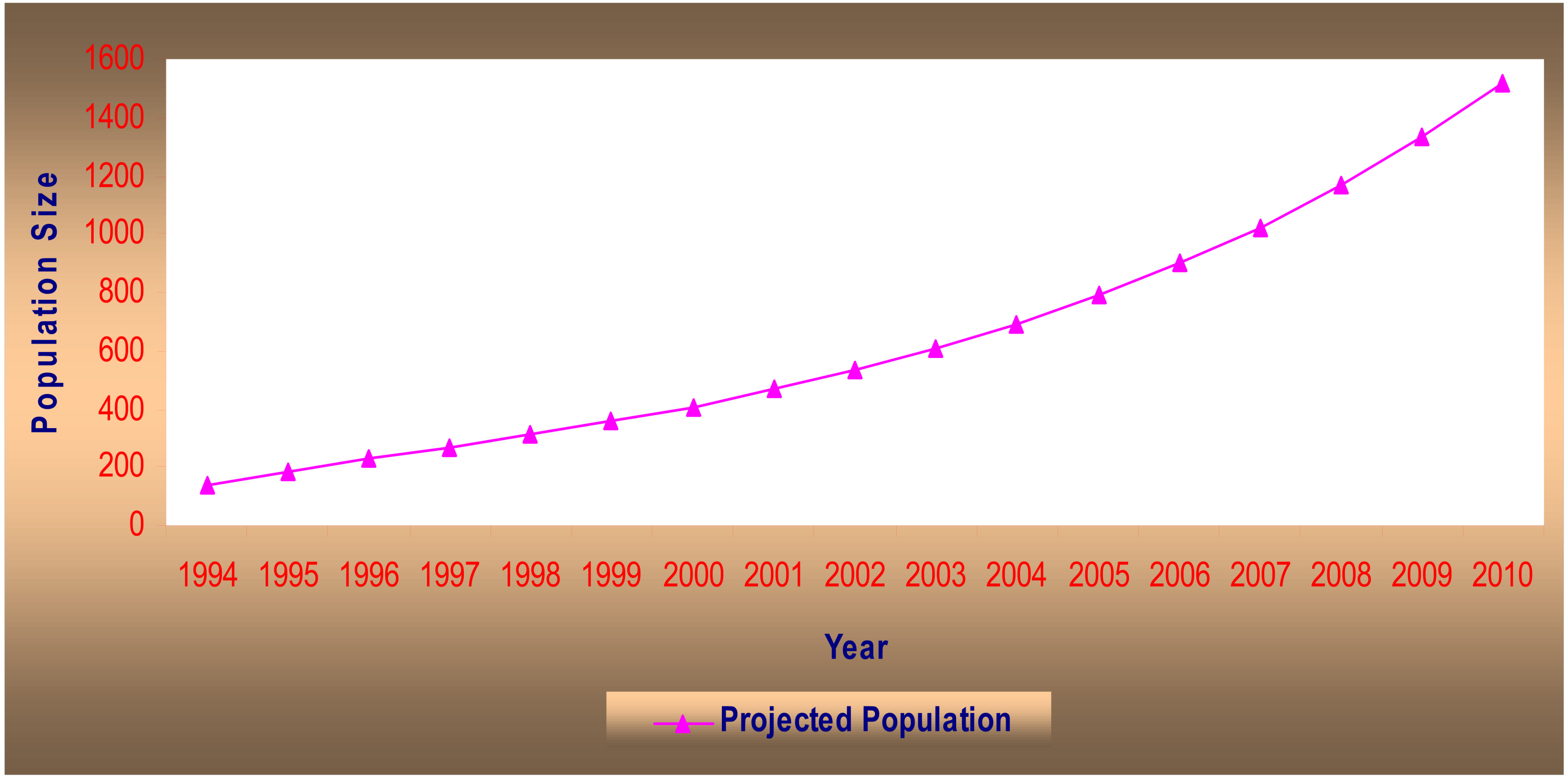

Figure No. 3(a)

PROJECTED POPULATION OF GROUP-II WITHOUT HARVESTING

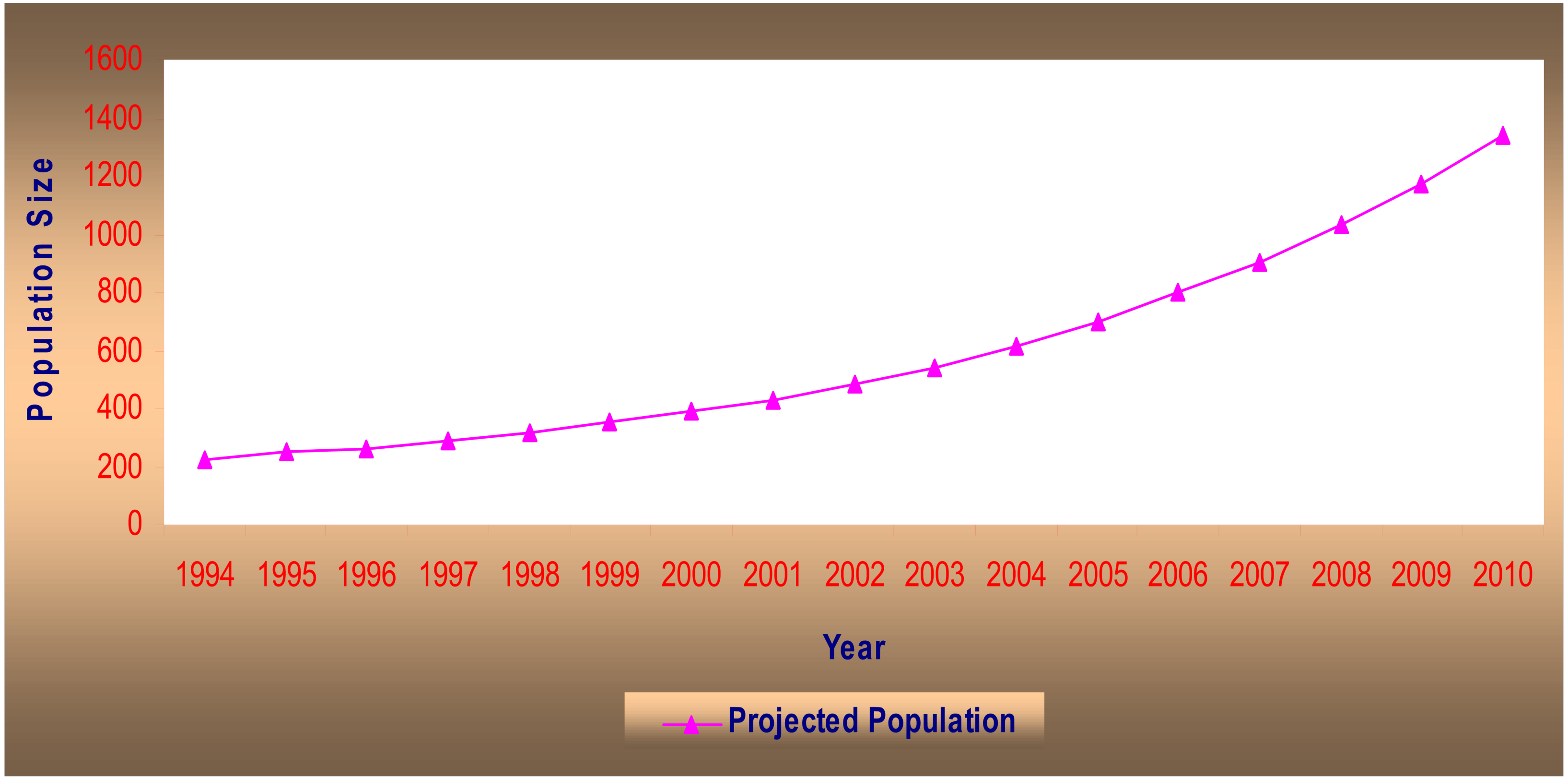

Figure No. 3(b)

PROJECTED POPULATION OF GROUP-III WITHOUT HARVESTING

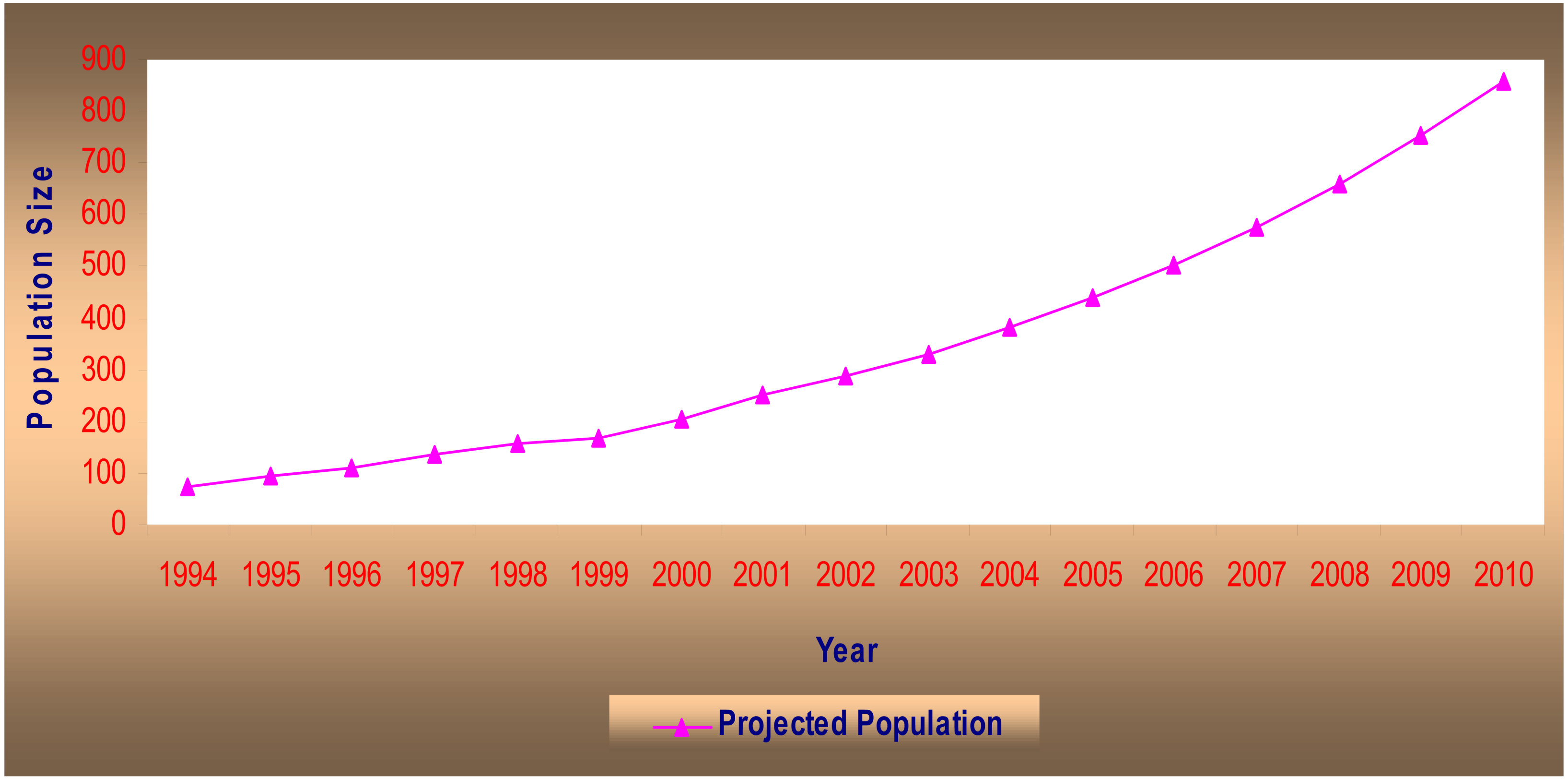

Figure No. 3(c)

From the above, it is clear that in the absence of harvesting, the cattle population increased beyond limits. Thus it supports the fact that the harvesting must be undertaken in order to control the population size with in limits.

SUMMARY

Since growth is an essential and important phenomenon in biological sciences so investigators are much interested in the study of growth process in all living organisms particularly the animal population. A growth process can be studied by constructing mathematical models by including variables which account for most of the variability in growth structure and inferences drawn from the model can be tested with the actual data pertaining to different fields of biological sciences. If there is

perfect agreement between the values predicted through the model and the actual data, the model may be used for future predictions. And if the predicted values are not in perfect agreement with the actual data the model may be modified in the light of discrepancies. Population growth models are important because of their many practical utilities in various fields of research. These models are able to provide projections of a population. The classical theory of population growth was given by Malthus (1798) and the systematic work in this direction has been initiated with the models of Bernardelli (1941), Lewis (1942) and Leslie (1945, 1948). Kapur (1985) developed a harvesting strategy for animal population.

Present work relates to development and illustration of an age dependent population growth model. Harvesting has been allowed in all the age-groups and inferences have been drawn about the population structure.

Chapter-1 gives the brief introduction about the population growth models and its applications in various fields. Chapter-2 is devoted to brief review of work done on population growth models and their applications in various fields. In Chapter-3, an age dependent population growth model has been developed. The model has also been developed by considering the growth, with and without harvesting. Section 3.1 gives a general introduction of the chapter. Section 3.2 gives the formulation of an age dependent population growth model and the population structure is given by

$$\underline{X}(t) = \underset{\sim}{Y}\, e^{\Lambda t} \underset{\sim}{Y}^{-1}\, \underline{X}(0)$$

Section 3.2.1 gives the conditions for growth, extinction and stability of population. Section 3.3.1 gives the population structure with single reproductive group. Section 3.3.2 provides the harvesting policy for stable population structure. Section 3.3.3 gives the population structure in the absence of harvesting.

Chapter-4 is devoted to the application of the models developed in Chapter-3 to a cattle population. Population projections have been found using the model developed with and without harvesting. Projected population has been shown by tables (4.2, 4.3 and 4.4) and graphs [1(a)-(c) to 3(a)-(b)]. The goodness of fit of the model has been tested by $_{2}$ at 5% level of significance.

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APPENDIX

Let $ax^3 + b\,x^2 + cx + d = 0$

Cubic equations have to be solved in several steps. First we define a variable '*f*':

$$f = \frac{(3c/a) - (b^2/a^2)}{3}$$

Next we define '*g*':

$$g = \frac{(2b^3/a^3) - (9bc/a^2) + (27d/a)}{27}$$

Then we define 'h':

$$h = (g^2/4) + (f^3/27)$$

If $h > 0$, there is only 1 real root and where $f=0$, $g=0$ and $h=0$, all 3 roots are real and equal and is solved by another method.

When $h \leq 0$, as is the case here, all 3 roots are real and we proceed as follows:

$$i = ((g^2/4) - h)^{1/2}$$

$$j = (i)^{1/3}$$

NOTE: The following trigonometric calculations are in radians

$k = \text{arc cosine}\ (-\ (g\ /\ 2i))$

$L = j * -1$

$M = \text{cosine}\ (K/3)$

$N = (\text{Square Root of } 3) * \text{sine}\ (K/3)$

$P = (b/3a) * -1$

$\lambda_1 = 2j * \text{cosine}(k/3) \quad -(b/3a)$

$\lambda_2 = L * (M + N) + P$

$\lambda_3 = L * (M - N) + P$

ABSTRACT

Title of thesis	:	**A population growth model with varying growth rates**
Name of the degree holder	:	**Jatinder Kumar**
Admission number	:	2004BS165M
Title of degree	:	Master of Science
Name of Major Advisor	:	**Dr. S.C. Gupta** Professor, Department of Maths & Statistics CCS Haryana Agricultural University, Hisar-125004
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The model of Lewis and Leslie (1945, 1948) has been extensively used for the study of population growth in various fields. However, complex growth structures require the use of more general models. The model of Kapur (1979) allows harvesting in the system is an initial step to move in this direction. But the need is being felt to develop more general models considering the effect of variable growth rates along with harvesting on the reproductive structure of living organisms.

In the present work, and age-dependent population growth model is proposed where birth, death and harvest rates are the functions of three population groups viz. pre-reproductive, more-reproductive and less-reproductive. The population structure with varying growth rate has been obtained. The condition for growth, extinction and stability for age-dependent cattle population has been found. Further the model has been illustrated by taking twelve year data on cattle growth. A harvest policy to ensure the stable population structure is also given. The population projections of cattle have been illustrated with tables and graphs.

MAJOR ADVISOR **SIGNATURE OF STUDENT**

HEAD OF THE DEPARTMENT