STUDIES ON BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

BY

ASIM KUMAR CHAKRABORTY

INDIAN AGRICULTURAL STATISTICS RESEARCH INSTITUTE
INDIAN AGRICULTURAL RESEARCH INSTITUTE
(I.C.A.R.)
NEW DELHI - 110 012

1996

STUDIES ON BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

To

My beloved parents

STUDIES ON BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

BY

ASIM KUMAR CHAKRABORTY

A Thesis
submitted to the Faculty of Post-Graduate School,
Indian Agricultural Research Institute, New Delhi,
in partial fulfilment of the requirements for
the award of the degree of

DOCTOR OF PHILOSOPHY
IN
AGRICULTURAL STATISTICS

INDIAN AGRICULTURAL STATISTICS RESEARCH INSTITUTE
POST-GRADUATE SCHOOL
INDIAN AGRICULTURAL RESEARCH INSTITUTE
(I.C.A.R.)
NEW DELHI 110 012

1996

STUDIES ON BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

BY

ASIM KUMAR CHAKRABORTY

A Thesis
submitted to the Faculty of Post-Graduate School,
Indian Agricultural Research Institute, New Delhi,
in partial fulfilment of the requirements for
the award of the degree of

DOCTOR OF PHILOSOPHY
IN
AGRICULTURAL STATISTICS

1996

APPROVED BY :

CHAIRMAN :

(P.R. SREENATH)

MEMBERS :

V. K. Gupta

(V. K. GUPTA)

(H.V.L. BATHLA)

(MAHESH KUMAR)

ACKNOWLEDGEMENTS

With profound gratitude, I express my sincere thanks to Dr. P. R. Sreenath, Principal Scientist and Head of Division, DE and AED, Indian Agricultural Statistics Research Institute, New Delhi and Chairman of my Advisory Committee whose continous encouragement, constructive criticism and guidance led me to pursue this comprehensive investigation.

I am grateful to Dr. V. K. Gupta, National Fellow, Dr. H. V. L. Bathla, Principal Scientist and Sh. R. Gopalan , Principal Scientist at I.A.S.R.I., New Delhi, members of my advisory committee for their invaluable help, assistance and encouragement throughout the course of this study.

I earnestly wish to thank the Director, Indian Agricultural Statistics Research Institute, New Delhi for providing necessary facilities at I.A.S.R.I. for carrying out this research work. Thanks are also due to Professor, I.A.S.R.I., New Delhi and staff of T.A.C., I.A.S.R.I., New Delhi.

I am highly thankful to the Director and Dean, I.A.R.I., New Delhi and staff of P.G.School.

My thanks are also due to Dr. S. Jaggi, Scientist, Dr. R. Parsad , Scientist and Dr. P. K. Batra, Scientist at I.A.S.R.I., New Delhi for their constant encouragement and willing co-operation whenever sought for during the course of this study.

My gratitude goes to my friends, Pradip, Kalyan, Nirmal, Debu and Krishanu for lending their helping hand in my days of tough-going and encouraging incessantly;

to Debasisda and Yogender Sharma for their reassuring love and encouragement; to Amrit, Hemanta, Kuntal, Anup and Sree for their unceasing encouragement and pleasant company. Thanks are also due to Kalraji, Dhandapani, Lal Mohan, Shyamal, Bunty, Dipu and others whose love and happy company made the things easy during my long stay at Sukhatme Hostel, I.A.S.R.I.

The Senior Research Fellowship awarded by Indian Agricultural Statistics Research Institute (I.C.A.R.), New Delhi, during the tenure of my studies is gratefully acknowledged.

I bow to the love and kind blessings of my parents and to the wishes and pleasantries of my kins. Most of all I wish to thank Sh. Santosh Kumar, Computer Graphics and Data Analysis Lab, A6/51, Krishi Niketan, Paschim Vihar, New Delhi-63, for his extreme patience through the endless changes and neat typing of the manuscript.

Asim Kumar Chakraborty

August 23, 1996
New Delhi 110 012

(ASIM KUMAR CHAKRABORTY)

INDIAN AGRICULTURAL STATISTICS RESEARCH INSTITUTE
LIBRARY AVENUE. NEW DELHI 110 012

Dr. P.R.SREENATH,
PRINCIPAL SCIENTIST
and Head of Div. DE & AED

CERTIFICATE

This is to certify that the thesis entitled " **STUDIES ON BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS**" submitted in partial fulfilment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN AGRICULTURAL STATISTICS** of the Post-Graduate School, Indin Agricultural Research Institute, New Delhi is a record of a *bona fide* research carried out by **SHRI ASIM KUMAR CHAKRABORTY** under my guidance and supervision and no part of the thesis has been submitted for any other degree or diploma.

The assistance and help received during the course of this investigation have been duly acknowledged by him.

August 23, 1996
New Delhi 110 012

P R
(P. R. SREENATH)
CHAIRMAN
ADVISORY COMMITTEE

CONTENTS

CHAPTER 1

INTRODUCTION AND REVIEW OF LITERATURE

1.1 Introduction

Experiments are performed by people in nearly all walks of life. Designing an experiment means deciding how the observations or measurements should be taken to answer a particular question/problem in a valid, efficient and economical way. The design and the analysis go hand-in-hand; they are inseparable in the sense that if an experiment is properly designed, then there will exist an appropriate way of analysing the data. From an ill-designed experiment no valid conclusions can be drawn. Thus a scientific approach to designing an experiment is essential.

Depending on the nature of the treatments under study and the nature of comparisons required among them experimental designs can be broadly classified into three categories(Das and Giri, 1988) . These are :

1. Varietal trials /unifactor experiments.
2. Factorial experiments/multifactor experiments and
3. Biological assays.

This thesis considers the first category of experimental design viz. unifactor experiments. The treatments in a unifactor experiment may vary qualitatively : varieties, crops, seeding and tilling practices, various formulations of fertilizers and pesticides etc. Also treatments may have quantitative grades fixed for the tested factors

: rates of seeding etc. The main purpose of these experiments is to compare the treatments in all possible pairs.

1.2 Block Designs

The three basic principles of experimental design, namely, the indispensability of replication and that of randomisation and the desirability of local control, were developed by R.A. Fisher.

Experiments in all fields of research have at least one feature in common, i.e., the variability of experimental material. This variability may vary in size, but its omnipresence can not be denied. When the variability is small and taking the observations is not costly, an elaborate experimental design is not required.

Heterogeneity of experimental material, if not suitably taken care of in designing of experiment, is likely to over-shadow the real treatment differences making them remain undetected, unless they are large enough. Blocking is the technique used to bring about homogeneity of experimental units within blocks for the case of one-way elimination of heterogeneity, so that treatment contrasts are estimated, making use of the intra-block information, with higher efficiency. The simplest and the most widely used block design is Randomised Complete Block Design (RCBD), wherein the treatments are randomly allocated to experimental units within blocks such that every treatment appears exactly once in each block.

With the increase in the number of treatments, the block size increases and consequently the purpose of local control is defeated due to the following two reasons: (i) unavailability of one replicate of units which are relatively homogeneous and (ii) the greater heterogeneity is introduced in the experimental error, inflating it and thus vitiating the conclusions to be drawn.

In such cases, one has to resort to Incomplete Block Designs (IBD). An incomplete block design is a design in which there is at least one block which is incomplete in the sense that it does not contain all the treatments. Different patterns of allocation of treatments to blocks have given rise to different types of incomplete block designs viz., Balanced Incomplete Block Designs (BIBD), Partially Balanced Incomplete Block Designs (PBIBD) etc.

1.2.1 Block Designs with Unequal Block Sizes

The literature on block designs with equal block sizes is abundant. In many practical situations however block designs with unequal block sizes (non-proper) may be required. Kageyama (1976) pointed out that with the advent of high speed computers, block designs with unequal block sizes may be particularly useful in large experiments both in industry and agriculture. Non-proper block designs add flexibility to the experimentation and may avoid waste of experimental material as well. The need for blocks of different sizes in biological experiments has been noted by Pearce (1964).

Consider an experiment where experimental units are individual organisms that exist in groups, like animals in a litter or trees planted in rows and columns before the experiment was conceived. Although such groups can be divided if need arises but leads to loss of efficiency. To achieve equal block sizes unwanted plots can be discarded but this can be very wasteful. However, it is not difficult to design experiments with varying block sizes.

Consider a class $D(v,b_1, b_2,\ldots, b_p, k_1, k_2, \ldots, k_p)$ of connected and non-proper block designs with v treatments and b_j blocks of size k_j, $(j = 1,2,\ldots, p)$. For characterisation and construction of designs with unequal blocks we can refer to

John (1964), Kulshreshtha, Dey and Saha (1972), Hedayat and Federer (1974), Kageyama (1976), Patterson and Williams (1976), Patterson, Williams and Hunter (1978), Gupta and Jones (1983), Jones , Sinha and Kageyama (1987) , Sinha and Jones (1988) etc. The optimality aspects of these designs under homoscedastic, fixed effects additive model has been studied by Lee and Jacroux (1987 a, b, c), Pal and Pal(1988), Dey and Das (1989), Gupta and Singh (1989) , Gupta, Das and Dey (1991). Under heteroscedastic model Gupta (1995) has studied the optimality of non-proper block designs.

1.2.2 Balancing in Block Designs

In comparative experiments, all the of v_{c_2} paired comparisons are of equal interest to the experimenter. It is, therefore, desirable that these comparisons are made with same precision. Balancing is an important and desirable statistical property to take care of this fact in block designs. In literature, the term balance has been used in several senses, viz., variance balance, efficiency balance, pairwise balance, X^{-1} balance, partial balance, partially efficiency balance etc. An excellent review on terminological tangle on different concepts of balance is given by Preece (1982). Definition of some types of balancing in block designs are given in the following. For this purpose we consider a block design with v treatments, b blocks, replication vector **r** and block size vector **k**, treatment x block incidence matrix **N**, diagonal matrices **R** and **K** having elements as in **r** and **k** respectively and total number of units being n.

1.2.2.1 Variance Balance

Definition 1.1 : A connected block design is said to be variance - balanced if it permits the estimation of all the normalised treatment contrasts with equal variance.

Definition 1.2 : A connected block design is variance - balanced if and only if all the non-zero eigen values of its information matrix , **C** , are equal.

Definition 1.3 : A connected block design is variance - balanced if its **C** - matrix has all the diagonal elements equal and all its off diagonal elements equal , i.e. **C** is of the form

$$\mathbf{C} = a\,\mathbf{I} + b\,\mathbf{J}\text{ , a and b are scalars.}$$

$$= \theta\ (\mathbf{I} - \frac{1}{v}\,\mathbf{J}).$$

where θ is the only non-zero eigen value with multiplicity (v -1) . In fact $\theta = [\,n - \text{trace}\,(\mathbf{N}\,\mathbf{K}^{-1}\,\mathbf{N}'\,)\,] / (v - 1)$. Here **I** is an identity matrix of order v and **J** is a matrix with elements as unity.

Puri and Nigam (1976) characterised variance balanced block designs in terms of the matrix $\mathbf{P} = \mathbf{N}\,\mathbf{K}^{-1}\,\mathbf{N}'$.

Most commonly used RBD is a variance - balanced design. In the class of incomplete block designs (IBD) , the most important binary, proper, and equireplicate variance-balanced design is the Balanced Incomplete Block Design (BIBD) introduced by Yates (1936). (1936)

BIB designs are not available for every parametric combinations and even if it exists , it demands too many experimental material. This led Bose and Nair (1939) to introduce Partially Balanced Incomplete Block (PBIB) designs as a generalization of BIB designs. In these designs all the elementary treatment contrasts $(\tau_i - \tau_j)$ are not estimated with equal variance , the variance of their estimate depends rather on the association of treatments i and j i.e., whether they are 1st, 2nd or . . ., mth associates.

1.2.2.2 Efficiency Balance

This concept is due to Jones (1959) and the term efficiency-balance was coined by Puri and Nigam (1975) and Williams (1975).

The efficiency of any treatment contrast is defined as the ratio of the variance of the estimate of the treatment contrast as estimated through an orthogonal design with same replication vector to the variance of the same contrast, assuming that both the designs have the same variance per plot, say, σ^2. A design is said to be orthogonal iff the incidence matrix N is of the form $\mathbf{N} = \mathbf{r}\,\mathbf{k}'/n$.

Definition 1.4 : A connected block design is said to be efficiency- balanced if every contrast of the treatment effects is estimated through the design with same efficiency factor.

If $\mathbf{p}'\tau$ is the contrast then the variance of its estimate, $V_d(\mathbf{p}'\hat{\tau}) = \sigma^2 \mathbf{p}'\mathbf{C}^{-}\mathbf{p}$ and $V_0(\mathbf{p}'\hat{\tau}) = \sigma^2 \mathbf{p}'\mathbf{R}^{-1}\mathbf{p}$. The suffixes d and o denote respectively the given block design and the orthogonal design. Then efficinecy of the contrast $\mathbf{p}'\tau$ is

$$E = \frac{\mathbf{p}'\mathbf{R}^{-1}\mathbf{p}}{\mathbf{p}'\mathbf{C}^{-}\mathbf{p}}.$$

For an efficiency balanced design, the C -matrix is of the form $\mathbf{C} = \theta\ (\mathbf{R} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}')$

and $\mathbf{C}^{-} = \theta^{-1}\mathbf{R}^{-1}$ is a g - inverse of C. Therefore, the efficiency factor becomes

$$E = \theta \text{ and } \theta = \frac{n - trace\left(\mathbf{N}\mathbf{K}^{-1}\mathbf{N}'\right)}{n - \sum_{i=1}^{v} r_i^2/n}$$

More details of efficiency balanced block designs can be found in Jones (1959), Calinski (1971), Puri and Nigam (1975), Williams (1975) , Kageyama (1980,1981) , Dey and Gupta (1986) , Nigam, Puri and Gupta (1988) etc.

1.2.2.3 Partially Efficiency Balanced (PEB) Designs

The Partially Efficiency Balanced (PEB) designs constiutue the most general class of connected block designs. Puri and Nigam (1977) gave definition of this class of designs.

Consider a connected block design D with v treatments arranged in b blocks and n plots.

Let $\mathbf{M}_0 = \mathbf{M} - \mathbf{1}\mathbf{r}'/n$ where $\mathbf{M} = \mathbf{R}^{-1}\mathbf{N}\mathbf{K}^{-1}\mathbf{N}'$, $\mathbf{R}^{-1}$ and $\mathbf{K}^{-1}$ are the inverse of the matrices **R** and **K** respectively.

According to Puri *et al.* (1977) a design D (v,b,r,k) is said to be a Partically Efficiency Balanced (PEB) design with m efficiency classes, if,

(i) there exists a set of v-1 linearly independent contrasts s_{ij}, $j = 1,2,\ldots,\rho_i$, $i = 1,2,\ldots,m$; such that ρ_i of them satisfy the equation $\mathbf{M}_0 s_{ij} = \mu_i s_{ij}$ $j = 1,2\ldots,\rho_i$, $i = 1,2,\ldots,m$ so that the efficiency factor associated with every contrast of the ith class is $1-\mu_i$, where μ_i's are the eigen values of $\mathbf{M}_0$ with multiplicities ρ_i , $\sum_{i=1}^{m} \rho_i = v-1$ and

(2) there exist mutually orthogonal idempotent matrices $\mathbf{L}_i^{v \times v}$ of rank ρ_i such that

$$\mathrm{M}_0 = \sum_{i=1}^{m} \mu_i L_i \quad \text{and} \quad \sum_{i=1}^{m} \mathbf{L}_i = \mathbf{I} - \mathbf{1}.r'/n$$

Pal (1980) has proved that condition (1) implies condition (2), the eigen values μ_i's , i = 1 , 2 , . . ., m are , of course, non-zero and distinct.

1.3 Designs with Nested Rows and Columns.

In many field and laboratory experiments, blocking alone fails to remove heterogeneity among experimental units due to the presence of several factors other than treatments which influence the response under study. Thus, one may have to use two-way elimination of heterogeneity, three-way elimination of heterogeneity, and so on. For some situations where heterogeneity persists within the blocks due to factors nested within it, Nested Block Designs have been suggested in literature.

Kleczkowski (1960) devised a form of Nested Incomplete Block Design (NIBD) with $v = 8$ treatments for a series of experiments in which bean plants, in two primary leaves stage, were inoculated with sap from tobacco plants with tobacco necrosis virus. The treatments were eight different virus concentrations. Each leaf had two inoculations, one for each half leaf. Ignoring the leaf positions, plants formed the block (of size 4) and leaves formed the sub-block (of size 2) of a Nested Balanced Incomplete block Design (NBIBD). Preece (1967) has for the case of two-way elimination of heterogeneity, one nested within the other, introduced a NBIBD with parameter ($v, r, b_1, k_1, \lambda_1, b_2, k_2, \lambda_2, m$).

Definition 1.5 : A nested balanced incomplete block design (NBIBD) is a design with v treatments, each replicated r times , with two systems of blocks , such that,

(i) the second system is nested within the first with each block from the first system (block) containing exactly m blocks from the second system (sub-blocks),

(ii) ignoring the second system leaves a BIBD with b_1 blocks each of k_1 units with λ_1 concurrences,

(iii) ignoring the first system leaves a BIBD with b_2 blocks each of k_2 units with λ_2 concurrences.

The parameters satisfy the following identities

$v r = b_1 k_1 = m b_1 k_2 = b_2 k_2;$

$\lambda_1 (v - 1) = r (k_1 -1) ; \lambda_2 (v-1) = r (k_2-1)$

, so that $(v-1)(\lambda_1 - m\lambda_2) = r (m-1)$.

Preece , Pearce and Kerr (1973) considered designs with three mutually orthogonal and fully crossed classification of plots for the elimination of three-way heterogeneity. The study of nested designs in a general framework was initiated by Srivastava (1978).

Consider the case of agroforestry experiments involving evaluation of crop species / varieties for their comparative performance when grown along with a given tree species. The tree species would be adding to the heterogeneity in the growing condition for the crop species / varieties being evaluated since (i) their stand (both number and vigour) in different plots (blocks) will be subjected to high variation and (ii) the tree-row will have differential effects in either directions. Thus a block with a single tree-row can be taken to have column (directional) effects and row (tree-row or tree- row segments) effects nested within it. These row and column effects within a block will be crossed between themselves. Such situations in the experimentation would require special consideration.

The case of experimental designs which are incomplete blocks with nested rows and columns, such that within each block row x column classification is orthogonal was considered by Singh and Dey (1979), and they introduced the Balanced Incomplete Block Designs with Nested Rows and Columns (BIBRCD). Such designs are useful for eliminating the additional heterogeneity in two-ways within a block, which may not be due to same source and /or of same magnitude within each block.

Following the definition of BIBD, Singh and Dey (1979) considered block designs with nested rows and columns in v treatments and b blocks , each block containing pq units, where each block is further arranged into p rows and q columns. Thus there are b sets (blocks) of arrangements involving p rows and q columns each. They considered the following model

$$y_{ijlm} = \mu + \tau_i + \beta_j + \rho_{l(j)} + \chi_{m(j)} + \varepsilon_{ijlm} \tag{1.1}$$

$$i = 1,2,\ldots,v;$$

$$j = 1,2,\ldots,b;$$

$$l = 1,2,\ldots,p;$$

$$m = 1,2,\ldots,q,$$

Where

Y_{ijlm} : is the response from the unit receiving i^{th} treatment in the $(l,m)^{th}$ cell of the j^{th} set; μ , the general mean ; τ_i , the effect of i^{th} treatment ; β_j , the effect of the j^{th} set/block ; $\rho_{l(j)}$, the effect of the l^{th} row in the j^{th} set ; $\chi_{m(j)}$, the effect of m^{th} column in the j^{th} set; ε_{ijlm}, uncorrelated error with zero expectation and variance σ^2.

Let $\mathbf{N} = ((n_{ij..}))$ be the (v × b) treatments × blocks, $\mathbf{N}_1 = ((n_{ijl.}))$ be the (v × bp) treatments x rows within blocks and $\mathbf{N}_2 = ((n_{ij.m}))$ the (v × bq) treatments × columns within blocks incidence matrices of the design, where n_{ijlm} (= 0 or 1) is the number of times i^{th} treatment occurs in the l^{th} row and m^{th} column of the j^{th} block.

and $n_{ij..} = \sum_{l=1}^{p} \sum_{m=1}^{q} n_{ijlm} = \sum_{l=1}^{p} n_{ijl.} = \sum_{m=1}^{q} n_{ij.m}$ is the number of times ith treatment occures in jth block.

$n_{ijl.} = \sum_{m=1}^{q} n_{ijlm}$ is the number of times ith treatment occurs in lth row of jth block. $n_{ij.m} = \sum_{l=1}^{p} n_{ijlm}$ is the number of times ith treatment occurs in mth column of jth block.

For the analysis, the fixed effect homoscedastic model (1.1) can be rewritten as

$$Y = \mu\, \mathbf{1}_n + \mathbf{D}'_\tau \tau + \mathbf{D}'_\beta \beta + \mathbf{D}'_\rho \rho + \mathbf{D}'_\chi \chi + \varepsilon$$

with $E(\varepsilon) = \mathbf{0}$ and $D(\varepsilon) = \sigma^2 . \mathbf{I}_n$

where $n = bpq$, $\mathbf{Y}$ (n x 1) is the observational vector, $\mathbf{1}_n$ is the (n x 1) vector of unities; $\mathbf{D}_\tau$(v x n) , $\mathbf{D}_\beta$ (b x n) $\mathbf{D}_\rho$ (bp x n) and $\mathbf{D}_\chi$ (bq x n) are the known design matrices for treatments, blocks, rows within blocks, columns within blocks respectively ; $\tau\,(v \times 1)$,$\beta\,(b \times 1)$,$\rho(bp \times 1)$ and $\chi\,(bq \times 1)$ are column vectors of unknown parameters and **T**, $\mathbf{B}_1$, $\mathbf{B}_2$ and **B** are the vectors of totals of treatments, rows within blocks, columns within blocks and blocks respectively ; $\varepsilon^{(n \times 1)}$ is the error vector ; the operators E and D denote the expectation and dispersion respectively ; σ^2 is unknown scalar quantity and $\mathbf{I}_n$ is an $(n \times n)$ identity matrix .

The reduced normal equations are $\mathbf{C}\,\tau = \mathbf{Q}$

$$\mathbf{C} = \mathbf{R} - \frac{1}{q}\mathbf{N}_1\mathbf{N}_1' - \frac{1}{p}\mathbf{N}_2\mathbf{N}_2' + \frac{1}{pq}\mathbf{N}\mathbf{N}' = \mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_0$$

$$\mathbf{Q} = \mathbf{T} - \frac{1}{q}\mathbf{N}_1\mathbf{B}_1 - \frac{1}{p}\mathbf{N}_2\mathbf{B}_2 + \frac{1}{pq}\mathbf{N}\mathbf{B}$$

Where C_1 , C_2 , C_0 are the C -matrices considering only row , column and block classification respectively, Q is the vector of adjusted treatment totals.

1.3.1 Characterization of Nested Row -Column Design

Definition 1.6 : A block design with nested rows and column in said to be connected it it permits the estimation of all the v_{C_2} elementary treatment contrasts.

Definition 1.7: A block design with nested rows and columns is said to be connected if rank (C) = v -1.

Definition 1.8 : A connected block design with nested rows and columns is said to be variance-balanced if it permits the estimation of all the normalised treatment contrasts with equal variance.

Definition 1.9 : A necessary and sufficient condition for a connected block design with nested rows and columns to be variance balanced is that C has the form

$$C = \theta \left(I - \frac{1}{v} J\right),$$

where θ is the unique non-zero eigen value of **C** with multiplicity v-1.

The binary and proper (in the sense of equal block, row and column sizes) block designs with nested rows and columns have been defined by Singh and Dey (1979) as Balanced Incomplete Block Designs with Nested Rows and Columns (BIBRCD).

Definition 1.10 : An arrangement of v treatments in b blocks each of p rows and q columns, such that $p q < v$, is called a Balanced Incomplete Block Design with Nested Rows and Columns (BIBRCD) if the following conditions are satisfied.

(i) every treatment occurs at most once in each block,

(ii) every treatment occurs exactly in r blocks and

(iii) $p N_1 N'_1 + q N_2 N'_2 - N N' = a \mathbf{I}_v + \lambda \mathbf{J}_{v,v}$, for some integer a and λ such that $\lambda(v-1) = r(p-1)(q-1)$. These designs are denoted as BIBRCD (v,b, r, p, q , λ).

The class of designs with pq = v were considered by Cheng (1986) and were called as Balanced Complete Block Designs with Nested Rows and Columns (BCBRCD). A number of methods of construction of BIBRCD and BCBRCD are now available in literature. A brief summary of these results is as follows.

The systematic method of construction of BIBRC designs using BIB designs when block size is a perfect square is given by Singh and Dey (1979) while introducing these designs. Some of the designs were evolved by the use of method of differences and by trial and error.

BIBRC designs developed by Agrawal and Prasad (1982 , 1983) were mainly based on the use of the method of differences such that each of the incidence matrices N, N_1, N_2 are those of BIB designs. However, these conditions are sufficient and not necessary.

The designs constructed by Singh and Dey (1979) and Agrawal and Prasad (1982, 83) exist for a few combinations of parameter and most of them demand many blocks. To overcome this, Ipinyomi and John (1985) gave a series of nested row - column designs based on generalised cyclic designs. They have listed the efficient generalised nested row-column designs for $5 \leq v \leq 15$, $p \leq 3$, $q \leq 7$ and $r \leq v$. These designs are A- optimal and some of them are balanced.

Cheng (1986) presented a method of construction of BIBRC designs in many of which none of the N, N_1 and N_2 matrices are the incidence matrices of BIB designs. Methods of construction of BIBRC and BCBRC designs with the help of initial blocks using available BIBD and BIBRC along with few trial and error solutions were presented by Sreenath (1989).

Uddin and Morgan (1990) have given a technique for the construction of BIBRC designs based on the method of differences. They have listed 149 designs for $v \leq 101$ and $3 \leq p \leq q$ out of which 80 were new.

Sreenath (1991) has extended the two theorems on method of symmetrically repeated differences due to Bose (1939) for the construction of BIBRC designs. The designs given by him put aside the formation of BIBD for rows and columns seperately. Uddin (1992) constructed BIBRC for $v = s^2 + s + 1$, where s is an odd prime or prime power.

Saha and Mitra (1992) have also given methods of construction of BIBRC designs for v being an odd prime or prime power exploiting the method of repeated symmetric differences. For a review and further methods of construction requiring smaller number of replications, Sreenath(1996) can be referred. He has also provided a catalogue of these design for $v \leq 30$ and $pq \leq 9$.

Some of the authors have discussed the properties of these designs, and always from the efficiency point of view. For example, Cheng (1986) gives an efficiency table for his designs and Ipinyomi and John (1985) have tabulated a number of efficient designs for certain parametric combinations.

1.4 Optimality Aspects of Block Designs

A large variety of designs available in literature contain a special feature that they are symmetric in treatment of the parameters of interest. This is so perhaps due to the fact that (a) the information matrices of these designs lead to ease of analysis (b) the designs had aesthetic appeal to mathematicians and often had algebraic and geometric representations for easy perception and (c) these yielded statistical estimates, which seemed, as accurate as possible, for the given number of observations. However, with the advent of high speed computers, it was felt that ease of analysis does not necessarily guarantee that the design is good.

A measure of goodness of a design is optimality, which must be given consideration. In a given class of designs attempt should be made to choose a member which is 'good' according to some well-defined criterion. This has led to the study of optimality of experimental designs.

Smith (1918) appears to be the first to introduce a specific optimality criterion in regression designs. Wald (1943) proposed two optimality criteria pertaining to designs with two-way elimination of heterogeneity.

The credit to modern theory of optimality studies goes to the decision theory school of U.S. Statistics, founded by Abraham Wald. The idea of 'risk' developed formally by Wald and arising out of the earlier work of Neyman and Pearson, was the important innovation of that school. There were parallel developments in utility theory, mathematical programming and mathematical economics, so that early history of the subjects were interwoven. Together with Wald, Jacob Wolfowiz and Jack Carl Kiefer were leading members of the school. They started the second great advance in the science of experimentation which lead to the development of an important branch of statistical theory, the 'theory of optimal designs'. However, it was Kiefer (1958) who, for the first time, formalised the optimality criteria in a general set up and presented in a unified way. He attributed different names satisfying different criteria which are universally accepted.

Experiments are generally performed to estimate or test hypothesis about some unknown parameters of a given model. We restrict attention to a setting, in which there is a class D of designs available to the statistician and in which, for each design d $\in$ D, the observations collected via design d obey a standard fixed effect model,

$$E(Y) = X_d \theta, \quad D(Y) = \sigma^2 I_n$$

here Y is the n - component vector of observations, θ is an m- component vector of unknown parameters and X_d is an n x m matrix of known elements ; E and D respectively denote the expectation and dispersion (variance - covariance) matrix.

Suppose $\hat{\theta}_d$ is the best linear unbiased estimator (BLUE) of θ using a design d and let V_d be the dispersion matrix of $\hat{\theta}_d$. It is then reasonable to define an optimality criterion as a meaningful function of V_d. It is to note that if all components of θ are estimable using a design d, then V_d is positive definite and $D(\hat{\theta}_d) = \sigma^2 (X_d' X_d)^{-1}$

Thus we want to choose a design whose information matrix $X_d' X_d$ is large (equivalently $(X_d' X_d)^{-1}$ is small) in some sense.

In practice, one is often interested in certain components of θ, rather than the entire vector θ. Let the vector be partitioned as $\theta = (\theta_1', \theta_2')$ and accordingly, X_d be partitioned as $X_d = (X_{d1} \; X_{d2})$, so that the model becomes

$$E(Y) = X_{d1} \theta_1 + X_{d2} \theta_2$$

here θ_1 contains the parameters of interest, say v in number, and θ_2 contains the nuisance parameters. Then the analogue of $X_d' X_d$ is Bose's C- matrix viz.

$$C(\theta_1)_d = X_{d1}' X_{d1} - X_{d1}' X_{d2} (X_{d2}' X_{d2})^- X_{d2}' X_{d1}, \tag{15}$$

where for a matrix A, A^- denotes an arbitrary generalised inverse (g - inverse).

In comparative design of experiments interest centres around contrasts of treatment effects, so that θ_1 consists of treatment effects and θ_2 consists of nuisance parameters, like block effects (in the context of one - way heterogeneity setting), row and column effects (in the context of row-column designs) etc. For such experiments, the inference problem can be specified as

$$\pi : \eta = \mathbf{P}'\tau ,$$

where **P** is a p × v matrix with zero row sums and τ is a vector of v treatment effects. Thus η contains p treatment contrasts. With reference to π, we call design d acceptable if all components of η are estimable using d. Let C_Π, be the class of all acceptable designs with reference to the problem Π. The problem Π is said to be non-singularly estimable iff rank (**P**) = p, and as a non-singularly estimable full rank problem if rank (P) = v-1. Clearly, for a full rank problem Π , C_Π consists of only such designs {d} for which rank ($\mathbf{C}_d$) = v-1. Such designs are known as connected designs. We confine our attention to only connected designs.

For any design d $\in C_\Pi$, let $\hat{\eta}_d$ denote the BLUE of η. Also let $\mathbf{V}_d$ denote the dispersion matrix of $\hat{\eta}_d$. It is then reasonable to define an optimality criterion as a meaningful function of $\mathbf{V}_d$. Some commonly used criteria are defined below.

Definition 1.11 : D-optimality : A design d* $\in C_\Pi$ is said to be D-optimal in C_Π if

$$\det (V_{d^*}) \leq \det (V_d)$$

for any other design d $\in C_\Pi$. Here 'det ' stands for determinant.This criterion was studied by Wald (1943) and applied by Mood (1946).

Definition 1.12 : A-optimality : A design $d^* \in C_\Pi$ is said to be A-optimal in $C\Pi$ if

$$\text{trace}\ (V_{d^*}) \le \text{trace}\ (V_d)$$

for any other design $d \in C_\Pi$.

This criterion was studied by Elfving (1952) and Chernoff (1953).

Definition 1.13 : E-optimality : A design $d^* \in C_\Pi$ is said to be E-optimal in C_Π

if $\lambda_{\max}(V_{d^*}) \le \lambda_{\max}(V_d)$

for any other design $d \in C_\Pi$. Where $\lambda_{\max}$ denotes the maximum eigen value. Thus this criterion considers minimax rule. This criterion was introduced by Ehrenfeld (1955). It is to be noted that if $P\tau$ represents a complete set of orthonormal treatment contrasts, the dispersion matrix of the BLUE of $P\tau$ is given by $\sigma^2 PC_d^- P' = \sigma^2 PC_d^+ P'$ where C_d^+ denotes the Moore-Penrose inverse of C_d. It can be shown that the eigen values of $PC_d^+ p$ are the same as the positive eigen values of C_d^+. Again the positive eigen values of C_d^+ are the reciprocals of non-zero eigen values of C_d. This observation led to define E-optimality criterion alternately as :

A design $d^* \in C_\pi$ is said to be E-optimal in C_π if the smallest eigen value of C_{d^*} is maximum among the competing designs. Each criterion has its own merit/statistical significance.

Let the observation vector Y follows multivariate normal distribution. Then $\hat{\eta}_d$ also follows multivariate normal distribution with mean vector η_d and dispersion matrix V_d. A 100 (1- α) % confidence region for η_d is the ellipsoid

$$(\eta_d - \hat{\eta}_d)' V_d (\eta_d - \hat{\eta}_d) \le \sigma^2 \chi^2_{\alpha,(v-1)} \qquad (1.6)$$

where σ^2 is the per observation variance, assumed known, and $\chi^2_{\alpha, v-1}$ is the 100 (1- α) percentile of a cental chi-square distribution with v-1 degrees of freedom. If σ^2 is unknown, the ellipsoid is given by

$$(\eta_d - \hat{\eta}_d)' \, V_d^{-} (\eta_d - \hat{\eta}_d) \leq v \, . \, s^2 \, . \, F_{\alpha, (v-1, n_e)} \quad \cdots\cdots\cdots\cdots \; (1.7)$$

where $F_{\alpha,(v-1,n_e)}$ is the 100 (1- α) percentile of central F distribution with v-1 and n_e degrees of freedom and s^2 is an unbiased estimor of σ^2.

Now the volume of (1.6) i.e., expected volume in (1.7) is proportional to the square root of det (V_d^{-}). Thus the D-optimality criterion chooses that design as the best for which the volume (expected volume) of the joint confidence ellipsoid is least. Hence, a D-optimal design is one for which the generalised variance of the BLUE of η is minimum in the class C_{Π}.

As trace is minimised in A-optimality criterion, it implies the choice of a design with minimum average variance of the estimates of η. The E-optimality criterion selects a design for estimating arbitrary normalised contrast wherein maximum variance of estimates isminimised. Under the assumption of normality of the observations, E-optimality criterion has another statistical interpretetion, viz, in hypothesis testing. Suppose we are interested in testing the hypothesis that all the components of τ are equal. Then the power function of the usual F test depends monotonically on a parameter $\delta = \sigma^{-2} \tau' C_d \tau$ and one may like to maximise the minimum power of the F test of size on the contour $\tau' \tau$ = c, a constant. This is equivalent to minimising the largest eigen value of V_d, which is E-criterion.

Definition 1.14 : MV-optimality : In many exploratory experiments, the primary interest is not to optimal estimate arbitrary treatment contrast, but rather to estimate the difference of type $\tau_i - \tau_j$ $(i \neq j)$, called elementary treatment contrasts, with maximum precision

A design d* $\in C\pi$ is said to be a minimum-variance (MV-optimal) design if

$$\max_{i \neq j} Var\ (\tau_i - \tau_j)_{d*} \leq \max_{i' \neq j'} Var\ (\tau_{i'} - \tau_{j'})_d \quad \text{for any other } d \in C\Pi .$$

The criteria of optimality just described are specific in nature. Since that time, some generalizations have been proposed. These generalised optimality criteria were called as "Functional" type of optimality.

Let $P\tau$ represents a complete set of orthonormal treatment contrasts. Then $A = \begin{bmatrix} v^{-1/2} \mathbf{1}' \\ \mathbf{P} \end{bmatrix}$ is an orthogonal matrix. Let B be a v × v symmetric matrix of rank (v-1) such that $\mathbf{B1} = \mathbf{0}$. Then it can be shown that non zero eigen values of **B** and $\mathbf{PBP}'$ are same. The BLUE of $P\tau$ is $P\hat{\tau}$ with dispersion matrix $D(P\hat{\tau})_d = \sigma^2 PC_d^+ P'$ where Cd^+ is the Moore-Penrose inverse of C_d with rank v-1 and zero row and column sums. Thus, instead of minimising a function of eigen values of PC_d^+P' to arrive at an optimal design, one may as well minimise the same function of non zero eigen values of Cd^+.

In view of the above, one may think of an optimality criterion as a function on the set of non negative definite symmetric matrices of order v with zero row sums. If $B_{v,o}$ is the set of all such matrices, then an optimality criterion is a function $\Phi : B_{v,o} \rightarrow (-\infty, \infty)$. A design is called Φ-optimal if it minimises $\Phi(C_d)$. Then A-, D- and E-optimality can be defined as follows :

$$\text{i) A-optimality : Min } \Phi_A (C_d) = \text{Min} \sum_{i=1}^{v-1} \mu_{di}^{-1}$$

ii) D-optimality : $\text{Min}\ \Phi_D(C_d) = \text{Min} \prod_{i=1}^{v-1} \mu_{di}$

iii) E-optimality : $\text{Min}\ \Phi_E(C_d) = \text{Min} (\max_i \mu_{di}^{-1})$

where μ_{di}'s are non - zero eigen values of C_d.

Kiefer (1974) introduced a family of optimality criteria, called Φ_p-optimality. Let

$$\Phi_p(C_d) = \left\{ \sum_{i=1}^{v-1} \mu_{di}^{-p} \right\}^{1/p}, \quad 0 < p < \infty$$

Definition 1.15 : A design d $\in C_\pi$ is said to be Φp - optimal if it minimises $\Phi_p(C_d)$ over a class of competing designs. The A-, D- and E- optimality criteria are special cases of Φ p- optimality criterion corresponding to $p=1$, $p \to 0$ and $p \to \infty$ respectively.

Kiefer (1975) introduced a very strong family of optimality criteria called universal optimality. Consider optimality functionals Φ defined over $B_{v,o}$, the class of all symmetric, non-negative definite matrices of order v with zero row sums, which satisfy

(i) Φ is matrix convex, i.e. for real α, $0 < \alpha < 1$,

$\Phi(\alpha C_1 + (1-\alpha) C_2) \leq \alpha \Phi(C_1) + (1-\alpha)\Phi(C_2)$ for any pair of matrices, C_1, $C_2 \in B_{v,o}$.

(ii) Φ (t.C) is non-increasing in the scalar $t \geq 0$ for all $C \in B_{v,o}$;

(iii) Φ is invariant with reference to the simultaneous permutations of rows and (the same on) columns of $C \in B_{v,o}$.

Definition 1.16 : A design d is called universally optimal if it minimizes Φ (c) over the class of competing designs for all Φ satisfying (i) - (iii) above.

It is to note that if a design is universally optimal, then it is A - , D - , E - , MV - and Φp - optimal as well. In general, it is not easy to identify universal optimal designs in a given class of competing designs. However, Kiefer (1975) gave a sufficient condition for universal optimality.

Let $C\pi$ be a given class of competing designs. A design d* $\in C\pi$ is universally optimal in $C\pi$ if

(i) C_{d*} is completely symmetric, i.e. $C_{d*} = e\, I_v + f\, J_v$ where Iv is v x v identify matrix, Jv is a v x v matrix of unities, e and f are scalars, and (ii) $\text{tr}\,(C_{d*}) = \max\limits_{d \in C_\pi} \text{tr}\,(C_d)$

The optimality criteria discussed so far are the ones that received much attention in the literature. However there are many more optimality criteria, viz., S-optimality (Shah, 1960), (M-S) - optimality (Eccleston and Hedayat, 1974), type 1 and type 2 optimality (Cheng , 1978), Schur optimality (Magda, 1979) , J-optimality (Majumdar, 1986), etc.

Major work on optimality theory is due to Kiefer (1958, 1959, 1971, 1974, 1975) who restricted his attention mostly to the problem of infering on any complete set of (v-1) orthonormal contrasts. However, there are situations, viz, test treatments control comparisions, where this restriction is no more required.

In practice, it is better to establish the Φ optimality of a design, which is difficult too. In these circumstances one has to stick to specific optimality criterion, viz, A - , D -, E - and MV - optimality.

The study shows that in usual block design set up, binary designs perform well. The optimality studies on nested row-column designs were taken up independently by

Chang and Notz (1990) and Bagchi, Mukhopadhayay and Sinha (1990). They started with studying optimality property of BIBRC designs, which are binary, existing in the literature. But surprisinly, they ended up proving that a class of non-binary designs (termed as BN-RC designs) perform very well, to the extent of being universally optimal under fixed effects model. In addition, Bagchi *et al* (1990) have studied optimality results for mixed effects model.

Definition 1.17: In a connected nested row-column design, d, if

$\mathbf{U} = N_1N_1' - \frac{1}{p} NN' = \mathbf{0}$ and N_2 is the incidence matrix of a balanced block design (BBD), then d is called a balanced nested row-column design and denoted by BN-RC (v, b, p, q).

Bagchi *et al.* (1990) have constructed a few series of BN-RC designs. They have studied the optimality of BN-RC designs under a mixed effects model with all nuisance factors random. Chang and Notz (1990) constructed some BN-RC designs from balanced block designs (BBD) by simple permutation method and they extended the construction method to the case where Generalized Youden Designs (GYD,a row - column design is a GYD if its rows and columns form a BBD when each row (or column) is considered as a block) or Pseudo-Youden Design (PYD, or lattice square designs, a row-column design with the number of rows being equal to the number of columns, and the rows and columns together form BBD) are given. Gupta (1992) and Chang and Notz (1994) too have constructed some BN-RC designs . All the designs developed by them were of equal block size, row size and column size.

1.5 Robustness of Block Designs

Optimal designs have beven developed and studied under, what may be called "ideal conditions". That is, if every thing goes right the optimal design is the best bet.

But under distrubance, it may so happen that the opitmum properties of the chosen design are totally lost.

Some factors that tend to disturb the ideal structures are

i) missing observations,

ii) presence of one (or more) outlying observations,

iii) presence of a systematic trend among observations within a block,

iv) inadequacy of the assumed model, e.g. fitting an incomplete model, correlation among observations, etc.

Presence of any one or more of these "disturbances" may render even an optimal design very poor. This realization motivated research workers to develop designs that are insensitive or robust against one or more of the distrubances mentioned above. A review on robustness of design is given by Dey et al.(1991). We are restricting ourselves for the case of unavailability of observation(s) keeping in view the problem being attempted. It ispossible that during the course of an experiment, some observations become unavailable because of unforseeable circumstances, e.g.

(i) destruction of some units by animals,

(ii) damage of some units by pest and/or deseases

(iii) death of plants or animals, etc.

The robustness of block designs against such non-availability of data has been investigated by Hedayat and John (1974), Most (1975), John (1976), Kageyama (1980), Ghosh (1982 a, 1982b, 1982c), Ghosh, Rao and Singhi (1983), Baksalary and Tabis (1987) , Dey and Dhall (1988) , Srivastava, Gupta and Dey (1990), Bhaumik and

Whittinghill III (1991) Gupta and Srivastava (1992) , Dey (1993) , Notz, Whittinghill and Zhu (1994)

In the literature robustness of block designs against missing observations has been investigated from different angle.

A criterion of robustness of incomplete block designs was introduced by Ghosh (1982b). According to this criterion (to be called Criterion 1), an incomplete block design is robust against the loss of t (≥ 1) observations if the residual design obtained by deleting these t observations remains connected. Ghosh (1982b) proved the robustness of BIBD to the unavailability of any (r-1) observations and all observations in any (r-1) blocks by this criterion. The paper by Ghosh (1982c) deals with the fact that different observations have different effects when they become unavailable. Using Criterion 1, Ghosh, Rao and Singhi (1983) investigated the robustness of m- associate class PBIB designs for m = 2 and m = 3. Ghosh (1981, 1982a) investigated the robustness of designs for two-way elimination of heterogeneiety also using Criterion 1. Baksalary and Tabis (1987) gave three sufficient conditions for binary incomplete block designs (IBD) to be robust to the unavailability of $\min_i (r_i)$ - 1 observations. Using the same criterion Notz, Whittinghill and Zhu (1994) presented results for finding or bounding max,B, the maximum number of arbitrary whole blocks of observations which can be removed from a block design. They also studied the maximum number of arbitrary and scattered observations that can become unavailable. These results they applied to balanced block designs (BBD) , reinforced BIBDs and balanced treatment incomplete block designs (BTIBD).

Another criterion of robustness (to be called Criterion 2) that has received attention, is in terms of the efficiency of the residual design. As per Criterion 2 , a design is said to be robust if the efficiency of the residual design relative to the original one is not too small. This efficiency , E, is defined as

$$E = \frac{\text{Hormonic mean of the non-zero eigen values of } C_{d*}}{\text{Hormonic mean of the non-zero eigen values of } C_{d}}$$

C_{d*} and C_d being the C-matrices of the residual (d*) and original designs respectively. This efficiency criterion of robustness is related to A-optimality criterion used for inferring on a complete set of elementary contrasts.

The papers by John (1976), Kageyama (1980), Dey and Dhall (1988), Whittinghill (1989), Srivastava, Gupta and Dey (1990) and Gupta and Srivastava (1992) are in this spirit.

John (1976) considered the situation in which all the observations pertaining to a single treatment in a BIBD are lost and concluded that the loss in efficiency due to non-availability of observations pertaining to a single treatment is small. Similar conclusions were obtained by Dey and Dhall (1988) in respect of augmented BIBDs.

Kageyama (1980) studied the complete loss of a treatment in an efficiency balanced design. He derived an upper and lower bound of efficiency for a connected residual design and proposed that the loss of efficiency even in the unbalanced case is, in general, small.

Constantine (1981) has established that if certain number of disjoint blocks in a BIBD are lost, the remaining structure is E-optimal.

Gupta and Srivastava (1992) has investigated the robustness of (i) binary balanced block (BBB) designs when all the observations in m (≥ 1) disjoint blocks are lost, (ii) resolvable balanced incomplete block designs (BIBD) when all the observations pertaining to one complete replicate are lost, and (iii) augmented BIB designs (derived from augmenting all the blocks by some new treatment) when all the observations in a block are lost. Dey (1993) considered both the criteria to study the robustness of arbitrary incomplete block designs with equal block sizes. He established necessary and sufficient conditions as per criterion 1 when the missing observations appear in the following patterns: (i) t (≥ 1) observations pertaining to the same treatment are misssing, and (ii) all the observations in a block are missing. Simple sufficient conditions for robustness according to criterion 1 are obtained and some classes of robust designs are identified. Dey also presents a lower bound to the efficiency of the residual design when a single observation is missing in an arbitrary incomplete block.

Let D (v, b, r, k, λ) be a BIBD. Let $\Omega = \{1, 2, \ldots, v\}$ denote the set of treatments and T be a subset of Ω with cardinality n ($\leq v - 2$). Let D_0 be the design obtained by deleting from D, all the experimental units receiving treatments in T. Then Hedayat and John (1974) made the following definitions.

Definition 1.17 : A BIB design D is said to be globally resistant of degree n if D_0 is variance - balanced with respect to the loss of any subset T of treatments with cardinality n.

Definition 1.18 : A BIB design D is said to be locally resistant of degree n if D_0 is variance balanced with respect to the loss of some subset (but not all) T of cardinality n. Most (1975), Chandak (1980) and Kageyama (1982) too have studied on resistant BIB designs.

1.6 Orientation of the Problem and Scope of the Thesis

A perusal of the available literature reveals that the existence of BN-RC designs heavily depend on the existence of Latin Square Designs (LSD), Youden Square Designs(YSD), Generalised Youden Designs (GYD) and Pseudo-Youden Designs (PYD). However, they exist only for a limited number of parameter combinations. Their existence also calls for larger blocks (due to the underlying stringent parametric relations) which demand for a large number of units (that are homogeneous under blocking factor) to be arranged in row-column designs, the factors of row and column classification being crossed together. Consequently there is a sharp increase in the replication of each treatment. Thus, in situations with larger block size it looses its practical utility, the experiment being costly and/or unmanageable. Also there may arise row-column interaction when many rows and columns are used. On the other hand, row-column interaction, even if it exists, in Nested Row-Column (NRC) designs with few rows and columns is likely not to be as severe as that with large number of rows and columns in a block (Chang and Notz, 1994).

These aspects led Sreenath (1996) to develop and catalogue BIB-RC (v, b, p, q) designs with block size, $pq \leq 9$ and with smaller number of replications.

Thus to reduce replication of treatments and to obviate the severity of row-column interaction there is a need to develop BN-RC designs which are universally optimal with smaller block sizes and fewer replications that are useful and managable. Keeping these aspects in mind our quest was to develop BN-RC designs with smaller block sizes throughout the thesis.

The equality in block sizes of BN-RC designs has its intuitive and aesthetic appeal. The available methods of construction of block designs are suitably modified and used for the construction of BN-RC designs with equal block sizes. In chapter 2 we have discussed the construction of BNRC design with $pq \leq 9$ with the help of available methods of construction of BIBRC designs and presented BN-RC designs for all v with block size,$pq = 8$, row size,$q = 4$ and column size,$p = 2$. Chapter 2 also includes some BN-RC designs with equal blocks for $v = 6, 7, \ldots, 13$, with block sizes $2 \times q$ for $q \geq 4$ with minimum replications. It also includes some general results on the construction of BN-RC with equal block sizes.

BNRC designs with equal blocks has many a drawbacks, viz., it's inflexible, may cause increase in replication of treatments, which is uncalled-for. This class of designs are more aesthetic than practical. This aspect necessitated to develop BN-RC designs with unequal blocks which provide necessary dynamism, flexibility and saving of a considerable amount of experimental units were possible. This class of BN-RC designs with unequal block sizes is entirely new in literature. Using the concept of balancing in block designs we have presented some methods of construction of Partially Balanced Nested Row-column (PBN-RC) designs, Variance Balanced Nested Row-column (VBN-RC) designs and BN-RC designs with unequal row (and hence block) sizes in chapter 3. Some methods of construction of BN-RC designs with $p=2$ and a combination of column sizes , $3 \leq q \leq 7$ are also presented in chapter 3.

To carry out an experiment with strictly following nested structure throughout demands high amount of experimental material which is a great restriction to an experimenter and at times unnecessary. Given a set of experimental material, it may happen that few of the total units , having homogeneity among themselves, can be arranged into simple blocks needing one-way elimination of heterogeneity only. Whereas , the rest of the units which are more heterogeneous are to be arranged in blocks with nested rows and columns and needing three-way elimination of heterogeneity. Thus, without imposing nested nuisance throughout the design, a mixture of two systems of blocking can be visualised for carrying out the experiment with lesser number of experimental units in hand. Designs with two systems of blocking is completely a new concept and have not been exposed in literature and has scope for improvisation taking into account the nature of experimental material. In Chapter 4, we have presented some consturctions of this type of balanced designs with two blocking systems. The saving in experimental material is remarkably high and this type of designs provide dynamism at its zenith and the restriction of imposing nested classification throughout the design, which may be uncalledfor, gets relaxed. The number of experimental units required under this type of design is least compared to the other two classes of designs described in chapter 2 and chapter 3.

The non-availability of observation(s) in an experiment is a usual phenomena. There is a need to study the effect of loss of observations on the estimability of the effects of interest and efficiency of the residual designs under missing observation(s). A perusal of the existing literature reveals that most of the studies on robustness against the unavailability of observation(s) relates to binary block designs. There is no study on the robustness against missing observations(s) of Block Designs with Nested Row and Columns. In chapter 5, we have presented, for the first time, a study on robustness of BN-RC designs against non-availability of all observations in row(s), column(s) and block(s).

A summary of the results obtained is provided after the chapter 5. Appendix 1 containing some important results in matrix theory and Appendix 2 containing the tables on relative efficiencies of residual designs are given. The thesis ends with the list of references cited in the text and some related topics though not cited as well.

CHAPTER 2

CONSTRUCTION OF BALANCED NESTED ROW- COLUMN (BN-RC) DESIGNS WITH EQUAL BLOCK SIZES

2.1 Introduction

Heterogeneity of experimental material, if not suitably taken care of in designing of an experiment is likely to overshadow the real treatment differences making them remain undected, unless they are large enough. Blocking is a technique used to bring about homogeneity of experimental units within block for the case of one-way elemination of heterogeneity so that the treatment contrasts are estimated, making use of the intra-block information, with higher efficiency. Due to the presence of multiple factors of heterogeneity, blocking alone may fail to remove such heterogeneity. Many a field and laboratory experiments may require three blocking factors in which two of them (row and column factors) are nested in the third (block factor). Since rows and columns are nested within blocks, these designs are called Nested Row-Column (NRC) designs.

A number of authors have studied various aspects of these designs in the row-column design setting. Singh and Dey (1979) presented the analysis of these designs in a situation where a two-way (row-column) structure is present and it is nested within another factor called block. They defined Balanced Incomplete Block Design with Nested Rows and Columns (BIBRC design) and presented some methods of construction. In some subsequent works (Agrawal and Prasad (1982, 1983), Uddin (1992), Uddin and Morgan (1990), Sreenath (1989, 1991, 1996) etc.) various methods of construction of BIBRC designs are presented. These designs are binary in blocks.

The optimality of Nested Row-Column (NRC) designs has been studied recently. Independently, Chang and Notz (1990) and Bagchi, Mukhopadhyay and Sinha (1990) have obtained a class of non-binary designs (termed as BN-RC designs) that perform very well, to the extent of being universally optimal under fixed effects model.

Consider a design d with nested rows and columns having v treatments, each being replicated r times, arranged in b blocks. The treatments in each block are arranged in p rows and q columns. From the model (1.2) in chapter 1, let $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}$ denote the treatment-row, treatment-column and treatment-block incidence matrices respectively. Let the component designs corresponding to $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}$ are respectively d_1, d_2 and d_0. Then the intra-block coefficient matrix of design d is

$$\mathbf{C} = r\,\mathbf{I} - 1/q\,\mathbf{N}_1\mathbf{N}_1' - 1/p\ \mathbf{N}_2\mathbf{N}_2' + 1/pq\ \mathbf{N}\mathbf{N}' = \mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_0,$$

Where $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_0$ denote the intra-block co-efficient matrices for the row-component design d_1, column-component design d_2 and block-component design d_0 respectively.

Definition 2.1 :

If a Nested Row-Column (NRC) design d is such that

i) $\mathbf{U} = \mathbf{N}_1\mathbf{N}_1' - 1/p\,\mathbf{N}\mathbf{N}' = \mathbf{0}$ i.e., $\mathbf{C}_1 = \mathbf{C}_0$, and

ii) $\mathbf{N}_2$ is the incidence matrix of a Balanced Block Design (BBD),

then d is called a Balanced Nested Row-Column (BN-RC) design and denoted by BN-RC (v,b,p,q).

Chang and Notz (1990) obtained BN-RC designs through Latin Square Designs (LSD), , Youden Square Designs (YSD), Generalised Youden Designs(GYD) and Pseudo-Youden Designs (PYD). Bagchi *et al* .(1990) developed methods of construction of BN-RC designs through BBD, YSD , Cyclic differences and by the use of another BN-RC designs. In addition , Bagchi *et al.* studied optimality results for mixed effects model in NRC's. Chang and Notz (1994) have presented a general method of construction of BN-RC designs and studied optimality of NRC's which do not have maximum trace of the information matrix (i.e., do not follow sufficient condition of universal optimality given by Kiefer (1975) but can be Φ_{α}-optimal, for some α, under certain conditions.

A large number of designs offered by the authors, mentioned above, depend heavily on the existence of LSD, YSD, GYD and PYD. However, they exist only for a limited number of parameter combinations. And in all the designs the number of rows and columns within each block are quite high (due to parametric relations). This asks for a large number of experimental material (which are homogeneous under the factor of blocking) that are to be accommodated in a row *vs* column crossed classification. Consequently, replication of each treatment and demand for total experimental units increase sharply. Which is a great constraint to an experimenter, if not lavish access to resources is agreed to. Also, with the increase in number of rows and columns, there may crop up row × column interaction. On the other hand BN-RC designs with fewer rows and columns within each block, the row × column interaction, even if exists, is not likely to be as severe as that with large blocks. Thus , the BN-RC designs with large block sizes have limited practical use.

To obviate these difficulties, BN-RC designs with smaller block sizes (i.e., fewer rows and columns) are needed. Gupta(1992) presented some methods of construction of BN-RC designs with $p = 2$, $q \geq v$. Our endeavour to keep the block size small along with total experimental material at the minimum level motivated us to attempt construction of BN-RC Designs with their N_2 as the incidence matrix of a BIBD (v, b, r, k, λ), which is a particular type of BBD for $k < v$, otherwise it calls for unnecessary repetition and consequent increase on the demand of resources.

Notations: We shall denote a BIBD with v treatments, arranged in b blocks, each of size k, each treatment appearing in r blocks and every pair of treatments appearing in λ blocks by BIBD (v, b, r, k, λ). Consequently BN-RC design with v treatments, arranged in b blocks, each having p rows and q columns, such that N_2 is an incidence matrix of a BIBD (v, bq, r, p , μ) will be denoted as BN-RC (v, b, r, p, q, μ). Unless othewise stated symbols will be denoted by elements of mod (v) i.e., 0, 1, 2, . . . , v-1.

·2.2 Some BN-RC Designs with Equal Block Sizes.

We now give below some general methods of construction of certain series of BN-RC designs.

Theorem 2.1 : Given v, an odd number, there always exists a BN-RC (v, $\frac{1}{2}$ (v-1), v-1, 2, v,1).

Proof: Form the i^{th} block of the design with j^{th} column elements as (j-1) and (j-1 + i) mod (v) respectively in two rows for $i = 1, 2, \dots, \frac{1}{2}(\nu - 1)$ and $j = 1, 2, \dots, v$.

i.e., ith block, $B_i = \begin{bmatrix} 0 & i & \dots & v-1-i & v-i & v-i+1 \dots & v-1 \\ i & i+1 \dots & v-1 & 0 & 1 & \dots & i-1 \end{bmatrix}$

It can be easily seen that the design so constructed is the BN-RC (v,$\frac{1}{2}(\nu-1)$, v-1, 2, v, 1).

Note 2.1 : When v is even the blocks B_i with $i = 1, 2, \ldots, v\text{-}1$ results in a BN-RC (v, v-1, 2 (v-1), 2, v,2). This has been reported by Gupta (1992) for both v = even and odd.

Note 2.2 : This design is an improvement over the one reported by Gupta (1992) as the number of replications is halved.

Example 2.1 : Let v = 5. Then Theorem 1 yields a BN-RC (5, 2, 4, 2, 5, 1) :

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}$$

Theorem 2.2 : The existence of D_1, a BIBD (v_1 , b_1 , r_1 , k_1 , λ_1), and of D_2, a BN-RC (k_1 , b_2 , r_2 , p , q , λ_2), implies the existence of D, a BN-RC ($v = v_1$, $b = b_1 b_2$, $n = r_1 . r_2$, p , q , $\mu = \lambda_1 \lambda_2$).

Proof : Let $j_1, j_2, \ldots, j_k$ be the symbols occurring in the j^{th} block of D_1. Using these as the symbols in D_2, we develop b_2 blocks of D. Repeating this procedure for all the b_1 blocks of D_1 we obtain $b_1 b_2$ blocks of D.

It can be seen that the D as obtained above is a nested row-column design with v_1 symbols/treatments, arranged in $b_1 b_2$ blocks each further arranged in p rows and q columns. Each of the treatments is replicated $r_1 r_2$ times. Since D_2 is a BN-RC, the contribution to $\mathbf{C}_1$ and $\mathbf{C}_0$ matrices of D from every block of the group of b_2 blocks as obtained from each of the blocks of D_1 will be equal and hence $\mathbf{C}_1 = \mathbf{C}_0$ for D. Obviously $\mathbf{N}_2$ of D will be the incidence matrix of a BIBD (v_1, $b_1 . b_2 . q$, $r_1 r_2$, p , $\lambda_1 \lambda_2$). Hence proved.

Corollary : If $\exists$ a BIBD (v, b, r, k,λ) it is always possible to construct

(i) a BN-RC (v, $\frac{1}{2} b (k - 1)$, r (k-1), 2, k, λ) for odd k , and

(ii) a BN-RC (v, b (k-1), 2 r (k-1), 2, k, 2 λ) for even k.

Example 2.2 : Consider D_1, a BIBD (10, 18, 9, 5, 4) having blocks as

(∞ , 0, 1, 4, 6)
(∞ , 1, 2, 5, 7)
(∞ , 2, 3, 6, 8)
(∞ , 3, 4, 7, 0)
(∞ , 4, 5, 8, 1)
(∞ , 5, 6, 0, 2)
(∞ , 6, 7, 1, 3)
(∞ , 7, 8, 2, 4)
(∞ , 8, 0, 3, 5)

(0, 1, 2, 4, 8)
(1, 2, 3, 5, 0)
(2, 3, 4, 6, 1)
(3, 4, 5, 7, 2)
(4, 5, 6, 8, 3)
(5, 6, 7, 0, 4)
(6, 7, 8, 1, 5)
(7, 8, 0, 2, 6)
(8, 0, 1, 3, 7)

and D_2, a BN-RC (5, 2, 4, 2, 5, 1) with blocks

as $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}$

Take the first block from D_1 with contents (∞ , 0, 1, 4, 6). Using the contents of this block in D, form two blcoks in D as

$\begin{bmatrix} \infty & 0 & 1 & 4 & 6 \\ 0 & 1 & 4 & 6 & \infty \end{bmatrix}$ and $\begin{bmatrix} \infty & 0 & 1 & 4 & 6 \\ 1 & 4 & 6 & \infty & 0 \end{bmatrix}$

In each of these two blocks $\mathbf{C}_1 = \mathbf{C}_0$ as two rows in each block contain the same set of elements.

Repeating this procedure for the rest of the blocks in D_1 we get the BN-RC

(10, 36, 36, 2, 5, 4)

Theorem 2.3 : If there exists D_1, a BIBD (v, vs, ks, k, μ) with s initial block solutions (without involving invariant element) then there always exists D, a BN-RC (v, s, ks, k, v, μ), for v belonging to single finite field.

Proof : Consider the j^{th} initial block of D_1. Let $j_1, j_2, \ldots, j_k$ be the contents of this block. The j^{th} block of D is obtained with its (m,l)-th element as (j_m + 1) reduced mod (v) for m = 1,2, . . . , k ; l = 1, 2, ... , v, and j = 1, 2, . . . , s. The columns within blocks of D will be the design D_1, a BIBD (v, vs, ks, k, μ). Since each of the treatments/symbols 0, 1, . . . , v-1 will be occuring exactly once in each of rows of any block of D, we have $\mathbf{C_1} = \mathbf{C_0}$ for D. Hence proved.

Example 2.3 : Consider a BIBD (13, 26, 6, 3, 1) with its inital blocks (0, 1, 4) and (0, 2, 7). From the first initial block we form the block

$$B_1 : \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 \end{bmatrix}$$

and from the second initial block another block B_2 is formed as

$$B_2 : \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 \\ 7 & 8 & 9 & 10 & 11 & 12 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

These two blocks together prodides a BN-RC (13, 2, 6, 3, 13, 1)

Note 2.3 : Some times it is possible to construct a BN-RC design with the help of an initial block solutions in the case of v = m.n symbols denoted with the m elements of a module M and associating n elements to each of them.

Example 2.4 : Consider a BIBD (21, 42, 12, 6, 3) with its six initial block solutions as

(i) $(0_1, 5_1, 1_2, 4_2, 2_3, 3_3)$; (iv) $(0_1, 1_1, 3_1, 0_2, 1_2, 3_2)$

(ii) $(0_2, 5_2, 1_3, 4_3, 2_1, 3_1)$; (v) $(0_2, 1_2, 3_2, 0_3, 1_3, 3_3)$

(iii) $(0_3, 5_3, 1_1, 4_1, 2_2, 3_2)$; (vi) $(0_3, 1_3, 3_3, 0_1, 1_1, 3_1)$

(mod 7)

We first form mutually exclusive and equisized groups of the initial blocks of the BIBD such that when the initial blocks in a group are placed in columns, after rearrangement within the columns, if necesary , symbols from different classes occur equally frequently in all the rows of such an arrangement. Then in each of the groups, each member (rearranged initial block of the given BIBD) is developed following the convention of block designs. The cases where no initial block solutions are left out of reckon in any of the groups provide the required BN-RC designs.

Here the first three initial blocks without any rearrangement form a group and the next three form another group since in these all the classes in six rows occur equally frequently

Column → Row ↓	First group			Second group		
	1	2	3	1	2	3
1	1	2	3	1	2	3
2	1	2	3	1	2	3
3	2	3	1	1	2	3
4	2	3	1	2	3	1
5	3	1	2	2	3	1
6	3	1	2	2	3	1

Then the following two blocks

$$\begin{bmatrix} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 \\ 5_1 & 6_1 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_3 & 6_3 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 \\ 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 & 0_3 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 \\ 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 & 4_3 & 5_3 & 6_3 & 0_3 & 1_3 & 2_3 & 3_3 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 & 3_1 \\ 2_3 & 3_3 & 4_3 & 5_3 & 6_3 & 0_3 & 1_3 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 \\ 3_3 & 4_3 & 5_3 & 6_3 & 0_3 & 1_3 & 2_3 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 & 0_3 \\ 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_3 & 4_3 & 5_3 & 6_3 & 0_3 & 1_3 & 2_3 \\ 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_3 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 \\ 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 & 0_3 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 \\ 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_3 & 4_3 & 5_3 & 6_3 & 0_3 & 1_3 & 2_3 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 \end{bmatrix}$$

results into BN-RC (21, 2, 12, 6, 21, 3).

Before giving the next theorem on the construction of the design we consider the following preliminaries.

Let $v = 2qt + 1$ be a prime or prime power and x be a primitive root of GF(v). The non-zero elements of GF(v) can be represented by $x^0 = 1, x, x^2, \dots, x^{2qt-1}$. Then $x^{2qt} = 1 \Rightarrow x^{qt} = -1$ (since $x^{qt} \neq 1$)

Consider a set of q elements

$$(x^i, x^{2t+i}, x^{4t+i}, \dots, x^{2(j-1)t+i} \dots, x^{2(q-1)t+i})$$

Then the sum of the first j elements in the above sequence is

$$x^i (x^{2jt}-1) / (x^{2t}-1) = x^i . y_j \quad \text{mod (v), say}$$

where $y_j = (x^{2jt}-1) / (x^{2t}-1)$; $j = 1, 2, \dots, q$

$y_1 = 1$ and $y_q = 0 = y_0$.

Clearly $x^i y_j$ is an element of GF (v) since $x^{2t} \neq 1$. Now let us consider the following initial block

$$B_i : \begin{bmatrix} 0 & x^i.y_1 & x^i.y_2 \ldots. & x^i.y_{j-1} \ldots. & x^i.y_{q-1} \\ x^i.y_1 & x^i.y_2 & x^i.y_3 \ldots & x^i.y_j \ldots. & x^i.y_q = 0 \end{bmatrix}$$

In this block we see that the elements in the two rows are same and the rows are binary.

The two differences arising out of the two elements in the jth column of this block are $x^i(y_j - y_{j-1}) = x^{2(j-1)t + i}$ and $x^i(y_{j-1} - y_j) = x^{qt + 2(j-1)t + i}$,for $j = 1, 2, \ldots q$

Obviously the differences $x^{2(j-1) + i}$ for $j = 1, 2, \ldots, q$ are distinct

Similarly the differences $x^{qt + (j-1)t + i}$ for $j = 1, 2, \ldots, q$ are also distinct

However, for some values of m and s we may have

$$x^{2st + i} = x^{qt + 2mt + i}$$

$$\Rightarrow \quad 2st = qt + 2mt \mod (2qt)$$

$$\Rightarrow \quad q = 2(s-m) \mod (2qt) \quad \text{i.e., q is even.}$$

If q is even $m = s - \frac{q}{2}$ or $s = m + \frac{q}{2}$

Thus for even q these two differences will be equal. Which implies that a difference is repeated twice in the columns of a block.

For an odd q, by complete enumeration it can be verified that among all the differences arising out the columns of the t initial blocks of B_i's, $i = 0, 1, \ldots, t-1$, every non-zero element of GF (v) appears exactly once. And for an even q, by complete enumeration it can be seen that among all the differences arising out of the columns of the 2t initial blocks of B_i's , i = 0, 1, ... , 2t - 1 , every non-zero element of GF (v) appears twice.

In view of these revealations we state the following theorem.

Theorem 2.4 : Given v = 2qt + 1, a prime or prime power, there exists

(i) a BN-RC (v, b = t.v, r = v-1, 2, q, 1) when q (> 1) is an odd number, and

(ii) a BN-RC (v, b = 2 tv, r = 2 (v-1), 2, q, 2) when q is an even number.

Proof: From the discussion made above, clearly,

(i) for q, an odd number , when B_i's in (A) are developed (following the convention of block designs) under mod (v) for , i = 0, 1, . . . , t-1, yields a BN-RC (v, b = t. v, r = v-1 , 2, q, 1) and,

(ii) for q, an even number, when B_i's , i = 0, 1, . . . , 2t-1, are developed under mod (v) results into a BN-RC (v, b = 2tv , r=2(v-1), q, 2).

Example 2.5 : (i) Consider q = 3 and t = 2. Then v = 13 is a prime number and 2 is a primitive root of GF (13). The elements of GF (13) are 0, 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7. Consider the following two initial blocks.

$$B_0 : \begin{bmatrix} 0 & 1 & 4 \\ 1 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B_1 : \begin{bmatrix} 0 & 2 & 8 \\ 2 & 8 & 0 \end{bmatrix}$$

Developing these two initial blocks under mod (13) yields a BN-RC (13, 26, 12, 2, 3, 1)

(ii) Consider q = 6 and t = 1 Then v = 13 is a prime number and primitive root and elements are same as that in the previous example. Consider the following two initial blocks.

$$B_0 : \begin{bmatrix} 0 & 1 & 5 & 8 & 7 & 3 \\ 1 & 5 & 8 & 7 & 3 & 0 \end{bmatrix} \quad \text{and} \quad B_1 : \begin{bmatrix} 0 & 2 & 10 & 3 & 1 & 6 \\ 2 & 10 & 3 & 1 & 6 & 0 \end{bmatrix}$$

Developing these two initial blocks under mod (13) yields a BN-RC (13, 26, 24, 2, 6, 2)

These two examples warrant a judicious assignment of q and t for same v to have a BN-RC with lesser experimental units.

2.3 Some Considerations on BN-RC Designs with Smaller Block Sizes

We have given some methods of construction of certain series of BN-RC designs. As stated earlier in section 2.1, these designs with smaller block sizes are more useful and appropriate since we are not accounting for the row x column interactions within each of the blocks in our model. Further, the grouping of experimental units in the blocks with nested rows and columns with a crossed classification of rows and columns in each block becomes difficult unless we have a large number of units to choose from for making such arrangement with larger number of rows and columns within blocks. As such we propose to study the methods of construction of BN-RC designs with smaller number of rows and columns in this section. We thus consider the cases, for $pq \leq 12$ with $p \leq q$. i.e., (p , q) = (2 , 2), (2 , 3), (2 , 4), (3 , 3), (2 , 5), (2 , 6) and (3 , 4). To have a BN-RC design we require that $C_1 = C_0$, which inturn implies that within each of the blocks of the design, the row contents are to be same (i.e. within a block if a row contain q symbols in the order $(j_1, j_2 \dots j_q)$ then each of the remaining rows is a permulation of it) and d_2, the column-component design should form a BIBD. In a block there should occur more than one distinct symbols, otherwise this block provides no information. Similarly a column with only one distict symbol contributes nil information and deletion of this column does not affect to the validity of $\mathbf{C_1} - \mathbf{C_0} = \mathbf{0}$. We, therefore, avoid construction of such designs.

i) Construction of BN-RC Designs with $p = q = 2$

If x_1 and x_2 are two distinct symbols in a blcok, the arrangement of the symbols in rows and columns of the blocks should necessarily be as

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} \qquad (*)$$

Such arrangement ensures $\mathbf{C_1} - \mathbf{C_0} = \mathbf{0}$. Thus, for d_2, the column-component designs to form a BIBD, it is necessary that each of the v_{C_2} distinct pairs are to be considered and each such pair are to be organised in a block of type ($*$). This is possible when each block of an irreducible BIBD with block size 2 is further subjected to form Latin sqaure design (LSD). The resulting design is a BN-RC (v, v_{C_2}, 2(v-1), 2, 2, 2,). This design is also reported by Chang and Notz (1990) and Bagchi *et al* .(1990).

(ii) Construction of BN-RC designs with $p = 2$, $q = 3$.

Arguing the same way as for $p = q = 2$, it can be shown that in each block of these designs there should appear three distinct symbols, x_1, x_2 and x_3, say, and the only possible arrangement in a block would be as

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{bmatrix}$$

It is to notice that columns within each block forms a BIBD (3, 3, 2, 2, 1) and each row form a Randomised block Design (RBD) with three symbols. Since d_2, the column- component design of the BN-RC (v,b,r,2,3, μ), is to be a BIBD (v, 3b, r, 2, μ) , it can be seen that d_1, the row component design should form a BIBD (v, 2b, r,3, 2μ) a s below.

The designs d_1 and d_2 are binary block disigns with v symbols each replicated r times. corresponding to each occurrence of a pair of symbols in d_2 they occur twice in d_1. Thus,

$$C_2 = \frac{\mu v}{2} (I - \frac{1}{v} J)$$

$$C_1 = r.I - \frac{1}{3}\left\{(r - 2\mu) I + 2\mu J \right\} = \frac{2\mu v}{3} \left(I - \frac{1}{v} J \right) \text{ , since } r = \mu (v - 1)$$

Thus, a BN-RC design with p = 2 and q = 3 from D_1, a BIBD (v, b, r, 3, μ) and each block of D_1 being further subjected to form a BIBD (3, 3, 2, 2, 1) or D_2 a YSD with 3 symbols and blcok size 2 . This design is also reported by Bagchi *et al* . (1990) and Chang and Notz (1990).

iii) Construction of BN-RC designs with p = q = 3.

Arguing in a similar way as for the case p = q = 2, a BN-RC with p = q = 3 exists if there exists D_1, a BIBD (v, b, r, 3, μ) and a LSD with 3 symbols. This desgin is also reported by Chang and Notz (1990) and Bagchi *et al*. (1990).

iv) Construction of BN-RC designs with p = 3 and q = 4.

It can be shown that from ablock with p = 3 and q = 4 maximum overall information is available if four distinct symbols appear in it and columns in each of the blcoks form a BIBD (4,4,3,3,2) i.e., if each of the blocks form a YSD with 4 symbols.

Following the same argument as the case for p = 2 and q = 3, a BN-RC with p = 3 and q = 4 exists if ∃ a BIBD with block size 4 and YSD with 4 treatments. This design is also reported by Bagchi *et al* .(1990) and Chang and Notz (1990). Thus we have completed the methods of construction of BN-RC designs with (p,q) = (2,2), (2,3), (3,3) and (3,4). In the following sections we consider the methods of construction of BNRC designs with (p,q) = (2,4), (2,5) and (2,6).

2.4 Some Consideration on the Method of Differences

For the construction of BN-RC designs with p = 2 and q = 4 we consider the cases v = 8t + i for i = 0,1, . . . , 7. And for the construction of BN-RC designs with (p,q) = (2,5) and (2, 6) we consider v = 6,7, . . . , 13. We make use of the method of differences in their construction. It involves choice of sets of q such differences whose sum is 0(mod v).

Let us denote $x_1 + x_2 + \ldots + x_j = y_j$ mod (v). Then using a set of q differences, D: (x_1, x_2, ... x_q) an initial block of BN-RC is obtained as

$$\begin{bmatrix} 0 & y_1 & y_2 & \cdots & y_{j-1} & \cdots & y_{q-1} \\ y_1 = x_1 & y_2 & y_3 & \cdots & y_j & \cdots & y_q = 0 \end{bmatrix}$$

The differences arising out of the columns of this inital block are $\pm x_1$, $\pm x_2$, . . . , $\pm x_q$. These intial sets are developed cyclically (partial or complete) under mod(v) to obtain distinct blocks in the sense that one can not be obtained from the other with re-ordering of columns. Unless otherwise mentioned an initial block is to be developed completely following the convention in block designs. The symbol P.C. means **Partial Cycle** and P.C. (m) means partial cycle upto m blocks i.e. developing an initial block with addition of 0, 1, 2,. . . , (m-1) in turn with the elements to form m blocks. In each case elements in a block are reduced to mod (v) of the initial block.

This method of construction ensures that $C_1 = C_0$ for the design obtained and d_2, the column-component design, is an equireplicated, proper and binary block design. If the method of construction ensures all the distinct differences (1,2,... ,v-1) occurring equally frequently over the inital blocks then d_2, the column-component design, will be a BIBD and hence the design so constructed will be a BN-RC design.

It is to be noted that in a BN-RC $(v, b, r, p = 2, q, \mu)$ the column component design d_2, forms a BIBD $(v, qb, r, 2, \mu)$. And our quest is to attain it with minimum experimental units. Since each block contain q pairs of treatments (in q columns), $\mu = 1$ if v_{C_2} is divisible by q, i.e., $\frac{v(v-1)}{2.q}$ is an integer and r has to be necessarily an even number (as in a block if a treatment appears it appears twice, once in each row, so that $C_1 = C_0$, thus enabling placement of q of v_{c_2} distinct pairs in each of b blocks. Otherwise a suitable minimum multiple (μ_{min}, say) of v_{c_2} pairs of treatments has to be taken to that effect.

2.5 Construction of BN-RC Designs with p = 2 and q = 4

In view of the observations in section 2.4, a BN-RC $(v, b, r, 2, 4, \mu)$ with $\mu = 1$ exists if $\frac{v(v-1)}{8}$ is an integer and r is an even number, otherwise a suitable minimum multiple (μ_{min}, say) has to considered. For the construction of BN-RC designs with $p = 2$ and $q = 4$ we consider the cases $v = 8t + i$, for $i = 0, 1, \ldots, 7$. The following list reveals μ_{min} that is required for the existence of a BN-RC with $p = 2$ and $q = 4$ for different v with minimum number of experimental units.

v	μ_{min}
8t	2
8t+1	1
8t+2	4
8t+3	4
8t+4	2
8t+5	2
8t+6	4
8t+7	4

We have achieved μ_{min} as enlisted above for each of v in the following.

v = 4 : The following blocks provide a BN-RC(4,3,6,2,4,2).

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

v = 5 : The following blocks provide a BN-RC(5,5,8,2,4,2)

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 & 1 & 3 \\ 4 & 1 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 4 & 2 \\ 2 & 4 & 2 & 0 \end{bmatrix}$$

v = 6 :

The following sets of differences

(i) One set of type, D_1 : [1, 2, 1, 2]. Repeat it twice and

(ii) One set of type , D_2 : [3, 3, 3, 3] P.C. (3)

mod (6) provides a BN-RC (6, 15, 12, 2, 4, 4)

v = 7 :

The following sets of differents

(i) one set of type, D_1 : [2, 2, 4, 6]

(ii) one set of type, D_2 : [3, 3, 2, 6]

(iii) one set of type, D_3 : [1, 1, 2, 3]

mod (7)

results into BN-RC (7, 21, 28, 2, 4, 4)

Now we construct the designs seperately for each v in the form of the proof of theorems for $t > 0$.

Case 1 : v = 8t

Theorem 2.5 : There always exist a BN-RC designs with parameters $v = 8t$, $b = 2t(8t-1)$, $r = 2(8t-1)$, $p = 2$, $q = 4$, $\mu = 2$.

Proof: Consider the following (2t + 1) sets of differences .

i) t sets of type, D_{1i} : [x , v-(x + 1) , v-(x + 2) , x-3]

where $x = 4i - 3$ for $i = 1, 2, \ldots, t$.

ii) (t-1) sets of type, D_{2j} : [y, v-(y + 1), v- (y + 2) , y + 3]

where $y = 2j - i$ for $j = 1, 2, \ldots, t-1$

iii) one set of type , D_3 : [2t-1, 2t + 1, 6t + 1, 6t -1] P.C.(4t)

iv) one set of type , D_4 : [2t, 2t, 2t, 2t] P.C.(2t)

mod (8t)

Using these sets of differences we construct the corresponding initial blocks as in section 2.4 and develop them cyclically. The sets D_3 and D_4 are developed partially, i.e., P.C.(4t) and P.C.(2t) respectively. The differences accounted for in different category are as follows :

In (i) each of the differences except 4t appears once and difference 4t appears twice (since it is its own complement)

In (ii) all the differences except (4t, 2t-1, 2t + 1, 6t + 1, 6t-1 and 6t) appear exactly once.

In (iii) the differences (2t-1, 2t + 1, 6t + 1, 6t-1) appear once. Complementary differences appearing in this category are so arranged that a P.C.(4t) ensures the happening.

In (iv) all the differences are equal and developing it in quarter of a complete cycle i.e., P.C.(2t) ensures that difference of 2t and 6t appears once.

Thus, in the 2t + 1 intial blocks each of the differences appear exactly twice. Hence proved

Example 2.6 : Let $t = 2$, $v = 16$. Then following sets of differences

Category	Sets of Differences	Initial Blocks	No. of blocks developed
i) D_{11} :	[1, 14, 13, 4]	$\begin{bmatrix} 0 & 1 & 15 & 12 \\ 1 & 15 & 12 & 0 \end{bmatrix}$	16
and D_{12} :	[5, 10, 9, 8]	$\begin{bmatrix} 0 & 5 & 15 & 8 \\ 5 & 15 & 8 & 0 \end{bmatrix}$	16
ii) D_{21} :	[1, 14, 10, 7]	$\begin{bmatrix} 0 & 1 & 15 & 9 \\ 1 & 15 & 9 & 0 \end{bmatrix}$	16
iii) D_3 :	[3, 5, 13, 11] P.C.(8)	$\begin{bmatrix} 0 & 3 & 8 & 5 \\ 3 & 8 & 5 & 0 \end{bmatrix}$	8

iv) D_4 : $[4, 4, 4, 4]$ P.C. (4) $\begin{bmatrix} 0 & 4 & 8 & 12 \\ 4 & 8 & 12 & 0 \end{bmatrix}$ 4

yield a BN - RC (16, 60, 30, 2, 4, 2).

The block developed using D_3 are :

$$\begin{bmatrix} 0 & 3 & 8 & 5 \\ 3 & 8 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 9 & 6 \\ 4 & 9 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 10 & 7 \\ 5 & 10 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 6 & 11 & 8 \\ 6 & 11 & 8 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 & 12 & 9 \\ 7 & 12 & 9 & 4 \end{bmatrix} \begin{bmatrix} 5 & 8 & 13 & 10 \\ 8 & 13 & 10 & 5 \end{bmatrix} \begin{bmatrix} 6 & 9 & 14 & 11 \\ 9 & 14 & 11 & 6 \end{bmatrix} \begin{bmatrix} 7 & 10 & 15 & 12 \\ 10 & 15 & 12 & 7 \end{bmatrix}$$

and the using D_4 are .

$$\begin{bmatrix} 0 & 4 & 8 & 12 \\ 4 & 8 & 12 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 9 & 13 \\ 5 & 9 & 13 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 10 & 14 \\ 6 & 10 & 14 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 & 11 & 15 \\ 7 & 11 & 15 & 3 \end{bmatrix}$$

Case 2 : $v = 8t + 1$

Theorem 2.6 : There always exists a **BN-RC** design with parameters, $v = 8t + 1$, $b = t(8t + 1)$, $r = 8t$, $p = 2$, $q = 4$, $\mu = 1$.

Proof : The sets of difference D_i : $[x, v\text{-}(x + 1), v\text{-}(x + 2), x + 3]$

where $x = 4i\text{-}3$ for $i = 1, 2, \ldots, t$ mod $(8t + 1)$ provide the required design.

The i^{th} differences account for successive 4 distinct differences along with their complementary 4 differences 4i-3, 4i-2, 4i-1 and 4i exactly once. Thus, on complete enumeration it can be found that in these t initial blocks all the differences (1,2, . . . ,8t) appear exactly once.

Case 3 : $v=8t+2$

Theorem 2.7 : There always exists a **BN-RC** ($v=8t+2$, $b=(8t+2)(4t+1)$, $r=4(8t+1)$, $p=2$, $q=4$, $\mu=4$).

Proof : Let us consider the following $4t+1$ initial sets of differences.

i) t sets of type, D_{1i} : $[x, v-(x+1), v-(x+2), x+3]$where $x=4i-3$ for $i=1,2,\ldots,t$. Repeat each of them 4 times and

(ii) one set of type, D_2 : $[4t+1, 4t+1,4t+1,4t+1]$

mod $(8t+2)$

In (i) all the differences except $(4t+1)$ appears once when D_{1i},$t=1,2\ldots,t$ is considered once. Where as in (ii) the difference $(4t+1)$ appears 4 times. Hence on repeating the differences in (i) 4 times each the BN-RC design can be constructed.

Case 4 : $v=8t+3$

Theorem 2.8 : There always exists a **BN-RC** design with parameters ($v=8t+3$, $b=(4t+1)(8t+3)$, $r=4(8t+2)$, $p=2$, $q=4$, $\mu=4$).

Proof: Consider the following sets of differences .

(i) t sets of type, D_{1i} : $[x, v-(x=1), v-(x+2), x+3]$

where $x=4i-3$ for $i=1,2,\ldots,4$. Repeat each of them twice.

(ii) t sets of type, D_{2j} : $[y, v-(y+1), v-(y+2), y+3]$

where $y = 4j-2$ for $j=1,2,\ldots,t$. Repeat each of them twice.

(iii) one set of type, D_3 : $[1, 4t+1, 8t+2, 4t+2]$

Thus, from (i) and (ii) all of the differences except (1, 8t + 2, 4t + 1 and 4t + 2) are accounted for 4 times whereas the differences (1, 8t+2, 4t+1, and 4t+2) are accounted for twice. In (iii) the differences (1, 8t+2, 4t+1 and 4t+2) appear twice. Hence proved.

Case 5 : v=8t+4

Theorem 2.9 : There always exists a **BN-RC** design with parameters $v=8t+4$, $b=(2t+1)(8t+3)$, $r=2\,(8t+3)$, $p=2$, $q=4$ and $\mu=2$.

Proof: The following sets of differences

(i) t sets of type, $D_{1i} = [x, v-(x+1), \frac{v}{2}+(x+1), \frac{v}{2}-x]$

where $x = 2i-1$ for $i=1,2,\ldots,t$. Repeat each of them twice.

(ii) one set of type, D_2: $[2t+1, 4t+2, 6t+3, 4t+2]$ P.C.(4t+2)

(iii) one set of type, $D_3 = [2t+1, 2t+1, 2t+1, 2t+1]$ P.C.(2t+1)

mod (8t+4)

provide the design.

In (i) all the differences except (2t+1, 6t+3, 4t+1, 4t+3 and 4t+2) are accounted twice. In (ii) difference (4t+2) is accounted twice and differences (2t+1 and 6t+3) accounted for once. And in (iii) differences (2t+1) and (6t+3) are accounted for once. Hence the theorem.

Case 6 : v=8t+5

Theorem 2.10: There always exists a **BN-RC** ($v=8t+5$, $b=(2t+1)\,(8t+5)$, $r=2(8t+4)$, $p=2$, $q=4$, $\mu=2$)

Proof: Consider the following sets of differences.

(i) $(t+1)$ sets of type, D_{1i} : $[x, \ v-(x+1), \ v-(x+2), \ x+3]$ where $x=4i-7$ for $i = 1,3, \ldots, t-1$ and $t>1$ Repeat each of them twice.

(ii) one set of type, D_2: $[4t-3, 4t+7, 4t+6, 4t]$

(iii) one set of type, D_3: $[4t+2, 4t+4, 4t-7, 4t-3]$

and (iv) one set of type, D_4: $[4t+2, 4t+4, 4t-7, 4t-3]$

mod $(8t+5)$.

All the differences are accounted for twice in the given sets. Hence proved.

Case 7 : $v=8t+6$

Theorem 2.11 : There always exists a **BN-RC** ($v=8t+6$, $b = (2t+3)(8t+5)$, $r = 4(8t+5)$, $p=2$, $q=4$, $\mu=4$).

Proof: Consider the following sets of differences.

(i) $2t+1$ sets of type, D_{1i} : $[i, \ v/2-i, \ v-i, \ v/2+i]$

for $i = 1, 2, \ldots, 2t+1$. Repeat each of them twice.

(ii) one set of type, D_2 : $[4t+3, 4t+3, 4t+3, 4t+3]$ P.C.$(4t+3)$.

mod $(8t+6)$.

In (i) all the differences except $4t+3$ is accounted for 4 times. And in (ii) the difference $(4t+3)$ is accounted 4 times.

Hence proved.

Cade 8 : $v=8t+7$

Theorem 2.12 : There always exists a **BN-RC** ($v=8t+7$, $b=(4t+3)(8t+7)$, $r=(4t+3)(8t+7)$, $p=2$, $q=4$, $\mu=4$).

Proof : Consider the following sets of differences

(i) (t-1) sets of type, D_{1i} : $[x, v-(x+1), v-(x+2), x+3]$

where $x = 4i-7$ for $i = 1, 2, \ldots, t$ and $t > 1$. Repeat each of them 4 times .

(ii) Four sets of type, D_{2j} : $[4t+1-j, 4t+5+j, 4t+4+j, 4t+4-j]$

where $j = 1, 2, 3, 4$

(iii) Two sets of type, D_{3k} : $[4t+1+k, 4t+7-k, 4t+7+k, 4t-1-k]$ for $k=1,2$ and

(iv) one set of type, D_4 : $[4t-3, 4t+3, 4t+10, 4t+4]$

mod $(8t+7)$.

In (i) all the differences except those included in (ii), (iii) and (iv) occur 4 times. And taking (ii), (iii) and (iv) the rest of the differences appear 4 times. Hence proved.

2.6 Construction of BN-RC Designs with $p=2$, and $q=5, 6$ and 7

Following the observations made in section 2.5 we consider the construction of some **BN-RC** designs in this section for $v=6,7,\ldots, 13$ with $p=2$ and $q=5, 6$ and 7. The designs mentioned so require the minimum possible experimental units in their specific category. Some of the designs constructed systematically and a few by trial and error method. In the following we deal each of $v= 6, 7, \ldots, 13$ seperately.

Case 1 : v=6, p=2 and q=5

The following is a trial and error solution of **BN-RC** (6,6,10,2,5,2)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 4 & 5 & 0 \\ 4 & 5 & 0 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 4 & 5 & 0 & 1 \\ 4 & 5 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 5 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 5 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}$$

Case 2 : v=7, p=2

a) q =5

The following initial blocks provide a **BN-RC** (7,21,30,2,5,5)

$$\begin{bmatrix} 0 & 1 & 3 & 5 & 2 \\ 1 & 3 & 5 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 4 & 3 \\ 1 & 2 & 4 & 3 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 4 & 5 & 1 & 3 \\ 4 & 5 & 1 & 3 & 0 \end{bmatrix} \quad \text{mod } (7)$$

b) q =6

The following initial block result into BN-RC (7,7,12,2,6,2)

$$\begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 5 \\ 1 & 3 & 0 & 4 & 5 & 2 \end{bmatrix} \quad \text{mod } (7).$$

Case 3 : v=8, p=2

a) q =5

The following initial blocks provide a **BN-RC** (8,56,70,2,5,10)

$$\begin{bmatrix} 0 & 1 & 3 & 6 & 2 \\ 1 & 3 & 6 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 6 & 5 & 7 \\ 1 & 6 & 5 & 7 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 7 & 4 & 6 \\ 1 & 7 & 4 & 6 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 6 & 3 & 5 \\ 1 & 6 & 3 & 5 & 0 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} 0 & 1 & 5 & 2 & 4 \\ 1 & 5 & 2 & 4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 7 & 2 & 4 \\ 1 & 7 & 2 & 4 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 & 7 & 4 \\ 1 & 2 & 7 & 4 & 0 \end{bmatrix} \quad \text{mod } (8)$$

b) q =6

The following blocks provide a BN-RC (8,28,42,2,6,6)

(i) $\begin{bmatrix} 1 & 2 & 3 & 5 & 6 & 7 \\ 5 & 6 & 7 & 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 & 5 & 6 & 0 \\ 5 & 6 & 0 & 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 6 & 7 & 0 \\ 6 & 7 & 0 & 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 5 & 3 & 7 & 4 & 0 \\ 5 & 1 & 7 & 3 & 0 & 4 \end{bmatrix}$

(ii) the initial block $\begin{bmatrix} 0 & 1 & 2 & 4 & 6 & 3 \\ 1 & 2 & 4 & 6 & 3 & 0 \end{bmatrix}$ mod (8), taken thrice.

This is a trial and error solution.

c) q=7

The follwoing initial block provides a **BN-RC** (8,8,14,2,7,2)

$$\begin{bmatrix} 0 & 1 & 3 & 6 & 7 & 2 & 4 \\ 1 & 3 & 6 & 7 & 2 & 4 & 0 \end{bmatrix} \text{ mod } (8)$$

Case 4 : v=9, p=2

a) q=5

The following sets of differences provide a **BN-RC** (9,36,40 2,5,5)

(1,2,3,8,4), (1,2,5,64), (1,7,3,3,4) and (1,2,3,7,9) mod (9).

Note 2.4: ∃ a BIBD (9,18,10,5,5) and a **BN-RC** (5,2,4,5,1) Hence following the theorem 2.2 a **BN-RC** (9,36,40,2,5,5)

b) q =6

There exist a **BIBD** (9,12,4,3,1) and a **BIBD** (3,3,2,2,1) thereby implying the existence of a BN-RC (9,12,8,2,3,1). Now, by taking two blocks at a time of the **BN-RC** (9,12,8,2,3,1) we form 6 blocks. These blocks provide **BN-RC** (9,6,8,2,6,1).

Case 5 : v=10, p=2

a) q=5

The following is a trial and error solution to **BN-RC** (10,8,18,2,5,2)

$$\begin{bmatrix}1&4&7&9&0\\4&7&9&0&1\end{bmatrix}\quad\begin{bmatrix}3&6&9&1&2\\6&9&1&2&3\end{bmatrix}\quad\begin{bmatrix}5&8&1&3&4\\8&1&3&4&5\end{bmatrix}\quad\begin{bmatrix}7&0&3&5&6\\0&3&5&6&7\end{bmatrix}$$

$$\begin{bmatrix}9&2&5&7&8\\2&5&7&8&9\end{bmatrix}\quad\begin{bmatrix}1&4&7&1&6\\4&7&1&6&1\end{bmatrix}\quad\begin{bmatrix}3&6&9&3&8\\6&9&3&8&3\end{bmatrix}\quad\begin{bmatrix}5&8&1&5&0\\8&1&5&0&5\end{bmatrix}$$

$$\begin{bmatrix}7&0&3&7&2\\0&3&7&2&7\end{bmatrix}\quad\begin{bmatrix}9&2&5&9&4\\2&5&9&4&9\end{bmatrix}\quad\begin{bmatrix}2&3&4&8&0\\3&4&8&0&2\end{bmatrix}\quad\begin{bmatrix}4&5&6&0&2\\5&6&0&2&4\end{bmatrix}$$

$$\begin{bmatrix}6&7&8&2&4\\7&8&2&4&6\end{bmatrix}\quad\begin{bmatrix}8&9&0&4&6\\9&0&4&6&8\end{bmatrix}\quad\begin{bmatrix}0&1&2&6&8\\1&2&6&8&0\end{bmatrix}\quad\begin{bmatrix}1&3&5&7&9\\3&5&7&9&1\end{bmatrix}$$

$$\begin{bmatrix}1&5&9&3&7\\5&9&3&7&1\end{bmatrix}\quad\begin{bmatrix}4&8&2&6&0\\8&2&6&0&4\end{bmatrix}$$

b) q =6

A **BN-RC** (10,15,18,2,6,2) can be obtained in the following way.

There exist a BIBD (10,30,9,3,2) and a BIBD (3,3,2,2,1). Hence there exist a BN-RC (10,30,18,2,2). Now combining blocks of this BN-RC two by two we have the required design.

Case 6 : v=11, p=2

a) q =5

Following the method given in section 2.5, case 4 i.e., for $v = 8t+3$ and $t=1$, we can construct the design. The following set of differences provide the design.

(1,9,3,4,5) mod (11).

b) q =6

A **BN-RC** (11,55,60,2,6,6) can be constructed using the theorem 2.2 presented in section 2.2 since a **BIBD** (11,11,6,6,3) and a BN-RC (6,5,10,2,6,2) exist.

Case 7 : v=12, p=2

a) q =5

A **BN-RC** (12,132,110,2,5,10) can be constructed by the use of the theorem 2.2 presented in section 2.2 , since there exist a **BIBD** (12,22,11,6,5,) and a **BN-RC** (6,6,10,2,5,2).

b) q =6

A **BN-RC** (12,22,22, 2,6,2) can be constructed by the use of the theorem 2.2 given in section 2.2 since there exist a BIBD (12,44,11,3,2) and a BN-RC (3,1,2,2,3,1).

Case 8 : v=13, p=2, q=6.

A **BN-RC** (13,13,12,2,6,1) can be constructed from the following difference set (1,2,3,9,5,6) mod (13).

CHAPTER 3

CONSTRUCTION OF BALANCED NESTED ROW-COLUMN DESIGNS WITH UNEQUAL BLOCK SIZES

3.1 Introduction :

In Chapter 2 we have dealt with BN-RC designs where within each block the sizes of rows and columns of the two factors , that are nested within blocks and are crossed with each other, are constant. In this chapter we will deal with varying sizes of these two nested factors.

In the initial stages equisized block designs were developed from the point of view of analysis and this calss of designs have intuitive and aesthetic appeal. With the advent of computer, the analysis of a given design in no more a problem. After all in many a practical situations block designs with unequal block sizes may be required. Kageyama (1976) points out that block designs with unequal block sizes may be particularly useful in large experiments both in industry and agriculture. Non-proper designs add flexibility to the experimentation and may avoid waste of experimental material as well. The need for blocks of natural sizes has been noted by Pearce (1964).

Consider an experiment where experimental units are organisms that exist in groups, like animals in a litter or trees planted in rows before it was conceived. To achieve blocks of equal sizes some of the units can be discarded but this can be a very wasteful exercise. And in this process the sizes of the blocks get reduced thereby resulting in a loss of efficiency unless the same is compensated by the decrease in error variance. However, it is not difficult to design experiments with varying block sizes.

The situation in block designs with nested rows and columns is further complicated. In this set up, it is not likely to be easy to form/find equisized groups of experimental units (under blocking factor) providing for the first way of elimination of heterogeneity with their further arrangement in an array requiring elimination of heterogeneity in both rows and columns, which are crossed with each other. And for achieving equality in block sizes a large number of experimental units may need to be discarded. In such situations BN-RC designs with unequal sizes would be useful and will add the necessary flexibility and dynamism in experimentation.

In the context of block designs a statistical property, balancing, is used to develop unequisized block designs. We restrict ourselves to equireplicate variance-balanced designs. A necessary and sufficient condition of variance balanced block design is that its information matrix takes the form, $\mathbf{C} = \theta\ (\mathbf{I}\text{-}1/v\ \mathbf{J})$, where θ is a scalar. The methods of construction of variance balanced designs with unequal block sizes is found to be useful for developing BN-RC designs with unequal block sizes. Let us consider some aspects of balancing in block designs.

3.2 Balancing in Block Designs With unequal Block Sizes.

Let Di : i = 1,2, . . . , h, be an m-associate partially balanced block design (PBIB) with parameters $v, b_i, r_i, k_i, n_1, n_2, \ldots, n_m, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{mi}$ and denoted by PBIB/m $(v, b_i, r_i, r_i, k_i, n_1, n_2, \ldots, n_m, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{mi})$, in which two treatments belonging to j^{th} associate class concur in λ_{ji} blocks, $j = 1,2,\ldots, m$. It is understood that D_i's belong to the same association scheme. Then following Jones, Sinha and Kageyama (1987), an equireplicate variance balanced block design with parameters v, $b = \sum_{i=1}^{h} \alpha_i b_i$, $r = \sum_{i=1}^{h} \alpha_i r$ and block sizes of $\mathbf{k} = (k_1, k_2, \ldots, k_h)'$, exists if there are h positive integers α_1 , α_2 ,. . . , and α_h such that for all j = 1,2 . . . ,m

$$\sum_{i=1}^{h} \frac{\alpha_i \lambda_{ji}}{k_i} = \text{constant} \tag{3.1}$$

For two PBIB/2 designs, D_1: PBIB/2 ($v, b_1, r_1, k_1, n_1, n_2, \lambda_{11}, \lambda_{21}$) and D_2 : PBIB/2 ($v, b_2, r_2, k_2, n_1, n_2, \lambda_{12}, \lambda_{22}$) with same association scheme, the condition in (3.1) reduces to

$$\frac{\alpha_1 \lambda_{11}}{k_1} + \frac{\alpha_2 \lambda_{12}}{k_2} = \frac{\alpha_1 \lambda_{21}}{k_1} + \frac{\alpha_2 \lambda_{22}}{k_2}$$

$$\Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{k_1 (\lambda_{22} - \lambda_{12})}{k_2 (\lambda_{11} - \lambda_{21})} \tag{3.2}$$

This expression is useful for choosing values of α_1 and α_2 to get a variance balanced block designs of two sizes $k_1$and k_2 ($\neq k_1$) from two given PBIB/2 designs D_1 and D_2 with two different sizes k_1 and k_2 respectively.

3.3 BN-RC Designs with Unequal Block Sizes

Bagchi *et al.* (1990) defined BN-RC designs in the context of equality of block, row and column sizes. These designs are non-binary and universally optimal. Based on the premise of the title of this chapter, in the following we define a BN-RC design with unequal block sizes. Since the condition $\mathbf{U} = \mathbf{0}$, ensures that the row and block classifications plays no role in the information matrix of a BN-RC design, it is possible to visualize a situation where the column size is fixed but row and hence block sizes are different from block to block and yet both the conditions for BN-RC are satisfied. This class of designs will continue to be universally optimal provided the column component design, d_2 is so.

Consider a block design d with nested rows and columns in which v treatments, each replicated r times, are arranged in $s = \sum_{i=1}^{w} b_j$ blocks, where each of b_j blocks are further arranged into p rows and q_j columns, $1 \le j \le w$ and all q_j's are different.

Let $\mathbf{N}^{v \times s}$, $\mathbf{N}_1^{v \times sp}$ and $\mathbf{N}_2^{v \times \sum_{j=1}^{w} b_j q_j}$ denote the treatment-block, treatment-row and treatment-column incidence matrices respectively and let d_0, d_1, d_2 denote respectively the block-component, the row-component and the column-component designs of d. Let $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$ denote the intra-block coefficient matrix/information matrix of the component designs d_0, d_1 and d_2 respectively. The **U** matrix of the definition (2.1) of the BN-RC design will, in this case, be

$$\mathbf{U} = \mathbf{C}_1 - \mathbf{C}_0 = \mathbf{N}_1 \left(\mathbf{K} \otimes \mathbf{I}_p\right)^{-1} \mathbf{N}_1' - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}'$$

where $\mathbf{K}^{s \times s} = Diag\left(\ldots, q_j \, \mathbf{I}_{bj}, \ldots\right)$ is the diagonal matrix of row sizes of s blocks, $\otimes$ stands for Kronecker product of two matrices and $\mathbf{I}_p$ denotes the identity matrix of order p.

We can, therefore, extend the definition (2. 1) of BN-RC designs to the case of BN-RC designs with unequal block (column within block) sizes as below

Definition 3.1: A nested row-column design d, as detailed above, is called a Balanced Nested Row-Column (BN-RC) design with unequal block sizes (i.e. with unequal number of columns over blocks) if it satisfies the following two conditions (to be called as **Condition (A)**) :

1)$\mathbf{U} = \mathbf{0}$ i.e., $\mathbf{C}_1 = \mathbf{C}_0$, and

2) $\mathbf{N}_2$ (the incidence matrix of d_2, the column-component design) is an incidence matrix of a Balanced Block Design (BBD).

As it happened in the previous chapter here too we restricted d_2 to be a BIBD, which is a particular type of BBD because of the obvious reason -- saving in the requirement of experimental units.

3.3.1 Variance Balanced and Partially Balanced Nested Row-Column Designs

Before defining them formally let us look into their nomenclature and related notations.

Originally **BN-RC** designs were defined for a constant number of rows and columns within all the blocks of Nested Row-Column (NRC) designs. With varying number of columns within blocks, NRC's can still exist satisfying the underlying conditions of **BN-RC** designs. They are also universally optimal. We have named them as **BN-RC** designs with uneqaul block sizes , in conformity with the names in block designs. When we mention simply BN-RC design it refers to **BN-RC** designs with equisized blocks.

For a design satisfying only the first in Condition (A), we name it as a Variance Balanced Nested Row-Column (**VBN-RC**) or a Partially Balanced Nested Row-Column (**PBN-RC**) Design depending on the form of N_2, the incidence matrix of the column-component design d_2 of the Nested Row-Column Design d, as below.

i) If d_2 is a Variance Balanced Block (**VBB**) Design then d is called as a **VBN-RC** design, and

ii) if d_2 is a partially Balanced Incomplete Block Desgin with m-associate classes (**PBIB/m**) then d is called as a PBN-RC/m Design. Thus, we define a **PBN-RC/m** and a **VBN-RC** design as follows.

Definition 3.2 : PBN-RC/m Design:

An arrangement of v treatments, each replicated r times, in $s = \sum_{j=1}^{w} b_j$ blocks, each of the bj blocks being further arranged into p rows and q_j columns, $1 \le j \le w$, is called an m-associate class Partially Balanced Nested Row-Column (**PBN-RC**/m) Design if.

i) $\mathbf{U} = \mathbf{0}$, and

ii) N2 forms an incidence matrix of an m-associate class Partially Balanced Incomplete Block (PBIB/m) Design.

The parameters of a PBN-RC/m design will be similar to those of a PBIB/m design as the parameters of a BN-RC is to those of a BIBD. For example the parameters of PBN-RC/2 will be $\left(\mathrm{v, b, r, p, q, n_1, n_2}, \mu_1, \mu_2\right)$ corresponding to the parameters of PBIB/2 viz. $\left(v, b_1, r, k, n_1, n_2, \mu_1, \mu_2\right)$.

Following the notion of variance balancing with unequal block sizes in the context of block design we define a VBN-RC design in the following.

Definition 3.3 : VBN-RC Design :

An arrangement of v treatments, each replicated r times, in $s = \sum_{j=1}^{w} b_j$ blocks, where each of the b_j blocks is having further arrangement into p_j rows (not all p_j's are equal) and q_j columns for $1 \le j \le w$ is called a Variance Balanced Nested Row-Column (VBN-RC) design with parameters $\left(v, \mathbf{b}', r, \mathbf{p}', \mathbf{q}', \psi\right)$ if it satisfies the following conditions.

i) $\mathbf{U} = \mathbf{C}_1 - \mathbf{C}_0 = \mathbf{0}$, and

ii) N_2 forms an incidence matrix of a Variance Balanced Block (VBB) design with block sizes $\mathbf{p}' = (p_1, p_2, \ldots, p_w)$ which are not simultaneously equal. Here $\mathbf{b}' = (b_1, b_2, \ldots, b_w)$, $\mathbf{q}' = (q_1, q_2, \ldots, q_w)$ and ψ is the common sum of the weighted concurrences for each pair of treatments (Pearce, 1976). The information matrix of the VBN-RC design takes the form, $\mathbf{C} = \mathbf{C}_2 = \psi\left(\mathbf{I} - \frac{1}{v}\mathbf{J}\right)$

Note 3.1 : It is to be noted that if p_j's are equal for all $j = 1,2,\ldots,w$ then a VBN-RC design reduces to a **BN-RC** design with equal or unequal block sizes depending on the equality or unequality of q_j's. And q_j's do not come in the way of achieving balance of such a nested row column design.

3.4 Construction of PBN-RC/m Designs

Before we discuss the methods of construction of **BN-RC** designs with unequal block sizes and that of **VBN-RC** designs, we state few results on the construction of PBN-RC/m designs.

3.4.1 Construction of PBN-RC/2 Designs

For constructing **PBN-RC/2** designs with 2-associate classes we state the following.

Theorem. 3.1 The existence of a design, D_1 : **PBIB/2** $(v, b, r, k, n_1, n_2, \lambda_1, \lambda_2)$ and a design, M : **BN-RC** $(k, \beta, \rho, p, q, v, \mu)$ implies the existence of **a PBN-RC/2** design, d^*: **PBN-RC/2** $(v, b^* = b\beta,\ r^* = r.\rho, p, q, n_1, n_2, \mu_1^* = \mu\lambda_1, \mu_2^* = \mu\lambda_2)$ belonging to the association scheme of D_1.

Proof : Consider the j^{th} block of D_1: **PBIB/2**. Let $j_1, j_2, \ldots, j_k$ be the set of k symbols occuring in this set. With these k symbols form a design M_1: **BN-RC** (k, β, ρ, p, q, μ). Then for the symbols appearing in the jth block of D_1, the column-component design d_2 forms a **BBD** or a **BIBD** with parameters (k , βq ,ρ ,p ,μ). Repeating this procedure for the remaining blcoks of D_1, it can be seen that in the resulting design, d^*, total number of blocks, $b^* = b\beta$, each treatment being replicated $r^* = r\rho$ times and d_2^*, the column-component design of d^* belongs to same association scheme as that of D_1. And in d_2^*, two first associate treatments concur in $\mu\lambda_1$ columns and two second associate treatments concur in $\mu\lambda_2$ columns. Hence proved.

Example 3.1: Consider D_1: **PBIB/2** ($v=10$, $b=5$, $r=3$, $k=6$, $n_1=6$, $n_2=3$, $\lambda_1=2$, $\lambda_2 = 1$) a triangular design having the blocks as (5,6,7,8,9,10) , (10,9,4,2,3,8) , (4, 3, 1, 10, 7, 6) , (7, 2, 9, 5, 1, 4) and (1,8,3,6,5,2). There is a design, M_1: **BN-RC** ($k=6$, $\beta=6$, $\rho=10$, $p=2$, $q=5$, $\mu=2$) as given in section 2.6 of chapter 2.

Using the first block of D_1 with contents as (5,6,7,8,9,10) we write down the **BN-RC** , M_1 with the symbols appearing in this block as below,

$$\begin{bmatrix} 6 & 7 & 8 & 9 & 10 \\ 7 & 8 & 9 & 10 & 6 \end{bmatrix}, \begin{bmatrix} 7 & 8 & 9 & 10 & 5 \\ 8 & 9 & 10 & 5 & 7 \end{bmatrix}, \begin{bmatrix} 8 & 9 & 10 & 5 & 6 \\ 9 & 10 & 5 & 6 & 8 \end{bmatrix},$$

$$\begin{bmatrix} 10 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 10 \end{bmatrix}, \begin{bmatrix} 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 9 & 10 & 5 & 6 & 7 \\ 10 & 5 & 6 & 7 & 9 \end{bmatrix}$$

to obtain 6 blocks of a block design with nested rows and columns. It is to note that each pair of treatments from the first block of D_1 appears $\mu = 2$ times in the columns of the blocks thus formed. The contributions of any block to the **U** matrix is null.

Repeating this process for all the blocks of D_1 results into d^*, a **PBN-RC/2** ($v=10$, $b^*=3\ 0$, $r^*=30$, $p=2$, $q=5$, $n_1=6$, $n_2 =3$, $\mu_1^* = 4$, $\mu_2^* = 2$), since the concurrence of two treatments which appeared $\lambda_1 = 2$ or $\lambda_2 = 1$ time(s) in D_1 occurs μ times of their concurrences in d^* and the columns of d^* form a **PBIB/2** (10, 30, 30, 2, 6, 3, 4, 2).

Extending this method of construction we can construct an m-associate class **PBN-RC** design as given in the following. Here $m \geq 2$.

Theorem 3.2 . The existence of a design, D1: PBIB/m ($v, b, r, k, n_1, n_2, \ldots, n_m, \lambda_1, \lambda_2, \ldots \lambda_m$) and a design, M_1: **BN-RC** ($k, \beta, \rho, p, q, \mu$) implies the existence of a design, d^* : **PBN-RC/m** ($v, b^* = b\beta, r^* = r\rho, n_1, n_2, \ldots, n_m, \mu^{*\prime} = (\mu\lambda_1, \mu\lambda_2, \mu\lambda_m)$) with same association scheme as that of D_1.

<u>Proof</u> : Similar to the proof of theorem 3.1.

We have presented the method of construction of **PBN-RC/m** designs. They are having a constant block, row and column sizes.

Let us now consider the method of construction of Nested Row-Column (**BN-RC**) design with unequal row, and hence block sizes.

3.5 Construction of BN-RC Designs with Unequal Block Sizes.

For constructing **BN-RC** designs with two different row (and hence block) sizes we state the following.

Theorem 3.3 : Given two designs,

d_1^*: **PBN-RC/2** ($v, b_1^*, r_1^*, p, q_1, n_1, n_2, \mu_{11}^*, \mu_{21}^*$), and

d_2^*: **PBN-RC/2** ($v, b_2^*, r_2^*, p, q_2, n_1, n_2, \mu_{12}^*, \mu_{22}^*$)

with the same association scheme (for their column-component designs), there exists a design, d : **BN-RC** $\left(v , \mathbf{b}' = \left(\alpha_1 b_1^*, \alpha_2 b_2^*\right), r = \alpha_1 r_1^* + \alpha_2 r_2^*, p , \mathbf{q}' = \left(q_1 , q_2\right) , \mu\right)$, if it is possible to find two positive integers α_1 and α_2 such that

$$\alpha_1 \mu_{11}^* + \alpha_2 \mu_{12}^* = \alpha_1 \mu_{21}^* + \alpha_2 \mu_{22}^* = \mu \text{ , } a \text{ constant} \qquad (3.3)$$

Proof : By combining together α_1 copies of the design d_1* and α_2 copies of the design d_2*, the new design with $\alpha_1 b_1^*$ nested blocks of size p × q_1 and $\alpha_2 b_2^*$ nested blocks of size p× q_2 is obtained, in which if the condition (3.3) holds good, each pair of treatments occur equally frequently in the column-component design (which is proper, binary and equireplicated) of the newly formed design d. Hence proved.

The relationship in (3.3) can be expressed as

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_{22}^* - \mu_{12}^*}{\mu_{11}^* - \mu_{21}^*} \qquad (3.4)$$

Appealing to Theorem 3.1, with the help of a design, D_i: **PBIB/2** (v, b_i, r_i, k_i, n_1, n_2, λ_{1i} , λ_{2i}) and a design, M_i : **BN-RC** (k_i, β_i , ρ_i, p, q , μ_i)

the design

$$d_i^*\text{:}\mathbf{PBN\text{-}RC/2} \left(v , b_i^* = b_i \beta_i , r_i^* = r_i \rho_i , p , \mathbf{q}' = \left(q_1 , q_2\right) , n_1 , n_2 , \mu_{1i}^* = \mu_i \lambda_{1i} , \mu_{2i}^* = \mu_i \lambda_{2i} \right) \text{ can be obtained for i = 1,2}$$

And in terms of the parameters mentioned in the above, the relationship in (3.4) can be written as $$\frac{\alpha_1}{\alpha_2} = \frac{\mu_2}{\mu_1} \cdot \frac{\lambda_{22} - \lambda_{12}}{\lambda_{11} - \lambda_{21}} , \qquad (3.5)$$

Which is useful for choosing values of α_1 and α_2 From (3.5) it can be noted that

(i) if $\lambda_{11} > \lambda_{21}$ then we need that $\lambda_{12} < \lambda_{22}$, or

(ii) if $\lambda_{11} < \lambda_{22}$ then we need $\lambda_{12} > \lambda_{22}$.

In terms of the parameters of **PBIB/2** and **BN-RC** designs, i.e. D_1, D_2, M_1 and M_2 above, the parameters of the **BN-RC** designs with unequal blocks obtained are,

d: BN - RC (v , $\mathbf{b}' = (\alpha_1 b_1 \beta_1, \alpha_2 b_2 \beta_2)$, $r = \alpha_1 r_1 \rho_1 + \alpha_2 r_2 \rho_2$, p,

$\mathbf{q}' = (q_1, q_2)'$, $\mu = \alpha_1 \mu_1 \lambda_{11} + \alpha_2 \mu_2 \lambda_{12} = \alpha_1 \mu_1 \lambda_{21} + \alpha_2 \mu_2 \lambda_{22}$)

This method can be extended to construct BN-RC desgins with more than two unequal block sizes and by adopting PBIB/m, $m \geq 2$, as well. To construct such **BN-RC** designs we state the following theorem which can be proved in the similar way.

Theorem 3.4 Suppose for i = 1,2, . . ., h there exist a design,

D_i: PBIB/m (v, b_i, r_i, k_i, n_1, n_2, . . . , n_m , λ_{1i}, λ_{2i} . . . , λ_{mi}), and a design,

M_i : BN-RC (k_i, β_i , ρ_i p , q_i , μ_i)

(D_i's are based on the same association scheme for i = 1,2, . . . , h)

Then it is possible to construct a design,

d: BN-RC (v, $\mathbf{b}' = (\alpha_1 b_1 \beta_1, \ldots, \alpha_i b_i \beta_i, \ldots, \alpha_h b_h \beta_h)$,

$r = \sum_{i=1}^{h} \alpha_i r_i \rho_i$, p, $\mathbf{q}' = (q_1, q_2, \ldots, q_h)$, μ), if we can get h positive integers $\alpha_1, \alpha_2, \ldots, \alpha_h$ such that.

$$\sum_{i=1}^{h} \alpha_i \lambda_{ji} \mu_i = \mu \text{ , a constant, for all } j = 1,2, \ldots, m \tag{3.6}$$

A BN-RC design with unequal block sizes can be constructed from another BN-RC design with unequal block sizes. To construct such designs we state the following theorem which can be proved easily.

Theorem 3.5 : If $\exists$ a design D_1: BIBD $(v, b_1, r_1, k_1, \lambda_1)$, and a design D2 : BN-RC $(k_1, \mathbf{b}'_2 = (b_{21}, b_{22}), r_2, p, \mathbf{q}' = (q_1, q_2), \mu_1)$ then there always exists a design,

d: BN-RC $(v, \mathbf{b}' = b_1 \mathbf{b}'_2, r = r_1 r_2, p, \mathbf{q}' = (q_1, q_2), \mu = \lambda_1 \mu_1)$

Example 3.2 : Consider two triangular PBIB desings T9 and T44 given in Table X of Clatworthy (1973). consider the following two pairs of designs

T9 $(v = 10, b_1 = 10, r_1 = 3, k_1 = 3, \lambda_{11} = 1, \lambda_{21} = 0)$

& M_1: BN-RC $(k_1 = 3, \beta_1 = 1, \rho_1 = 2, p_1 = 2, q_1 = 3, \mu_1 = 1)$

and T44 : $(v = 10, b_2 = 6, r_2 = 3, k_2 = 5, \lambda_{12} = 1, \lambda_{22} = 2)$

& M_2: $(k_2 = 5, \beta_2 = 2, \rho_2 = 4, p_2 = 2, q_2 = 5, \mu_2 = 1)$

Hence to form a BN-RC for $v = 10$ with $q_1 = 3$ and $q_2 = 5$ from equation (3.5)

$$\frac{\alpha_1}{\alpha_2} = 1 \quad \text{Let } \alpha_1 = \alpha_2 = 1 .$$

Thus, single copy of T9 and T44 when developed by BN-RC's M1 and M2 respectively, results into BN-RC $(v = 10, \mathbf{b}' = (10, 12), r = 18, p = 2, \mathbf{q}' = (3,5), \mu, \rho = 2)$.

3.6 CONSTRUCTION OF VBN-RC DESIGNS

For construction of Variance Balance Nested Row-Column (VBN-RC) designs with two unequal column sizes from 2-associate PBIB designs we have the following.

Theorem 3.6 : Let there exist two PBIB/2 designs, D_i : PB1B / 2 (v, b_i, r_i, k_i, n_1, n_2, λ_{1i}, λ_{2i}) , i = 1 , 2 , with the same association scheme , and there exist two BN-RC designs M_i : BN-RC $(k_i, \beta_i, \rho_i, p_i, q_i, \mu_i)$, i = 1,2 .

Then there exists a design,

$$\text{d: VBN-RC}\,(\text{v}, \mathbf{b}' = \left(\alpha_1 b_1 \beta_1, \alpha_2 b_2 \beta_2\right), r = \alpha_1 r_1 \rho_1 + \alpha_2 r_2 \rho_2,$$

$$\mathbf{p}' = \left(p_1, p_2\right), \mathbf{q} = \left(q_1, q_2\right), \Psi)$$

if there exist two positive integers α_1 and α_2 such that

$$\frac{\alpha_1 \mu_1 \lambda_{11}}{p_1} + \frac{\alpha_2 \mu_2 \lambda_{12}}{p_2} = \frac{\alpha_1 \mu_1 \lambda_{21}}{p_1} + \frac{\alpha_2 \mu_2 \lambda_{22}}{p_2} = \Psi \tag{3.7}$$

Here Ψ is the common sum of weighted concurrences for each pair of treatments (Pearce, 1976)

<u>Proof</u> : For i = 1 and 2, from the given designs D_i and M_i we have a PBN-RC/2 design, d_i^*. By unionizing α_1 copies of the nested blocks of size $p_1 \times q_1$ of d_1^* with α_2 copies of the nested blocks of size $p_2 \times q_2$ of d_2^*, the the new design d has $\alpha_1 b_1 \beta_1$ blocks of size $p_1 \times q_1$ and $\alpha_2 b_2 \beta_2$ blocks of size $p_2 \times q_2$. The method of construction ensures that inthis design $\mathbf{C}_1 = \mathbf{C}_0$. Now, if the condition (3.7) holds good, the information matrix of d, i.e. $\mathbf{C}_2$ takes the form a **I** - b **J**, where

$$\text{a} = \frac{p_1 - 1}{p_1} \alpha_1 r_1 \rho_1 + \frac{p_2 - 1}{p_2} \alpha_2 r_2 \rho_2 + b \qquad \text{and}$$

$$\text{b} = \frac{\alpha_1 \mu_1 \lambda_{11}}{\text{p}_1} + \frac{\alpha_2 \mu_2 \lambda_{12}}{\text{p}_2} = \frac{\alpha_1 \mu_1 \lambda_{21}}{\text{p}_1} + \frac{\alpha_2 \mu_2 \lambda_{22}}{\text{p}_2} = \frac{\text{a}}{\text{v}}.$$ Hence the theorem.

The relationship in (3.7) can be expressed as

$$\frac{\alpha_1}{\alpha_2} = \frac{p_1}{p_2} \cdot \frac{\mu_2}{\mu_1} \cdot \frac{\lambda_{22} - \lambda_{12}}{\lambda_{11} - \lambda_{21}} \tag{3.8}$$

which is useful for choosing values of α_1 and α_2 for two given PBIB/2 designs and corresponding BN-RC designs made use of.

Example 3.2 : Consider two triangular PBIB designs T9 and T44 given in Table X of Clatworthy (1973). They belong to the same association scheme . Consider the following pairs of designs.

T9 : $(v=10, b_1=10, r_1=3, k_1=3, \lambda_{11} = 1, \lambda_{21} = 0)$

and M_1 : BN-RC $(k_1=3, \beta_1 = 1, \rho_1 = 2, p_1 = 2, q_1 = 3, \mu_1 = 1)$

T44 : $(v=10, b_2=6, r_2=3, k_2=5, \lambda_{12} = 1, \lambda_{22} = 2)$

and M_1 : BN-RC $(k_2=5, \beta_2 = 2, \rho_2 = 6, p_2 = 3, q_2 = 5, \mu_2 = 3)$

hence to form a VBN-RC for $v=10$ with $p_1=2$, $p_2=3$ and $q_1=3$ and $q_2=5$ from equaqtion (3.8) $\frac{\alpha_1}{\alpha_2} = 2$. Let $\alpha_1=2$ then $\alpha_2=1$.

Then 2 copies of T9 and one copy of T44 when developed by BN-RC's M_1 and M_2, respectively results into VBN-RC $(v=10, \mathbf{b}' = (20, 12), r=30, \mathbf{p}' = (2,3), \mathbf{q}' = (3,5), \Psi = 2.0)$ efficiency of this design, $E = \frac{v \cdot \Psi}{r} = 66.67\%$.

This method for constructing VBN-RC designs with two unequal column and hence block sizes from 2- associate PBIB designs can be extended to more than two unequal column sizes and by adopting m-associate class, $m \geq 2$, PBIB designs with the same association scheme. To construct such designs we state a theorem in the following (without proof) that can be proved in a similar fashion as that of Theorem 3.6.

Theorem 3.7 : For i = 1,2, . . ., h if there exist a design,

D_i: PBIB/m $(v, b_i, r_i, k_i, n_1, n_2, \ldots, n_m, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{mi})$, and a design,

M_i: BN-RC $(k_i, \beta_i, \rho_i, p_i, q_i, \mu_i)$,

$(\mathrm{D}_1, \mathrm{D}_2, \ldots, \mathrm{D}_{h'}$ are based on the same association scheme) then there exist a design,

$$\text{d: VBN-RC } (v, \mathbf{b}' = \left(\alpha_1 b_1 \beta_1, \ldots, \alpha_i b_i \beta_i, \ldots, \alpha_h b_h \beta_h\right), r = \sum_{i=1}^{h} \alpha_i r_i \rho_i,$$

$$\mathbf{p}' = \left(p_1, p_2, \ldots, p_n\right), \mathbf{q}' = \left(q_1, q_2, \ldots, q_n\right), \Psi)$$

if h positive integrs $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\sum_{i=1}^{h} = \frac{\alpha_i \mu_i \lambda_{ji}}{p_i} = \Psi \text{ , a constant for all } j = 1,2 \ldots m \qquad (3.9)$$

can be found.

Here Ψ is the common sum of weighted concurrences for each pair of treatments in the column-component design of d. The efficiency of the VBN-RC design is the ratio of average variance of treatment comparisons in this design to the average variance in a randomised block design with the same replication, i.e., the efficiency, $E = \frac{v\Psi}{r}$.

Note 3.2 : In these methods of construction we have first obtained PBN-RC desgins and taken suitable number of their copies and merged (added) them to obtain BN-RC designs with unequal block sizes and VBN-RC designs. In these cases if all p_i's are equal the resulting design is a BN-RC design with unequal block sizes provided all q_i's are not equal. If, however, all p_i's are not equal then the resulting design is VBN-RC whether or not all the q_i's are equal. If all p_i's as well as all q_i's are equal the design will naturally be a BN-RC design with equal block sizes. It is well -known that the union of designs D_1 : BIBD (v, b_1, r_1, k_1, λ_1) and D_2 : BIBD (v, b_2, r_2, k_2, λ_2) results into a variance balanced blocks (VBB) design. There is a similar happening in BN-RC designs also. The union of designs D'_1 : BN-RC (v, b_1, r_1, p_1, q_1, μ_1) and D'_2 : BN-RC (v, b_2, r_2, p_2, q_2, μ_2)results into a design d : VBN-RC (v, $\mathbf{b}' = (b_1, b_2)$, $r = r_1 + r_2$, $\mathbf{p}' = (p_1, p_2)$, $\mathbf{q}' = (q_1, q_2)$, $\Psi = \frac{\mu_1}{p_1} + \frac{\mu_2}{p_2}$) and for $p_1 = p_2$ this VBN-RC design d reduces to a BN-RC design.

3.7 Construction of BN-RC Desgins with Unequal Blocks for p =2.

To construct BN-RC designs with unequal blocks for p = 2 and q varying from 3 to 6 we used the method of symmetrically repeated differences. Following the convention described in section 2 .4 of chapter 2 , here we present these designs. As it happnded in section 2 .4 of chapter 2, here too, the contents of two rows, in each of the blocks, are same and thereby ensures $C_1 = C_0$. It only remains to be verified that all the differences occur equallly frequently over all the columns. Here also we looked for forming d_2, the column-component design, as a BIBD because of saving experimental units. In the following we can see that the varying block sizes results into saving of a considerable amount of experimental units. The following list displays minimum μ, the number of times each pair of treatments appeared in d_2, the column-component design of d, a BN-RC design with unequal blocks. We consider these designs in the series $v = 8t + i$, $i = 0, 1, 2, \ldots, 7$. It was possible to get a design with same or lower value of μ using unequal blocks as compared to equisized blocks. These results are presented below in a tabular form.

Table : 3.1 : Values of μ in the BN-RC designs constructed with equal/unequal block sizes.

v	8t	8t+1	8t+2	8t+3	8t+4	8t+5	8t+6	8t+7
equal block sizes	2	1	4	4	2	2	4	4
uneqal block sizes	2	1	2	1	2	1	2	1

The table shows that for an odd v, $\mu = 1$ and for an even v, $\mu = 2$. A plausible explantaion for $\mu = 2$ when v is even is that there is a difference v/2 which is its own complement, implying that when a pair of treatments with difference v/2 appears it appears twice automatically to satisfy the condition $C_1 = C_0$. Thus this seems to be the minimum achievable for these cases. In the construction of these BN-RC designs we have a combination of blocks of sizes 2 × 3, 2× 4, 2×5 and 2× 6 depending on v. In the following we construct these designs individually in the form of proofs of theorems. In each case we kept p = 2. It is known that from a set of q differences a block of 2 rows and q columns can be formed and a set of q differences generate x blocks of size 2× q when developed under Partial Cycle (x).

v = 4 : The following blocks provide a

$$\text{BN-RC}\,(v = 4, \mathbf{b}' = (2,1), r = 6, p = 2, \mathbf{q}' = (3,6), \mu = 2)$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 0 & 2 & 3 & 1 \end{bmatrix}$$

v = 5 : The following blocks provide a

$$\text{BN-RC}\,(v = 5, \mathbf{b}' = (2,1), r = 4, p = 2, \mathbf{q}' = (3,4), \mu = 1)$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 4 \\ 3 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

v = 6 : The following blocks provide a

$$\text{BN-RC}\,(v = 6, \mathbf{b}' = (6,3), r = 10, p = 2, \mathbf{q}' = (3,4), \mu = 2)$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 4 \\ 3 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 & 5 \\ 4 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 & 3 & 2 \\ 5 & 3 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 5 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 2 & 5 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

v = 7 : The following blocks provide a

$$\text{BN-RC}\,(v = 7, \mathbf{b}' = (5,1), r = 6, p = 2, \mathbf{q}' = (3,6), \mu = 1)$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 4 & 6 \\ 4 & 6 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 5 & 0 \\ 5 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 5 & 6 & 1 & 6 & 0 & 2 \\ 6 & 1 & 5 & 0 & 2 & 6 \end{bmatrix}$$

In the following we shall construct BN-RC designs with unequal block sizes for $t > 0$.

Case 1 v = 8t

Theorem 3.8 : There always exists a BN-RC design with parameters $v = 8t$, Clearly all the differences are accounted for once in the sets.

Case 3 $v=8t+2$

Theorem 3.10 : There always exists a BN-RC desgin with parameter $v=8t+2$, $\mathbf{b}'=[4(4t+1), 4(t-1)(4t+1), 4t+1]$, $r= 2(8t+1)$, $p=2$, $\mathbf{q}' = (3,4,6)$, $\mu=2$.

Proof : Consider the following sets of diffrences

(i) one set of type, D_1: $[1,2,8t-1]$ repeated twice

(ii)(t-1) sets of type, D_{2i} : $[x, v-(x+1), v-(x+2), x+3]$ repeated twice. where $x=4i$ for $i= 1,2,\ldots, t-1$ and $t>1$ and

(iii) one set of type, D_3: $[4t, 4t+1, 4t+2, 4t, 4t+1, 4t+2]$ P.C.(4t+1)

mod (v).

The differences accounted for in different sets are as below. In (i) the differences (1, 8t+1, 2, 8t,3, 8t-1) appear twice. In (ii) all the differences except (i, 8t+1, 2, 8t, 3, 8t-1, 4t, 4t+2 and 4t+1) appear twice. In (iii) the differences (4t, 4t+2 and 4t+1) appear twice. Hence proved.

Case 4 $v=8t+3$

Theorem 3.11 : There always exists a BN-RC design with parameters $v=8t+3$, $\mathbf{b}'=((8t+3)(t-1), 8t+3)$, $r= 8t+2$, $p=2$, $\mathbf{q}' = (4,5)$ and $\mu = 1$.

Proof : Consider the following sets of diffrences

(i) one set of type, D_1 : $[8t+2, 2, 8t, 4t+3, 4t+2]$.

(ii) (t-1) sets of type, D_{2i} : $[x, v-(x+1), v-(x+2), x+3]$, where $x = 4i$ for $i = 1,2,\ldots,$ $t-1$ and $t>1$

mod (v).

The differences accounted for in different sets are as below. In (i) the differences of type u and v-u, for $u = 1,2,3, 4t$ and $4t+1$ appear once. In (ii) rest of the differences appear once. Hence proved.

Case 5 $v=8t+4$:

Theorem 3.12 : There always exists a BN-RC design with parameters $v=8t+4$, $\mathbf{b}' = (8t+4, 4t(4t+2))$, $r = 2(8t+3)$, $p=2$, $\mathbf{q}' = (3,4)$ and $\mu = 2$.

Proof : Consider the following sets of diffrences.

(i) one set of type, D_1 : $[1, 2, 8t+1]$

(ii) one set of type, D_2 : $[1, 8t+2, 8t, 5]$

(iii) t sets of type, D_{3i} : $[x, v-(x+1), v-(x+2), x+3]$ where $x=4i-1$ for $i = 1,2,\ldots,t$.and

(iv) (t-1) sets of type, D_{4j} : $[y, v\,(y+1), v-(y+2), y+3]$

where $y = 4j+2$ for $j = 1,2, \ldots, t-1$ and $t > 1$

mod (v).

In (i) the differences $(1, 8t+3, 2, 8t+2, 3, 8t+1)$ appear once. In (ii) the differnces $(1, 8t+3, 2, 8t+2, 4, 8t, 5, 8t-1)$ appear once. In (iii) all the diffrences except $(1, 8t+3, 2, 8t+2$ and $4t+1)$ appear once and the difference $4t+1$ appears twice. In (iv) all the differences except (u, v-u for $u = 1,2,34,5$ and $4t+1$) appear once. Hence proved.

Case 6 $v=8t+5$.

Theorem 3.13 : There always exists a BN-RC design with parameters $v=8t+5$, $\mathbf{b}' = (2(8t+5), (t-1)(8t+5))$, $r = 8t+4$, $p=2$, $\mathbf{q}' = (3,4)$ and $\mu = 1$

proof : Consider the following sets of differences

(i) one set of type, D_1 : $[1, 4t-1, 4t+5]$

(ii) one set of type, D_2 : [2, 4t + 1, 4t + 2] and

(iii) (t-1) sets of type, D_{3i} : [x, v-(x + 1), v-(x + 2), x + 3] where x = 4j-1 for j = 1,2, . . . , t-1and t > 1.

mod (v).

In (i) the differences (1, 8t + 4, 4t-1, 4t + 6, 4t, 4t + 5) appear once. In (ii) the differences (2, 8t + 3, 4t + 1, 4t + 4, 4t + 2, 4t + 3) appear once. And rest of the differences appear once in (iii). Hence proved.

Case 7 V = 8t + 6

Theorem 3.14 : For an even t, there always exists a BN-RC design with parameters v = 8t + 6, $\mathbf{b}'$ = (8t + 6, (4t + 1) (4t + 3)), r = 2(8t + 5), p = 2, $\mathbf{q}'$ = (3,4) and μ = 2

Proof : Conssider the following sets of differences

(i)t/2 sets of type, D_{1i} : [x, v-(x + 1), v-(x + 23), x + 3] where x = 4i-3 for i = 1,2, . . . ,t/2
Repeat each of them twice

(ii) t/2 sets of type, D_{2j} : [y, v-(y + 1), v-(y + 2), y + 3]

where y = 2t-1 + 4j for j = 1, 2, . . . , t/2

(iii) one set of type, D_3 : [2t + 1, 2t + 2 , 4t + 3]and

(iv) one set of type, D_4: [2t + 1, 2t + 1, 2t + 2, 2t + 2] P.C. (4t + 3).

mod (v).

In (i) the differences of type (u_1 and v-u_1 for u_1 = 1,2, . . . , 2t) appear once.

In (ii) the differences of type (u_2 and v-u_2 for u_2 = 2t + 3, 2t + 4, . . . , 4t + 2) appear twice.

In (iii) the differences $(2t+1, 6t+5, 2t+2, 6t+4)$ appear once and the difference $4t+3$, being its own complement, appears twice. And finally in (iv) the differences $(2t+1, 6t+5, 2t+2$ and $6t+4)$ appear once due to P.C. $(4t+3)$. Hence proved.

Theorem 3.15 : For an odd t, there always exists a BN-RC design with parameters $v=8t+6$, $\mathbf{b}'=(8t+6,\ (4t+1)(4t+3))$, $r=2(8t+5)$, $p=2$, $\mathbf{q}'=(3,4)$ and $\mu=2$.

Proof : Consider the following sets of diffrences

(i) $\frac{t-1}{2}$ sets of type, D_{1i} : $[x, v-(x+1), v-(x+2), x+3]$

where $x= 4i-3$ for $i = 1,2,\ldots,(t-1)/2$ and $t>1$. Repeat each of them twice.

(ii) t-1/2 sets of type, D_{2j} : $[y, v-(x+1), v-(x+2), x+3]$

where $y= 2t+1+4j$ for $j= 1,2,\ldots, t-1/2$ and $t>1$. Repeat each of them twice.

(iii) one set of type, D_3 : $[2t-1, 2t, 2t+3, +4]$. Repaeat it twice

(iv) one set of type, D_4 : $[2t+1, 2t+2, 4t+3]$ and

(v) one set of type, D_5 : $[2t=1, 2t+2, 2t+1, 2t+2]$ P.C.$(4t+3)$. mod (v).

In (i) the differences of type (u_1 and $v-u_1$ for $u_1=1,2,\ldots,2t-2$) appear twice.

In (ii) the differences of type (u_2 and $v-u_2$ for $u_2 = 2t+5, 2t+6, \ldots, 4t+2$) appear twice

In (iii) the differences of type (u_3 and $v-u_3$ for $u_3 = 2t-1, 2t, 2t+3$ and $2t+4$) appear twice.

In (iv) the differences $(2t+1, 6t+5, 2t+2, 6t+4)$ appear once and $4t+3$, being its own complement, appears twice .

In (v) the differences $(2t+1, 6t+5, 2t+2, 6t+4)$ appear once since P.C. $(4t+3)$ is used. Hence proved.

Case 8 $v=8t+7$

Theorem 3.16 : There always exists a BN-RC design with parameters $v=8t+7$, $\mathbf{b}'=(8t+7, t(8t+7))$, $r=8t+6$, $p=2$, $\mathbf{q}'=(3,4)$ and $\mu=1$.

Proof: Consider the following sets of differences.

(i) one set of type, D_1: $[1,2,8t+4]$ and (ii) t sets of types, D_{2i}: $[x, v-(x+1), v-(x+2), x+3]$ where $x = 4i$ for $i = 1,2,\ldots,t$.

In (i) the differences $(1, 8t+6, 2, 8t+5, 3, 8t+4)$ appear once. In (ii) rest of the differences appear once. Hence proved.

CHAPTER 4

CONSTRUCTION OF BALANCED DESIGNS WITH TWO SYSTEMS OF BLOCKING

4.1 Introduction

Till now we have dealt with block designs having two nested factors (that are crossed with each other) within each of the blocks. In this chapter we will deviate from these nested blocking thoughout the design and study the designs having few simple blocks (without any nesting factor) and a large number of blocks with nested row-column structure.

With the purpose of recommending a design for a given set of experimental units, an overall assessment is made on the existing variability among them. On the basis of the assessment a statistican decides whether one- or two- or more-way elimination of heterogeneity is warranted and recommends to adopt a design which provides for such elimination for the maximum number of ways. For example, the given experimental units may be heterogeneous to warrant a two-way elimination of heterogeneity for most of the units and some of them may deserve one-way elimination only by the formation of blocks. In such a situation, the statisticians have been recommending a latin square or a row-column design for the experiment. This recommendation may force an experimenter to discard some of the available units and there is a compulsion to look for another set of units that can be accommodated in the type of design recommended by the statistician. This exercise invites difficulty to the experimenter through the increase in the requirement of experimental units and some of the degrees of freedom (d.f.) for the error component are lost in accounting for the unnecessary elimination of the heterogeneity. Further the use of higher ways of elimination imposes restricted randomization, which is avoidable. For instance for a row-column design the corresponding randomizations are restricted to the sets of strips (i.e. rows and columns) and not to the individuals (their intersection). This uncalled-for restriction on randomization may impugn the unbiasedness in estimation and validity of the experiment.

For an experiment with 4 treatments when a Latin Square Design (LSD) is recommended there is a need to have two squares in order to estimate error with sufficient d.f., which turns out to be 15. However, if the experimental units permit, one can use two replications of a randomised block design (RBD) along with one Latin Square and obtain the error with 12 d.f., thus saving one third of the experimental resources.

This becomes more useful in situations requiring the use of incomplete block designs with nested rows and columns, when in some of the blocks the units are more homogeneous allowing the use of incomplete blocks unless one is attempting to use the inter-block and inter-sub-block information.

This process of adopting two systems of blocking can be extended to more than two systems also and thereby an agreement on the available experimental units and judicious control of heterogeneity is attainable. Thus, without pursuing a typical type of blocking thoughout the design but meeting the demand of a given situation, this type of deigns permit mixing of two or more types of blocking. In the process a considerable amount of saving of exdperimental material is possible and the compulsion of 'restricted randomization' (in pursuance of a particular type of blocking thoughout the design being in vogue till now) gets relaxed. This class of design provides dynamism to the experimentation at its zenith by rising to the demanding situation.

In the experimental design, the parameters of interest are the treatment parameters and nuisance parameters arise generally due to different blocking systems like blocks ; rows and columns ; blocks, columns within blocks, and rows within blocks; etc. When a union of two designs are used for the experiment, the treatment parameters or the parameters of interest will be common but the nuisance parameters due to blocking etc. will be different for different desgins.

Keeping this in view, let us consider two desgins Δ_1 and Δ_2 under the general linear model setup as below.

4.2 Model and Preliminaries

Consider, for a design Δ_i the general linear model setup

$$\mathbf{Y}_i^{n_i \times 1} = \mathbf{X}_{i1}^{n_i \times v} \tau^{v \times 1} + \mathbf{X}_{i2}^{n_i \times m_i} \beta_i^{m_i \times 1} + \varepsilon_i \qquad (4.1)$$

with $E(\varepsilon_i) = \mathbf{0}$ and $D(\varepsilon_i) = \sigma_i^2 \mathbf{I}_{n_i}$, for $i = 1, 2$.

Where $\mathbf{Y}_i$ is the observational vector, τ is the vector of parameters of interest (treatments), β_i is the vector of nuisance parameters in Δ_i, ε_i is the vector of random errors, $\mathbf{X}_{i1}$ and $\mathbf{X}_{i2}$ are incidence matrices of observations vs parameters of interest and observations vs nuisance parameters respectively in Δ_i, $i = 1,2$. Since the experimental units are different for the two desgins, we assume that $\text{cov}(\varepsilon_1, \varepsilon_2) = \mathbf{0}$.

Then the reduced normal equations for parameters of interest (treatments) are

$$\mathbf{C}_{\Delta_i} \tau = \mathbf{Q}_{\Delta_i}, \quad \text{for } i = 1, 2 \qquad (4.2)$$

where $$\mathbf{C}_{\Delta_i} = X_{i1}' [I - X_{i2} (X_{i2}' X_{i2})^{-} X_{i2}'] \mathbf{X}_i \qquad (4.3)$$

$$\mathbf{Q}_{\Delta_i} = X_{i1}' [I - X_{i2} (X_{i2}' X_{i2})^{-} X_{i2}'] Y_i \qquad (4.4)$$

and A^- denotes a generalised inverse (g-inverse) of A.

For the design Δ, which is the union of the two designs Δ_1 and Δ_2 (i.e., $\Delta = \Delta_1 \cup \Delta_2$), we will have the linear model as

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{21} & 0 & X_{22} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \qquad (4.5)$$

which is in the form $\mathbf{Y} = \mathbf{X}\beta + \varepsilon$

with $E(\varepsilon) = \mathbf{0}$ and $D(\varepsilon) = \begin{bmatrix} D(\varepsilon_1) & 0 \\ 0 & D(\varepsilon_2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 \mathbf{I}_{n_1} & 0 \\ 0 & \sigma_2^2 \mathbf{I}_{n_2} \end{bmatrix}$

Then the reduced normal equations for the parameters of interest from model (4.5) are

$$C_\Delta \tau = Q_\Delta \qquad (4.6)$$

$$\text{where } C_\Delta = \frac{1}{\sigma_1^2} C_{\Delta_1} + \frac{1}{\sigma_2^2} \cdot C_{\Delta_2} \qquad (4.7)$$

$$\text{and } \quad Q_\Delta = \frac{1}{\sigma_1^2} Q_{\Delta_1} + \frac{1}{\sigma_2^2} \cdot Q_{\Delta_2}$$

It is to be noted that C_Δ is the weighted sum of C_{Δ_1} and C_{Δ_2}, the weights being the inverse of variances.

If we have information on the relative values of σ_1^2 and σ_2^2 it appears possible to choose the design Δ_1 and Δ_2 suitably so that the design $\Delta = (\Delta_1 \cup \Delta_2)$ will have some desirable properties, like variance balance. For example, for the similar blocking systems, we have several examples of union of two PBIB/2 designs with the same symbols/treatments resulting in a balanced block design with unequal block sizes . In fact the method of construction of Variance Balanced Incomple Block Designs (VBB) depend heavily on this fact with the another factor of taking copies of the two or more of the involved equisized block portions in order to offset the unequal replications and/or unequal block sizes for adjustment of the sums of weighted concurrences of different pairs of treatments to become equal in the final design.

Example 4.1 : Refer to the following two triangular PBIB designs listed in Table X of Clatworthy (1973).

T24 with parameters ($v=36, b_1=84, r_1=7, k_1=3, n_1=14, n_2=21, \lambda_{11} = 1 , \lambda_{21} = 0$)

and T85 with parameters ($v=36$, $b_2=28$, $r_2=7$, $k_2=9$, $n_1, =14$, $n_2 =21$, $\lambda_{11} = 1 , \lambda_{21} = 2$)

They are based on the same association scheme. It is to be noted that in T24 block size is 3, whereas in T85 block size is 9, thrice of that of T24. Jones, Sinha and Kageyama (1987) have shown that by adding 3 copies of the design T85 with the design T24 results into a Variance Balanced Block (VBB) design with two block sizes, the parameters of the VBB design being ($v=36$, $\mathbf{b}'=(84,84)$, $r=28$, $k1=3$, $k_2=9, \omega =0.67$). Here ω is the common sum of the weighted concurrences for each pair of treatments.

Given a set of experimental units, one may find few units that can be placed into simple blocking system, whereas for the rest of the units it may happen that row-column or a nested row-column classification is required to be adopted for local control. In the portion of incomple block design with nested row-column classification we ensure that it satisfies the first conditions as given in (A) of section **3.2** so that the information matirces of block-component design and that of the row-component design are equal. We call this portion of incomplete block design with nested rows and columns as Δ_1 and the portion of incomplete blocks only as Δ_2. We obtain the design with two systems of blocking so that the blocks of Δ_2 alongwith the columns of Δ_1 form a BIBD. Thus in the derived design Δ (which is the union of Δ_1 and Δ_2), the second condition as given in (A) of section **3.2** is also satisfied and thereby possess some desirable statistical properties.

As noticed in table 3.1 our effort to reduce the requirement of units to the minimum was fruitful for an odd v through BN-RC designs with unequal block sizes. But for an even v, the requirement is still high. Through the mixture of two blocking mixture systems, proposed in the following, it is possible to overcome the hurdles posed for an even v and avail a design with the minimum number of experimental units.

Note 4.1 Since for each block of Δ_1, its contributions to $\mathbf{C}_1$ and $\mathbf{C}_0$ are being kept equal, it is possible to replace this nested row-column block by its columns as blocks, if the experimental units permit, and add to Δ_2 and the two conditions in (A) of section **3.2** will continue to be satisfied.

Since for an odd v the requirement of experimental units has reached the lowest level through BN-RC designs with unequal block sizes, we will not consider this case here. In the following we make use of the method of symmetric differences for the construction of designs with two systems of blocking. Since $\mathbf{U}=\mathbf{0}$ for the portion Δ_1, it remains to verify that all the differences are occurring equally frequently in the columns of design $\Delta \left(= \Delta_1 \cup \Delta_2\right)$.

Note 4.2 : A set of differences in Δ_1 (with $p=2$), the incomplete block design with nested rows and columns, when used to generate an initial block will continue to follow the convention porposed in chapter 2 (section **2.4**). An initial block in Δ_1 as well as in Δ_2 is to be developed following the same rule known for block designs.

In the following we present some designs with two blocking systems - (i) blocks with nested row-coumn structure, forming design Δ_1 and (ii) simple block structure, forming design Δ_2.

4.3 Construction of Balanced Designs with Two Systems of Blocking for even v and p=2 in the (Nested) Row-Column Design

We discuss below the construction of these designs for v in the series 8t+2m, for m=0,1,2,3.

Case 1 : v =8t.

The following sets of differences in Δ_1

(i) One set of type, D_1 : [1,2, 8t-3], and

(ii) (t-1) sets of type, D_{2i} : [x, v - (x + 1), v - (x + 2), x + 3] where x = 4i for i = 1,2 . . . ,t-1, and (0, 4t) P.C.(4t) the initial block in Δ_2 when developed mod(v) provide a design where columns as blocks of Δ_1 along with the blocks of Δ_2 form a BIBD $\left(v, \frac{1}{2} v (v-1), v-1, 2, 1\right)$. This happens due to the fact that all the differences except 4t appear once in the columns of Δ_1 and all the pairs with the difference 4t appear once in the blocks of Δ_2.

In this design we will be having 8t nested incomplete blocks of size 2 x 3, 8t (t-1) nested incomplete blocks of size 2 x 4 and 4t incomplete blocks of size 2.

Case 2 : v=8t + 2

The following sets of differences in Δ_1

i) t sets of type, Di : [x, v - (x + 1), v - (x + 2), x + 3] where x = 4i for i = 1,2, . . . , t and in Δ_2 the initial block [0, 4t + 1] P.C.(4t + 1) when developed under mod (v) provide a design where columns from Δ_1 and blocks of Δ_2 form a BIBD$\left(v, \frac{1}{2}v(v-1), v-1, 2, 1\right)$.

In this design we will be having t (8t+2) nested incomplete blocks of size 2 x4 and (4t+1) incomplete blocks of size 2.

Case 3 : v=8t+4

a) t=odd : The following sets of differences for Δ_1

(i) $\frac{t-1}{2}$ sets of type, D_{1i} : [x, v - (x+1), v - (x+2), x+3]

where x=4i-3 for i=1,2,. . ., $\frac{t-1}{2}$ and t>

(ii) $\frac{t-1}{2}$ sets of type, D_{2j} : [y, v - (y+1), v - (y+2), y+3]

where y=2t+4j= for j=1,2,. . . , $\frac{t-1}{2}$ and t>1

(iii) one set of type, D_3 = [2t-1, 2t, 2t+2, 2t+3],

and (iv) one set of type, D_4 = [2t+1, 2t+1, 4t+2] P.C.(2t+1)

and for Δ_2 the following initial blocks

(a) $\begin{bmatrix} 4t+2 \\ 6t+3 \end{bmatrix}$ P.C.(4t+2), and

(b) $\begin{bmatrix} 2t+1 \\ 6t+3 \end{bmatrix}$ P.C.(2t+1).

when developed under mod(v) provide the design Δ and the columns from Δ_1 and blocks of Δ_2 from a BIBD $\left(v, \frac{1}{2}v(v-1), v-1, 2, 1\right)$.

In this design there are $t(8t+1)$ incomplete nested blocks of size 2 x 4, $(2t+1)$ incomplete nested blocks of size 2 x 3 and $(6t+3)$ blocks of size 2.

Example 4.2 : for $t=1$, $v=12$

Δ_1 : initial blocks $\begin{bmatrix}0&3&6\\3&6&0\end{bmatrix}$ P.C.(3), $\begin{bmatrix}0&1&3&7\\1&3&7&0\end{bmatrix}$ mod (12)

Δ_2: $\begin{array}{c|c|c|c|c|c|c|c|c} 6&7&8&9&10&11&3&4&5\\ 9&10&11&0&1&2&9&10&11\end{array}$

The columns from Δ_1 and blocks of Δ_2 form a BIBD (12,66,11,2,1). There are 3 nested block of size 2 x3, 12 nested blocks of size 2 x 4 and 9 blocks of size 2.

b) t = even : The following sets of differences for Δ_1,

(i) t/2 sets of type, D_{1i} : $[x, v - (x+1), v - (x+2), x+3]$

where $x = 4i-3$ for $i=1,2,\ldots,t$,

(ii) t/2 sets of type, D_{2j} : $[y, v - (y+1), v - (y+2), y+3]$,

where $y = 2t+4j-2$ for $j = 1,2,\ldots, t/2$, and

(iii) one set of type, D_3 : $[2t+1, 2t+1, 4t+2]$ P.C.$(2t+1)$

along with the following initial blocks for Δ_2

(a) $\begin{pmatrix}4t+2\\6t+3\end{pmatrix}$ P.C. $(4t+2)$, and

(b) $\begin{pmatrix}2t+1\\6t+3\end{pmatrix}$ P.C. $(2t+1)$.

when developed under mod (v) results into a design with two systems of blocking with $t(8t+4)$ nested blocks of size 2 x 4, $(2t+1)$ nested blocks of size 2 x 3 and $(6t+3)$ incomplete blocks of size 2.

In this design columns from Δ_1 and blocks from Δ_2 form a BIBD $\left(v, \frac{1}{2} v(v-1), v-1, 2, 1\right)$.

Example 4.3 : $t=0$, $v=4$.

$$\Delta_1 : \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} ; \quad \Delta_2 : \begin{array}{c|c|c} 2 & 3 & 1 \\ 3 & 0 & 3 \end{array}$$

The columns as block from Δ_1 and blocks from Δ_2 form a BIBD (4,6,3,2,1).

There is another design for $v=4$ with two blocking systems.

$$\Delta_1 : \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} ; \quad \Delta_2 : \begin{array}{c|c} 0 & 1 \\ 2 & 3 \end{array}$$

The columns of Δ_1 and blocks of Δ_2 form a BIBD (4,6,3,2,1).

Case 4 : $v = 8t+6$.

In Δ_1

(i) t sets of differences of type, D_{1i} : $[x, v-(x+1), v-(x+2), x+3]$

where $x=4i-1$ for $i = 1,2,\ldots,t$.

(ii) $(4t+3)$ nested blocks of type $\begin{bmatrix} 2j & 2j+1 & 2j+2 \\ 2j+1 & 2j+2 & 2j \end{bmatrix}$

where $j=0,1,\ldots,4t+2$

(iii) one nested block of type $\begin{bmatrix} 1 & 3 \ldots & 8t+5 \\ 3 & 5 & 1 \end{bmatrix}$

along with the following intial blocks for Δ_2

$$\begin{pmatrix} 0 \\ 4t+3 \end{pmatrix} \quad \text{P.C.}(4t+3)$$

results into the design with two blocking systems where the columns of Δ_1 and blocks of Δ_2 form a BIBD$\left(v, \frac{1}{2}v(v-1), v-1, 2, 1\right)$. There are t(8t + 6) nested blocks of size 2 x 4, (4t + 3) nested blocks of size 2 x 3, one nested block of size 2x(4t + 3) and (4t + 3) blocks of size 2.

Example 4.4 : For $t=0$, $v=6$.

$$\Delta_1: \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 5 & 0 \\ 5 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 5 & 3 & 1 \end{bmatrix}$$

$$\Delta_2: \begin{array}{c|c|c} 0 & 1 & 2 \\ 3 & 4 & 5 \end{array}$$

Columns from Δ_1 and blocks from Δ_2 form a BIBD (6,15,5,2,1). There are 4 nested blocks of size 2 x 3 and 3 blocks of size 2.

Example 4.5 : For $t=1$, $v=14$

$$\Delta_1: \begin{bmatrix} 0 & 2 & 5 & 9 \\ 2 & 5 & 9 & 0 \end{bmatrix} \text{ mod (14) and the following Nested Blocks}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \end{bmatrix}, \begin{bmatrix} 6 & 7 & 8 \\ 7 & 8 & 6 \end{bmatrix}, \begin{bmatrix} 8 & 9 & 10 \\ 9 & 10 & 8 \end{bmatrix}, \begin{bmatrix} 10 & 11 & 12 \\ 11 & 12 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 13 & 0 \\ 13 & 0 & 12 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 3 & 5 & 7 & 9 & 11 & 13 & 1 \end{bmatrix}$$

$$\Delta_2: \begin{array}{c|c|c|c|c|c|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

There are 14 nested blocks of size 2 x 4, 7 nested blocks of size 2 x 3, one nested block of size 2 x 7 and 7 blocks of size 2. The columns of Δ_1 and blocks of Δ_2 form a BIBD (14,91, 13,2,1).

Theorem : 4.1 If s is a prime power, it is always possible to construct a balanced design for $v=s^2$ as a mixture of Δ_1 : a row-column design with s rows and s^2 columns and Δ_2 : an incomplete block design in s blocks each of size s.

Proof The solution of this design is based on a complete set of Mutually Orthogonal Latin Squares (MOLS). We note that when s is a prime or prime power a BIBD $(s^2, s^2+s, s+1, s, 1)$ can always be constructed with the help of a complete set of MOLS of size $s \times s$.

Let the s^2 symbols/treatments be arranged in an $s \times s$ array A and let $L_1, L_2, \ldots, L_{s-1}$ be the complete set of MOLS of size s. Let the rows of Δ_1 be numbered as $0,1,\ldots,s-1$ and the columns be numbered as $1,2,\ldots,s^2$. Then the design Δ_1 can be obtained by forming an $s \times s^2$ array as follows:

First s columns are the s columns of A. For getting the rest of s^2-s columns in Δ_1, L_k's, $k=1,2,\ldots,s-1$, are to be superimposed on A in turn. Let the j^{th} symbol in the i^{th} row of L_k falls on a symbol, say, α, of A, $i=0,1,\ldots,s-1$; $k=1,2,\ldots,s-1$, $j=1,2,\ldots,s$. Let $i+k=p \pmod s$. Then $(p, ks+j)^{th}$ cell of Δ_1 is filled by the symbol α, for $0 \leq i \leq s-1 ; 1 \leq j \leq s ; 1 \leq k \leq s-1$.

The design Δ_2, an incomplete block design, is obtained by treating the rows of A as its blocks.

Obviously by this construction, we have ensured that in Δ_1 each of the s^2 symbols occur exactly once in each of the s rows. The columns of Δ_1 as blocks along with the blocks of Δ_2 account for the blocks of a BIBD $(s^2, s^2+s, s+1, s, 1)$. Hence proved.

Example 4.5 : $v = 3^2 = 9$

$$A:\begin{pmatrix}1&2&3\\4&5&6\\7&8&0\end{pmatrix}\qquad L_1=\begin{pmatrix}A&B&C\\B&C&A\\C&A&B\end{pmatrix}\qquad L_2=\begin{pmatrix}A&B&C\\C&A&B\\B&C&A\end{pmatrix}$$

$$\Delta_1:\begin{bmatrix}1&2&3&8&0&7&5&6&4\\4&5&6&1&2&3&0&7&8\\7&8&0&6&4&5&1&2&3\end{bmatrix}\qquad \text{and}\qquad \Delta_2:\begin{array}{c|c|c}1&4&7\\2&5&8\\3&6&0\end{array}$$

4.4 Construction of Balanced Designs from Initial Block Solutions of BIBD

While forming a block in Δ_1 from initial block solutions (each block is of size k) of a BIBD the convention we will be following is as below:

Case (i) BIBD's involving symbols from a single finite field

The initial blocks not involving the invariant treatment, ∞, are the candidates for getting the blocks in Δ_1. Let the contents of such a block be ($\alpha_1, \alpha_2, \ldots, \alpha_k$). Then the corresponding nested block in $\Delta_1$is obtained by forming an array of size k × s with its (i,j)th cell by the element $(\alpha_i + j)$ mod (s), for j = 0,1, . . . , s-1. This block of Δ_1 is represented by $\left[\left(\alpha_1, \alpha_2, \ldots, \alpha_k\right)' \text{ mod(s)}\right]$

Case (ii) BIBD's involving symbols of different classes

If in an initial block of the given BIBD all the elements belong to the same class it will continue to be treated as detailed in case (i). Otherwise by a judicious choice of few initial blocks a block in Δ_1 is to be generated. We first form mutually exclusive groups of the intial blocks of the BIBD such that when the initial blocks in a group are placed in columns, after rearrangement within the columns, if necessary, symbols from different classes occur equally frequently in all the rows of such an arrangement. Then in each of the groups, each member (rearranged initial block of the given BIBD) is developed as in case (i) above. The solutions left out are used to develop blocks of Δ_2.

Note 4.3 : If no initial block solution is left out, Δ_1 will form a BN-RC design with equal or unequal block sizes according as the sizes of the groups (number of distinct initial blocks included) are equal or unequal.

Example 4.6 : Consider the following initial block solutions of a BIBD in serial number 45 given in Raghavarao (1971).

$(\infty, 0_1, 0_2, 1_2, 2_2, 4_2)$ ....(1)

$(\infty, 0_1, 3_1, 5_1, 6_1, 0_2)$ ....(2)

$(0_1, 1_1, 3_1, 0_2, 2_2, 6_2)$ .... (3)

$(0_1, 1_1, 3_1, 1_2, 5_2, 6_2)$ ....(4)

$(0_1, 4_1, 5_1, 0_2, 1_2, 3_2)$ ....(5)

mod 7.

A block in Δ_1 can be formed by grouping 3rd and 4th initial blocks and developing each one under mod 7. It is to note that the third block and the rearranged fourth block when placed in two columns looks as

$$\begin{array}{cc} 0_1 & 1_2 \\ 1_1 & 5_2 \\ 3_1 & 6_2 \\ 0_2 & 0_1 \\ 2_2 & 1_1 \\ 5_2 & 3_1 \end{array}$$

where symbols from two classes occur once in each of the 6 rows.

The block Δ_1 developed from this group will be

$$\begin{bmatrix} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 \\ 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 \\ 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 \\ 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 \\ 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 3_1 & 4_1 & 5_1 & 6_1 & 0_1 & 1_1 & 2_1 \end{bmatrix}$$

and this block will be represented in Δ_1 by the two initial block solutions (3^{rd} and 4^{th}) of the given BIBD as

$$\left[\left(0_1, 1_1, 3_1, 0_2, 2_2, 5_2\right)', \ \left(1_2, 5_2, 6_2, 0_1, 1_1, 3_1\right)' \quad \text{mod } 7 \right]$$

It is to note that the rest of the three initial blocks are to be used to form block as no more grouping is possibole. The blocks of Δ_2 generated from the initial block solution (1) are

$$\begin{array}{c|c|c|c|c|c|c} \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 \\ 0_2 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 \\ 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 0_2 & 1_2 \\ 4_2 & 5_2 & 6_2 & 0_2 & 1_2 & 2_2 & 3_2 \end{array}$$

and it will be represented as $\left(\infty, 0_1, 0_2, 1_2, 2_2, 4_2\right)'$ mod 7.

In the following, we give some designs constructed from R (n), a BIBD series n as given by Raghavarao (1971) and D(n), a BIBD in serial number n as given by Das and Giri (1988).

The initial block solutions for a BIBD with v treatments are listed as difference sets which are to be developed under mod (s), $s \leq v$, as mentioned therein.

When s = v and no partial cycle is involved, only the design design Δ_1 is to be developed and it provides a BN-RC design. A partial cycle, when developed provides the blocks of design Δ_2. For s = v, we have the cases involving symbols and/or a number of classes. We are discussing these cases below.

4.4.1 From the initial block solutions of BIBD's given by Raghavarao

1) v = 8

Consider the initial blocks of R (15). Then if x is a primitive root of GF(7)

Δ_1 : $[\left(0, x^1, x^3, x^5\right)' \bmod (7)]$ and

Δ_2 : $\left(\infty, x^0, x^2, x^4\right)' \bmod 7$

provides a design with two blocking systems. The columns of Δ_1 alongwith blocks of Δ_2 form a BIBD (8,14,7,4,3). There is one nested block of size 4 x 7 and 7 incomplete blocks of size 4.

2) v = 12

Consider the initial blocks of R (34). Then

Δ_1 : $[\left(0, 1, 3\right)' \bmod 11]$, $[\left(0, 1, 5\right)' \bmod 11]$, $[\left(0, 4, 6\right)' \bmod 11]$ and

Δ_2 : $\left(\infty, 0, 3\right)' \bmod 11$.

results into a design with two systems of blocking. The columns of Δ_1 alongwith the blocks of d_2 forms a BIBD (12, 44, 11, 3,2). There are 3 nested blocks of size 3 x 11 and 11 blocks of size 3.

3) v = 12

Consider the initial blocks of R(35). Then

Δ_1 : $[\ (0,1,3,7)'$ mod 11$]$, $[\ (0,2,7,8)'$ mod 11$]$ and

Δ_2 : $(\infty,0,1,3)'$ mod 11

provide a design with two blocking systems. The columns of Δ_1 along with the blocks of Δ_2' forms a BIBD (12,33,11,4,3). There are two nested blocks of size 4 x 11 and 11 incomplete blocks of size 4.

4) v = 12

Consider the initial blocks of R(36). Then we can construct a design with two systems of blocks. The columns of Δ_1 and blocks of Δ_2 forming a BIBD (12,22,11,6,5). There is one nested block of size 6 x 11 and 11 incomplete blocks of size 6.

5) v = 15

Consider the initial blocks of R (42). Then

Δ_1 : $[\ (1_1,4_1,0_2)'$, $(1_2,4_2,0_3)'$, $(1_3,4_3,0_1)'$ mod 5 $]$

$[\ (2_1,3_1,0_2)'$, $(2_2,3_2,0_3)'$, $(2_3,3_3,0_1)'$ mod 5 $]$ and

Δ_2 : $(0_1,0_2,0_3)'$ mod 5

provide a design with two systems of blocks. The columns of Δ_1 alongwith blocks of Δ_2 forms a BIBD (15,35,7,3,1). There are two nested blocks of size 3 x 15 and 5 incomplete blocks of size 3.

6) $v = 5$

Considering the initial blocks of R(45) and forming the blocks as discussed in the begining of this section we have a design with two systems of blocks. The columns of Δ_1 and blocks of Δ_2 form a BIBD (15, 35, 14, 6, 5). There is one nested block of size 6 x 14 and 21 incomplete blocks of size 6.

7) $v = 21$

Consider the initial blocks of R (59). Then

$$\Delta_1: \quad [\,(1_1, 6_1, 0_2)', \ (1_2, 6_2, 0_3)', \ (1_3, 6_3, 0_1)' \ \bmod 7\,],$$

$$[\,(3_1, 4_1, 0_2)', \ (3_2, 4_2, 0_3)', \ (3_3, 4_3, 0_1)' \ \bmod 7\,],$$

$$[\,(2_1, 5_1, 0_2)', \ (2_2, 5_2, 0_3)', (2_3, 5_3, 0_1)' \ \bmod 7\,] \quad \text{and}$$

$$\Delta_2 : (0_1, 0_2, 0_3)' \qquad \bmod 7$$

results into a design with two blocking systems. The columns of Δ_1 alongwith the blocks of Δ_2 forms a BIBD (21, 70, 10, 3, 1). There are 3 incomple nested blocks of size 3 x 21 and 7 incomplete blocks of size 3.

Similary from R(63) for $v = 22$, R(69) for $v = 26$, R(72) for $v = 28$ and R(84) for $v = 45$ designs with two systems of blocks can be constructed.

Now we will construct some designs with two systems of blocking from the initial solutions to BIBD's by Das and Giri (1988).

4.4.2 From initial block solutions of BIBD's given in Das and Giri

1) v = 9

Consider the initial blocks of D(31). Then

$\Delta_1 : [\left(0, 1, 3\right)'$ mod 8] and

$\Delta_2 : (\left(\infty, 0, 4\right)'$ P.C. (4) ; mod 8

provide the design. The columns of Δ_1 along with blocks of Δ_2 form a BIBD (9,12,4,3,1). There is one nested block of size 3 x 8 and 4 incomplete blocks of size 3.

2) v = 16

Consider the initial blocks of D (33). Then

$\Delta_1 : [\left(0, 2, 3, 11\right)'$ mod 15]

and $\Delta_2 : (\left(\infty, 0, 5, 10\right)'$ P.C.(5) mod 15

results into a design with two systems of blocks. The columns of d1 and blocks of d_2 form a BIBD (16, 20, 5, 4, 1) There is one nested block of size 4 x 15 and 5 incomplete blocks of size 4 in the design.

3) **v = 25** : From the initial blocks of D (34) a design with one nested block of size 5 x 24 and 6 incomplete blocks of size 5 can be constructed. The columns of nested block and blocks of incomplete block design will form a BIBD (25,30,6,5,1)

4.5 Construction of Balanced designs for p = 3 and q = v in the (Nested) Row-Column Design

1) v = 6

$$\Delta_1 : \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 3 & 4 & 5 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 3 & 4 & 5 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \end{bmatrix}$$

$$\Delta_2 : \begin{array}{c|c} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{array}$$

The column of d1 and blocks of d2 form BIBD (6, 20, 10, 3, 4)

2. Design for v = 12

The columns of d1 and the blocks of d2 when combined form a BIBD (9, 12, 4, 3, 1)

$$\Delta_1 : \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 \\ 6 & 7 & 8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 & 1 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 0 & 1 & 2 \end{bmatrix}$$

$$\Delta_2 : \begin{array}{c|c|c|c|c|c|c|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 9 & 10 & 11 & 0 & 1 & 2 & 3 \end{array}$$

The columns of Δ_1 alongwith blocks of Δ_2 form a BIBD (12, 44, 11, 3, 2)

3) v = 15

$$\Delta_1 : \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}$$

$$\Delta_2 : \begin{array}{c|c|c|c|c} 0 & 1 & 2 & 3 & 4 \\ 4 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \end{array}$$

The columns from Δ_1 and blocks of Δ_2 form a BIBD (15, 35, 7, 3, 1).

CHAPTER 5

ROBUSTNESS OF BN-RC DESIGNS AGAINST THE UNAVAILABILITY OF OBSERVATIONS

5.1 Introduction

Missing observations or non-availability of some of the observations is a common occurence against which the experimental design need to be robust. With this aspect in view, we will study the robustness of BN-RC designs against the unavailability of observations in this chapter. This study, in a way, would help a user of BN-RC design to assess the loss that may occur when some observations are lost.

In the literature robustness of block designs against missing observations has been studied from different angles. According to Ghosh (1982 b), a block design is robust against the non-availability of any $t\,(\geq 1)$ observations if the block design remains connected on deletion of these t observations. This criterion (to be called Criterion 1) ensures that a robust design retains the estimability of all treatment contrasts in the residual design so obtained. Ghosh (1982 b) proved the robustness of a BIBD (v, b, r, k, λ) to the unavailability of any $(r-1)$ observations and all observations in any $(r-1)$ blocks by this criterion.

Another criterion (to be called Criterion 2) of robustness that has received attention, is in terms of the efficiency of the residual design. According to this criterion, a design is said to be robust if the efficiency of the residual design relative to the original one is not too small. This efficiency, E, is defined as

$$E = \frac{\text{Harmonic mean of the non-zero eigen values of } C_{d*}}{\text{Harmonic mean of the non-zero eigen values of } C_{d}}$$

C_d and C_{d*} being the C-matrices of the original design d and the residual design d* respectively.

It is to note that if rank $(C_{d*}) = v - 1 = \text{rank}(C_{d^*})$ then the design is robust according to Criterion 1. And if the loss of efficiency (1-E) is small (negligible) the design d is robust against the loss of observation(s). We will confine our study to the extent of finding the eigen values of C_{d*} and the efficiency of the residual design d* .

A perusal of the existing literature reveals that except the work of Ghosh (1981) all the studies of rebustness against the unavailability of observation(s) relates to block designs only. So far as we are aware, there is no study on robustness of BN-RC designs.

In the following a mention of robustness will mean the robustness against the unavailability of observations only

In this chapter we will be presenting some studies on robustness of BN-RC designs against the non-availability of block(s), row(s) and column(s). Since the C-matrix of a BN-RC design involves the C-matrices of the three component designs, dealing with the loss of scattered observations becomes complex.

Before going into the details, let us consider some more notaions in continuity with those used in the previous chapters.

5.2 Preliminaries and Notations

Let d be a connected BN-RC designs with parameters (v, b, r, p, q', μ). Then the information matrix of d is

$$C_d = C_{2d} = \frac{\mu v}{p}(I_v - 1/v\, J_{v,v}) \qquad (5.1)$$

Let B_j, R_{jl} and K_{jm} denote the j^{th} block, the l^{th} row within the j^{th} block and the m^{th} column within the j^{th} block for $1 \le j \le b$; $1 \le l \le p$ and $1 \le m \le q_j$. Let $C_{0d}(B_j)$, $C_{1d}(R_{jl})$ and $C_{2d}(K_{jm})$ denote the **C** - matrices of the j^{th} block, l^{th} row within the j^{th} block and the m^{th} column within the j^{th} block respectively of the design d. The residual design will be referred by d^*.

When observations in row(s) or column(s) of block B_j is missing, the reduced block, will be denoted by B_j^*, the reduced l^{th} row within the j^{th} block will be denoted by $\mathbf{R}_{jl}^*$ and $\mathbf{K}_{jm}^*$ will denote the reduced m^{th} column within the j^{th} block.

It is to note that in a BN-RC design, within a block same set of symbols appear in different rows with same frequency to ensure the equality of information matrices of d_1, the row-component and d_0,the block component designs respectively, i.e.

$$C_{od}(B_j) = \sum_{l=1}^{p} C_{1d}(R_{jl}) \qquad \text{for all } j = 1, 2, \ldots, b \tag{5.2}$$

Even if some of the rows become unavailable this equality holds good for the residual block. However, this will not be the case when some of the columns are missing. Hence when a complete block is missing this is equivalent to the loss of all the columns contained in that block and the $\mathbf{C}_d$ (given in (5.1)) gets reduced by the amount of information contained in the columns of the affected block.

Note 5.1 : In case all but one, say the first, row/column of a block, say the first without any loss of generality, are lost/missing, then it is as worse as missing the block (B_1) completely, as this reduces block (say B_1^*) is completely disconnected one and provides nil information. This can be revealed from the **C**- matrices of the component design of the affected block ;

(1) when only one row remains , $C_{0d}(B_1^*) = C_{1d}(R_{11})$ and $\sum_{m=1}^{q_1} C_{2d}(K_{1m}) = 0$ and

(ii) when only one column remains , $C_{0d}(B_1^*) = C_{2d}(K_{11})$ and $\sum_{l=1}^{p} C_{1d}(R_{1l}) = 0$

5.3 Robustness to the Unavailability of Block(s)

Following Ghosh (1982 b) we give a bound for the number of missing blocks.

5.3.1 A Bound for Number of Missing Blocks.

Given a design d : BN-RC (v, b, r, p, q, μ) with equal block sizes, the column-component design of it is a design D : BIBD (v, bq, r, p, μ). Clearly r is a multiple of p. Let any t blocks of the BN-RC design become unavailable.

Since the loss of a block in d is equivalent to the loss of a set of q blocks in D, following Ghosh (1982 b), the design d is robust (in terms of criterion 1) against the loss of t blocks in d, if

$$tq \leq r - 1 = \frac{bpq}{v} - 1$$

$$\Rightarrow t \leq \frac{bp}{v} - \frac{1}{q}$$

$$\Rightarrow t \leq \left[\frac{bp}{v} - \frac{1}{q}\right] \text{ , t being an integer}$$

where [x] denotes the greatest integer contained in x. Hence a BN-RC (v, b, r, p, q, μ) is robust against the non-availability of any $t \leq \left[\frac{bp}{v} - \frac{1}{q}\right]$ blocks in terms of connectedness.

In the following we shall consider the robustness against missing block(s) / row(s) / column(s) under Criterion 2.

It is to be noted that the information matrix of the residual design d^* can be written down as $\mathbf{C}_{d*} = \mathbf{C}_{2d} - \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\mathbf{F}$ is again a C-matrix with zero row and column sums. Two matrices in the R.H.S. above are commutative too since $\mathbf{C}_{2d}$ is of the form $a\mathbf{I} + b\mathbf{J}$. A proof of it is provided in the Appendix 1. The Appendix also contain some results, on matrix theory, we made use ofin working out the eigen values of $\mathbf{C}_{d*}$.

5.3.2 One Block Missing

Without any loss of generality we can assume that the first block of size p x q_1 containing the first $q_1 = q$ symbols /treatments from a BN-RC (v, b, r, p, $\mathbf{q}'$, μ) design d becomes unavailable. Then the information matrix of the residual design d^* is

$$\mathbf{C}_{d*} = \mathbf{C}_{2d*} = \mathbf{C}_{2d} - \sum_{m=1}^{q} \mathbf{C}_{2d}(K_{1m}) \tag{5.3}$$

Let $N_2(B_1)$ be the incidence matrix of q treatments in the q columns contained in the first block B_1. Then 5.3 reduces to $\mathbf{C}_{d*} = \mathbf{C}_{2d} - \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where **F** is of the form

$$F = p\,\mathbf{I}_q - 1/q\; N_2(B_1)\, N_2'(B_1) \tag{5.4}$$

Case 1 : If $N_2(B_1)$ corresponds to the incidence matrix of a symmetric BIBD (q, p, λ) which happens when the affected block of the BN-RC is a YSD (q, p, λ), we have

$$\mathbf{F} = \frac{\lambda q}{p}(\mathbf{I}_q - 1/q\,\mathbf{J}_{q,q}) \tag{5.4}$$

The eigen values (θ_i) and the corresponding multiplicies (α_i) of C_{d*} are

$$\theta_1 = \frac{\mu v}{p} \quad , \quad \alpha_1 = v - q$$

$$\theta_2 = \frac{\mu v - \lambda q}{p} \quad , \quad \alpha_2 = q - 1$$

$$\theta_3 = 0 \quad , \quad \alpha_3 = 1$$

It can be easily seen that θ_1 and θ_2 are positive and hence d^* will be a connected design if $q < v$, or $\lambda < \mu$.

The efficiency, E, of the residual design d^* is given by

$$E = 1 - [\lambda q (q - 1)] / [(v - q)(\mu v - \lambda q) + (q - 1)\mu v]$$

Case 2 : When $N_2(B_1)$ corresponds to an incidence matrix of a PBIB/m or any incomplete block design (IBD), let the **F** matrix of (5.4) have its non-zero eigen values as μ_i with corresponding multiplicites α_i, $i = 1,2,\ldots, m$, $\sum_{i=1}^{m} \alpha_i = q' - 1, q' \leq q$,

Then C_{d^*} has the eigen values (θ_i) with multiplecities (α_i) as below :

$$\theta_i = \frac{\mu v}{p} - \mu_i \ , \ \alpha_i \ \text{for} \ i = 1, 2, \ldots m.$$

$$\theta_{m+1} = \frac{\mu v}{p} \ , \ \alpha_{m+1} = v - q'$$

$$\theta_{m+2} = 0 \quad , \quad \alpha_{m+2} = 1 \ ,$$

The relative efficiency of d^* can be found from

$$E = (v - 1) / [v - q' + \frac{\mu v}{p} \sum_i{}' \frac{\alpha_i}{\theta_i}]$$

Where $\sum'$ implies summation taken over those i's for which $\theta_i \neq 0$.

Consider the BN-RC design in the series $v = 8t + i$, $0 \le i \le 7$, $p = 2$ and $3 \le q \le 6$ discussed in section 2 .5, 2.6 and 3.7 . We note that for a block B_1 having q distinct symbols in a row

i) of size 2 x 3, the columns of B_1 forms a symmetric BIBD(3,2,1). Then the two non-zero eigen values of **F** are 3/2,

ii) of size 2 x 4, the columns of B_1 forms a GD block design with parameter ($v = b = 4$, $r = k = 2, n_1 = 1, n_2 = 2$, $\lambda_1 = 0, \lambda_2 = 1$). Then the non-zero eigen values of **F** are 1 and 2 with multiplicities 2 and 1 respectively ,

iii) of size 2 x 5, the columns of B_1 forms a cyclic design. And the non-zero eigen values of **F** are $\frac{5 - \sqrt{5}}{4}$ and $\frac{5 + \sqrt{5}}{4}$ with multiplicities 2 each, and

iv) of size 2 x 6, the columns of B_1 forms a cyclic design. The non-zero eigen values of **F** are 2 , 3/2 and 1/2 with multiplicities. 1, 2 and 2 respectively.

5.3.3 More than One Block Missing

Consider a BN-RC (v, b, r, p, $\mathbf{q}'$, μ) design d. Let d^* be the residual design wherein s blocks $B_1, B_2, \ldots, B_s$ are missing . Let the j^{th} block B_j is of size p x q_j , $j = 1,2 \ldots, s$. Then the information matrix of d^* is

$$\mathbf{C}_{d*} = \mathbf{C}_{2d} - \sum_{j=1}^{s} \sum_{m=1}^{q_j} \mathbf{C}_{2d}(\mathbf{K}_{jm}) = \mathbf{C}_{2d} - \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \text{say} \tag{5.5}$$

5.3.3.1 Two Blocks Missing

The case of the two missing blocks which have no common treatments does not cause any problem in obtaining the eigen values of the **F** matrix since the non-zero eigen values of **F** will be the non-zero eigen values of the **C** -matrices of the two missing blocks taken individualy whether or not they are of the same size. As such this case needs no special discussion.

Let all the observations in two blocks B_1 and B_2 of the same size, p × q, are unavailable. Let m of the treatments appeared in both the blocks and the remaining n=q - m treatments are distinct/ do not match. Without any loss of generality we assume that the set of treatments $(1, 2, \ldots, n, n+1, \ldots, n+m=q)$ and $(n+1, n+2, \ldots, n+m=q, q+1, \ldots, q+n)$ appeared in B_1 and B_2 respectively.

Let for the i^{th} block, i= 1, 2, $N_2(B_i)$ is an incidence matrix of a symmetric BIBD (q, p, λ). Then $\mathbf{C}_{d*}$ can be written as in (5.5), where $\mathbf{F} = \frac{\lambda q}{p}\mathbf{A}$ and $\mathbf{A}$,a square matrix of order q+n is given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_n - \frac{1}{q}\mathbf{J}_{n,n} & -\frac{1}{q}\mathbf{J}_{n,m} & 0 \\ -\frac{1}{q}\mathbf{J}_{m,n} & 2(\mathbf{I}_m - \frac{1}{q}\mathbf{J}_{m,m}) & -\frac{1}{q}\mathbf{J}_{m,n} \\ 0 & -\frac{1}{q}\mathbf{J}_{n,m} & \mathbf{I}_n - \frac{1}{q}\mathbf{J}_{n,n} \end{bmatrix}$$

The eigen values (μ_i) with corresponding multiplicities(α_i) of $\mathbf{A}$ are

$\mu_0 = 0$, $\alpha_0 = 1$

$\mu_1 = 1$, $\alpha_1 = 2(n-1)$

$\mu_2 = 2$, $\alpha_2 = m-1$

$\mu_3 = \frac{m}{q}$, $\alpha_3 = 1$

$\mu_4 = \frac{q+n}{q}$, $\alpha_4 = 1$

Hence for $\mathbf{C}_{d*}$, the eigen values (θ_i) with corresponding multiplicities (α_i) are

$\theta_0 = 0$, $\alpha_0 = 1$

$$\theta_1 = \frac{\mu v - \lambda q}{p} \quad , \quad \alpha_1 = 2(n-1)$$

$$\theta_2 = \frac{\mu v - 2\lambda q}{p} \quad , \quad \alpha_2 = m-1$$

$$\theta_3 = \frac{\mu v - \lambda m}{p} \quad , \quad \alpha_3 = 1$$

$$\theta_4 = \frac{\mu v - \lambda(q+n)}{p} \quad , \quad \alpha_4 = 1$$

$$\theta_5 = \frac{\mu v}{p} \quad , \quad \alpha_5 = v - (q + n + 1)$$

And the relative efficiency of d^* can be found from

$$E = \frac{(v-1)p}{\mu v} \sum_i{}' \frac{\alpha_i}{\theta_i} ,$$

Where $\sum'$ implies summation taken over i's for which $\theta_i \neq 0$.

5.3.3.2 Unavailability of s Blocks containing Mutually Disjoint Sets of Treatments

Consider $B_1, B_2, \ldots, B_s$ are the s blocks having mutually disjoint set of treatments are unavailable. Let Bj is of size p x q_j, $1 \leq j \leq s$. Without any loss of generality we assume that ($\Phi_{j-1} + 1 , \Phi_{j-1} + 2 , \ldots , \Phi_{j-1} + q_j = \Phi_j$) are the set of treatments appearing in the j^{th} block B_j, $\Phi_0 = 0$ and $\sum_{i=1}^{s} q_i = \Phi_s \leq v$.

Then the matrix **F** in (5.5) takes the form $\mathbf{F}^{\Phi_s \times \Phi_s} = \text{Diag}(\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_s)$, a diagonal matrix of $\mathbf{F}_j$ matrices, $1 \leq j \leq s$. It is to note that $\mathbf{F}_j$ is a C- matrix for all j = 1, 2, . . . , s. Hence the eigen values of $\mathbf{F}_j$'s are the eigen values of **F** .

Now we consider the BN-RC designs in the series v = 8t + i, i = 0, 1, . . . ,7 discussed in section 2.5, 2.6 and 3.7 where a block is of size 2 x 3, or 2 x 4, or 2 x 5, or 2 x 6. Let

s_1 blocks of size 2 x 3, s_2 blocks of size 2 x 4, s_3 blocks of size 2 x 5 and s_4 blocks of size 2 x 6 are missing, $s_1 + s_2 + s_3 + s_4 = s$, say. Then following the discussion in case 2 of section 5.3.2, the eigen values (θ_i) with corresponding multiplieicities (α_i) of $\mathbf{C}_{d^*}$ are

$$\theta_0 = 0 \quad , \quad \alpha_0 = 1$$

$$\theta_1 = \frac{\mu v - 1}{2} \quad , \quad \alpha_1 = 2s_4$$

$$\theta_2 = \frac{\mu v - 2}{2} \quad , \quad \alpha_2 = 2s_2$$

$$\theta_3 = \frac{\mu v - 3}{2} \quad , \quad \alpha_3 = 2(s_1 + s_4)$$

$$\theta_4 = \frac{\mu v - 4}{2} \quad , \quad \alpha_4 = s_2 + s_4$$

$$\theta_5 = \frac{\mu v - 2k}{2} \quad , \quad \alpha_5 = 2s_3$$

$$\theta_6 = \frac{\mu v - (5 - 2k)}{2} \quad , \quad \alpha_6 = 2s_3$$

$$\theta_7 = \frac{\mu v}{2} \quad , \quad \alpha_7 = v - (2s + 2s_3 + s_2 + 3s_4 + 1)$$

where $k = \dfrac{5 - \sqrt{5}}{4}$.

And hence the relative efficiency of d*, $E = \dfrac{2(v - 1)}{\mu v \sum_i{}' \dfrac{\alpha_i}{\theta_i}}$.

where $\sum{}'$ implies summation taken over those i's for which $\theta_i \neq 0$

Efficiencies of the reduced BN-RC designs have been worked out and provided in the Appendix 2 . As can be seen from the Table 1 of Appendix 2 , the relative efficiency of the designs after missing column(s), increases with the increase in either v or μ and except for $v < 12$ the relative efficiency is greater than 93%. For the cases of $7 \le v \le 10$ the efficiency is more than 91% either for $\mu > 2$ or for $q > 5$.

5.4 Robustness to the Unavailability of Row(s)

5.4.1 One Row Missing

Without loss of any generality let all the observations in the first row of the first block (B_1) of size p x q_1 are missing from a BN-RC (v, b, r, p, $\mathbf{q}'$, μ) design d. Assuming that the first q_1 (= q) treatments appeared in B_1, the information matrices of the component designs of the residual design, d*, are as follows :

$$C_{1d^*} = C_{1d} - C_{1d}(R_{11})$$

$$C_{2d^*} = C_{2d} - \sum_{m=1}^{q} C_{2d}\ (K_{1m}) + \sum_{m=1}^{q} C_{2d}\ (K^*_{1m}) \quad \text{and}$$

$$C_{0d^*} = C_{0d} - C_{0d}(B_1) + C_{0d}(B^*_1) = C_{0d} - C_{1d}(R_{11}).$$

Where the notations carry the meaning as detailed in section 5.2. It is to note that

$\sum_{m=1}^{q} C_{2d^*}(k^*_{1m})$ is the information matrix of a new block design with $(v' = q = b', r' = p-1 = k')$ considering the new columns (of the affected block) as blocks.

Therefore, the co-efficient matrix of the residual design, after the loss of first row housed in the first block of the BN-RC design, d, can be written as

$$\mathbf{C}_{d^*} = \mathbf{C}_{2d^*} = \mathbf{C}_{2d} - \sum_{m=1}^{q} \mathbf{C}_{2d}(K_{1m}) + \sum_{m=1}^{q} \mathbf{C}_{2d}(K^*_{1m})$$

Case 1 : Let the missing row is from a block of size q x q which was an LSD with q treatments. It is well-known that the reduced columns in the affected block forms a symmetric BIBD (q, q-1, q-2). Then the C-matrix of the residual design is $C_{d*} = C_{2d} - \frac{q}{q-1}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $\mathbf{A} = \mathbf{I}_q - \frac{1}{q}\mathbf{J}_{q,q}$. Hence The eigen values (θ_i) with multiplicities (α_i) of C_{d*} are

$$\theta_1 = \frac{\mu v}{q} - \frac{q}{q-1} \quad , \quad \alpha_1 = q - 1$$

$$\theta_2 = \frac{\mu v}{p} \quad , \quad \alpha_2 = v - q$$

$$\theta_3 = 0 \quad , \quad \alpha_3 = 1$$

It is to note that $\theta_1 > 0$ for any q, μ and v. The relative efficiency of d* is

$$E = \frac{(v-1)[(q-1)\mu v - q^2]}{(v-1)(q-1)\mu v - (v-q)q^2}$$

Case 2 : Let the affected block originally is a YSD (q, p, λ_1) from a BN-RC (v, b, r, p, q, μ). Thus originally the columns of the block form a symmetric BIBD (q, p, λ_1)

(a) After the loss a typical row the new columns of the affected block still may exibit a symmetric BIBD (q, p-1, λ) where $\lambda < \lambda_1$. For q = 4t+3, a prime or prime power and p = $\frac{q+1}{2}$ there exist an intial block solution which on developing provides the constituent YSD (q, p, λ_1). It is to note that for q = 2p-1 there is a possibility that after the loss of a typical row the reduced columns within the block will still form a BIBD. For the pair of (p,q) as (4,7), (6,11) etc. there may arise this happening

Under this circumstance C_{d*} takes the form

$$C_{d*} = C_{2d} - \theta \begin{bmatrix} \mathbf{I}_q - {}^{1}\!/_{q}\,\mathbf{J}_{q,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} ,$$

where $\theta = \left(\frac{\lambda_1}{p} - \frac{\lambda}{p-1}\right) q = \frac{(p-\lambda_1)q}{p(p-1)}$, since $\lambda_1 - \lambda = 1$.

The non-zero eigen values of C_d are $\frac{\mu v}{p} - \theta$ and $\frac{\mu v}{p}$ with multiplicities (q-1) and (v-q) respectively. The relative efficiency of d*, $E = \frac{(v-1)(\mu v - p\theta)}{(v-1)\mu v - (v-q)p\theta}$

(b) After the loss of a row the reduced columns may exibit a PBIB design with m-associates classes. Letting its C -matrix as $F^{q \times q}$, then C_{d*} takes the form

$$C_{d*} = C_{2d} - \begin{bmatrix} \lambda_1/q\,(I_q - 1/q\,J_{q,q}) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix},$$

Once the contents of reduced columns are known we can find eigen values of **F** and hence eigen values of C_{d*} .

5.4.2 More than Two Rows Missing

Let the first s_j rows from the j^{th} block B_j of size p x q_j became unavailable, $1 \le j \le b, \sum_{j=1}^{b} s_j = s$. The information matrix of the residual design d^* can be found from

$$C_{d*} = C_{2d} - \sum_{j=1}^{b} \sum_{m=1}^{q_j} C_{2d}(K_{jm}) + \sum_{j=1}^{b} \sum_{m=1}^{q_j} C_{2d}(K^*_{jm}).$$

Here $\sum_{j=1}^{b} \sum_{m=1}^{q_j} C_{2d}(K^*_{jm})$ is the information matrix of the reduced column- component design, the columns within j^{th} block being of size p x (q_j- s_j) , $0 < s_j \le p-2$.

Case 1 : Consider a BN-RC (v, b, rq, q, q, λq), where each block is a LSD of order q. Let one row from each block is missing. Then each of the reduced blocks is a YSD (q, q-1, q-2). And the residual design d^* is a BN-RC (v, b, r (q-1), q-1, q, λ (q-2). Then

$\mathbf{C}_{d*} = \frac{\lambda(q-2)v}{q-1}(\mathbf{I}_v - 1/v\, \mathbf{J}_{v,v})$. Hence the relative efficiency of d^* is $E = \frac{q-2}{q-1}$

Case 2 : Consider a design d : BN-RC (v, b, rp, p, q, $\lambda\lambda_1$), where each block is a YSD (q, p, λ_1). Let after the loss of a row in a block it still exibits a YSD (q, p-1, $\lambda_1 - 1$). If this happens for loss of a row in every block of d, the residual design d^*: BN-RC (v, b, r (p-1), p-1, q, λ (λ_1- 1)) . Then $\mathbf{C}_{d*} = \frac{\lambda(\lambda_1-1)v}{p-1}(\mathbf{I}_v - 1/v\, \mathbf{J}_{v,v})$ The relatively efficiency of d^*, E $= \frac{p\,(\lambda_1-1)}{(p-1)\lambda_1}$.

5.5 Robustness to the Unavailability of Column(s)

5.5.1 One Column missing

Without loss of any generality we assume the following :

i) the affected column is the first column K_{11} of the first block B_1 having p rows and q coumns,

ii) 1, 2, . . ., s distinct symbols/treatments are appearing in the affected block and the i^{th} symbol appearing q_i times in each of the p rows of B_1,

iii) the symbols in the first column are distinct and they are first p symbols, namely, 1, 2, . . . , p , $p \le s$ and they are occuring in the 1^{st}, 2^{nd},. . . , p^{th} row respectively.

The assumptions suggests that the columns are binary whereas rows may be non-binary and blocks are non-binary. Then the C- matrix of the residual design, d^*, after the loss of a column is $\mathbf{C}_{d*} = \mathbf{C}_{1d*} + \mathbf{C}_{2d*} - \mathbf{C}_{0d*}$, where

$$\mathbf{C}_{1d*} = \mathbf{C}_{1d} - \sum_{l=1}^{p} \mathbf{C}_{1d}\,(R_{1l}) + \sum_{l=1}^{p} \mathbf{C}_{1d}\,(R_{1l}{}^*)$$

$$\mathbf{C}_{2d*} = \mathbf{C}_{2d} - \mathbf{C}_{2d}\,(K_{11}) \quad \text{and}$$

$$\mathbf{C}_{0d*} = \mathbf{C}_{0d} - \mathbf{C}_{0d}(\mathrm{B}_1) + \mathbf{C}_{0d}(\mathrm{B}_1^*)$$

R^*_{1l} is of size (q-1,) for all l = 1,2, . . ., p and B_1^* is of size p (q-1). It is to note that

$$\mathbf{C}_{\mathbf{od}}(\mathrm{B}_1) = \sum_{l=1}^{p} \mathbf{C}_{1d}(R_{1l}).$$

Hence $\mathbf{C}_{d*} = \mathbf{C}_{2d} - \mathbf{C}_{2d}(K_{11}) + \sum_{l=1}^{p} \mathbf{C}_{1d}(R^*_{1l}) - \mathbf{C}_{0d}(B_1^*)$

Now $\mathbf{C}_{2d}(K_{11}) = \begin{bmatrix} \mathbf{I}_p - 1/p\,\mathbf{J}_{p,p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

The replication vector in block B_1 can be partitioned as $\mathbf{r}_{(1)} = (\mathbf{q}_{(1)}, \mathbf{q}_{(2)})'$, where $\mathbf{q}_{(1)}' = (q_1, q_2, \ldots, q_p)$ and $\mathbf{q}_{(2)}' = (q_{p+1}, q_{p+2}, \ldots, q_s)$. Then the replication vector in B_1^* is $\mathbf{r}^*_{(1)} = \mathbf{r}_{(1)} - (\mathbf{1}_p, \mathbf{0})'$.

$$\sum_{l=1}^{p} \mathbf{C}_{1d}(R^*_{1l}) = \mathbf{R}^*_{(1)} - \frac{1}{q-1}\begin{bmatrix} \mathbf{q}_{(1)}\mathbf{1}_p' - \mathbf{I}_p \\ \mathbf{q}_{(2)}\mathbf{1}_p' \\ \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{q}_{(1)}\mathbf{1}_p' - \mathbf{I}_p \\ \mathbf{q}_{(2)}\mathbf{1}_p' \\ \mathbf{0} \end{bmatrix}' \quad \text{and}$$

$$\mathbf{C}_{0d}(B_1^*) = \mathbf{R}^*_{(1)} - \frac{1}{p\,(q-1)}\begin{bmatrix} p\,\mathbf{q}_{(1)} - \mathbf{1}_p \\ p\,\mathbf{q}_{(2)} \\ \mathbf{0} \end{bmatrix}\begin{bmatrix} p\,\mathbf{q}_{(1)} - \mathbf{1}_p \\ p\,\mathbf{q}_{(2)} \\ \mathbf{0} \end{bmatrix}'$$

where $\mathbf{R}^*_{(1)}$ is a diagonal matrix, elements in the diagonal being as that in $\mathbf{r}_{(1)}$.

Then on simplification $\mathbf{C}^*_{\mathbf{d}}$ are reduces to

$$\mathbf{C}_{d*} = \mathbf{C}_{2d} - \frac{q}{q-1}\begin{bmatrix} \mathbf{I}_p - 1/p\,\mathbf{J}_{p,p} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The non-zero eigen values of $\mathbf{C}_{d*}$ are $\frac{\mu v}{p}$ and $\frac{\mu v}{p} - \frac{q}{q-1}$ with multiplicities as v-p and p-1 respectively. The relative efficiency of d^* is

$$E = \frac{(v-1)\left\{(q-1)\mu v - pq\right\}}{(v-p)\left\{(q-1)\mu v - pq\right\} + (p-1)(q-1)\mu v}$$

5.5.2 More Than One Column Missing

Let s columns are missing. These s columns may consist of s_j columns (the first s_j column without any loss of genelity) from j^{th} block of size px q_j such that $\sum_{j=1}^{p} s_j = s$. Then the information matrix of

$$C_{d*} = C_{1d*} + C_{2d*} - C_{0d*}$$

$$= C_{2d} - \sum_{j=1}^{b} \sum_{m=1}^{s_j} C_{2d}(K_{jm}) + \sum_{j=1}^{b} \sum_{l=1}^{p} C_{1d}(R^*_{jl}) - \sum_{j=1}^{b} C_{0d}(B^*_j)$$

In the following the cases for s = 2 is discussed.

5.5.2.1 Two Columns Missing

Case 1: Consider the case of two columns, one each from two blocks, having disjoint set of treatments are missing. Let (1,2, . . . , p) be the set of treatments appeared in K_{11}, the first column in B_1 and (p + 1, p + 2, . . . , 2p) be the set of treatments in K_{21}. Then the first column in B_2 .

$$C_{d*} = C_{2d} - \begin{bmatrix} \frac{q_1}{(q_1-1)}\mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{q_2}{(q_2-1)}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{A} = \mathbf{I}_p - 1/p\, \mathbf{J}_{p,p}$

Then the eigen values (θ_i) with corresponding multiplicities (α_i) of $\mathbf{C}^*_d$ are

$$\theta_0 = 0 \quad , \quad \alpha_0 = 1$$

$$\theta_1 = \frac{\mu v}{p} \quad , \quad \alpha_1 = v - 2p + 1$$

$\theta_2 = \frac{\mu v}{p} - \frac{q_1}{q_1 - 1}$, $\alpha_2 = p - 1$

$\theta_3 = \frac{\mu v}{p} - \frac{q_2}{q_2 - 1}$, $\alpha_3 = p - 1$

The relative efficiency of d^*,

$E = \frac{(v-1)p}{\mu v \sum_i{}' \frac{\alpha_i}{\theta_i}}$, where $\sum{}'$ implies summations taken over those i's for which $\theta_i \neq 0$.

Case 2 : Consider the case of two columns having disjoint set of treatments, from the same block are missing. Assume that the columns are the first and the second column in the block and they contain two sets of disjoint treatments (1,2, . . . , p) and (p + 1, p + 2, . . . , 2p) respectively, $s \geq 2p$. Then it can be seen that

$$\mathbf{C}_{d*} = \mathbf{C}_{2d} - \frac{1}{q-2}\begin{bmatrix} \mathbf{F} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\mathbf{F} = \begin{pmatrix} q-1 & 1 \\ 1 & q-1 \end{pmatrix} \otimes (\mathbf{I}_p - 1/p\,\mathbf{J}_{p,p})$

The eigen values (θ_i) with corresponding multiplicities (α_i) of $\mathbf{C}_\mathbf{d}^*$ are

$\theta_0 = 0$, $\alpha_0 = 1$

$\theta_1 = \frac{\mu v}{p}$, $\alpha_1 = v - 2p + 1$

$\theta_2 = \frac{\mu v}{p} - 1$, $\alpha_2 = p - 1$

$\theta_3 = \frac{\mu v}{p} - \frac{q}{q-2}$, $\alpha_3 = p - 1$

Using which the relative efficiency of d^* can be worked out.

Case 3 : Consider the case of two missing columns from two different blocks, each of size p x q , wherein m treatments are common /matching and the rest of the n = p-m treatments are distinct in both the columns.

Without any loss of generality let (1, 2, . . . , n, n+1 . . . . n+m=p) and (n+1,n+2,. . . ,n+m=p,p+1,. . . ,p+n) be the set of treatments in the two columns, the columns being from the first and the second block respectively. Then

$$\mathbf{C}_{d*} = \mathbf{C}_{2d} - \frac{q}{q-1}\begin{bmatrix} \mathbf{A}_n & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}' & 2\mathbf{A}_m & \mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{A}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{A}_t = \mathbf{I}_t - 1/p\,\mathbf{J}_{t,t}$

hence the eigen values (θ_i) with corresponding multiplicities (α_i) of C_d^* are

$$\theta_0 = 0 \quad , \quad \alpha_0 = 1$$

$$\theta_1 = \frac{\mu v}{p} \quad , \quad \alpha_1 = v - (p + n)$$

$$\theta_2 = \frac{\mu v}{p} - \frac{q}{q-1} \quad , \quad \alpha_2 = 2(n - 1)$$

$$\theta_3 = \frac{\mu v}{p} - \frac{2q}{q-1} \quad , \quad \alpha_3 = m - 1$$

$$\theta_4 = \frac{\mu v}{p} - \frac{mq}{(q-1)p} \quad , \quad \alpha_4 = 1$$

$$\theta_5 = \frac{\mu v}{p} - \frac{(p+n)q}{(q-1)p} \quad , \quad \alpha_5 = 1$$

from which the relative efficiency of d^* can be worked out.

Case 4 : Consider the case when two columns from a block with m common (matched) treatments are missing. Let the first and the second column contain the set of treatments $(1, 2, \ldots, n, n+1, \ldots, n+m = p)$ and $(n+1, n+2, \ldots, n+m = p, P+1, \ldots, p+n)$ respectively. Then the **C**-matrix of the residual design d^* can be written as

$$\mathbf{C}_{d*} = \mathbf{C}_{2d} - \begin{bmatrix} \mathbf{A}_n & -\frac{1}{p}\mathbf{J}_{n,m} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{p}\mathbf{J}_{m,n} & 2\mathbf{A}_m & -\frac{1}{p}\mathbf{J}_{m,n} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{p}\mathbf{J}_{n,m} & \mathbf{A}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$+ \frac{1}{p(q-2)} \begin{bmatrix} \mathbf{J}_{n,n} & 2\mathbf{J}_{n,m} & \mathbf{J}_{n,n} & \mathbf{0} \\ 2\mathbf{J}_{m,n} & 4\mathbf{J}_{m,m} & 2\mathbf{J}_{m,n} & \mathbf{0} \\ \mathbf{J}_{n,n} & 2\mathbf{J}_{n,m} & \mathbf{J}_{n,n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} - \frac{1}{q-2} \begin{bmatrix} \mathbf{MM}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

Where $\mathbf{A}_t = \mathbf{I}_t - 1/p\, \mathbf{J}_{t,t}$ and **M** is the incidence matrix of $p+n$ treatments (that occurred in two columns) in p rows of size 2 occured in two affected columns.

SUMMARY

The present thesis concerns to the studies on Balanced Nested Row-Column (BN-RC) designs which have been found to be universally optimal. These designs are useful for the case where the experimental material is subjected to hetrogeneity due to three factors (or factor groups), the second and the third being fully crossed with each other and nested within the first. The available methods of construction are very limited and mostly make use of the properties of Latin Square, Youden Square and similar others for taking care of the nested factors, thus requiring a large amount of experimental material. The effort has been to get these designs for smaller number of experimental units. Taking note of the fact that larger block sizes could result in substantial row x column interaction, while it could be negligible in case of smaller block sizes, effort was more directed towards obtaining these designs with smaller block sizes. The thesis thus attempts to give the methods of construction of Nested Row-Column designs, balanced for pair-wise treatment comparisons using blocks of equal and unequal sizes as also a mixture of two blocking systems. The robustness of these designs against missing row(s) / column(s) / block(s) was also studied. The thesis is organized into 5 Chapters.

An introduction to block designs, some aspects of their balancing, optimality and robustness is provided in Chapter 1. A brief review of the available methods for constructing BIBRC and BN-RC designs are given in this Chapter. It concludes with the motivation, scope and salient features of the present investigations.

Chapter 2 deals with the construction of BN-RC designs with equal block, row and column sizes ; emphasis being on minimising block sizes and demand on experimental material. Some general methods of construction of BN-RC designs with the help of initial block solutions of BIBD's and the method of differences along with a method of their construction using an available BIBD and a BN-RC design are presented in this

Chapter. It also includes construction of BN-RC designs with p=2 and q=4 for all v in the series 8t+i for i=0, 1, . . . , 7 and some BN-RC designs with p=2 and q=5, 6 and 7 for v = 6, 7, . . . , 13.

Construction of BN-RC designs with unequal block sizes has been dealt with in Chapter 3. Using the concept of balancing in block designs we have presented few methods for constructing Partially Balanced Nested Row-Column (PBN-RC) designs, Variance Balanced Nested Row-Column (VBN-RC) designs and BN-RC designs with unequal (now and hence) block sizes. Some methods of construction of BN-RC designs with p= 2, 3 and a combination of column sizes q in the range from 3 to 7 are also presented in this Chapter.

A general discussion on design with unequal sizes is presented in Chapter 4. This process of mixing of unequisized blocks is extended to evolve a design by adopting two systems of blocking-- (1) blocks with nested rows and columns and (2) simple blocks. Construction of balanced designs with two blocking system from initial block solutions of BIBD, MOLS and by the method of differences are presented in this Chapter.

Robustness of BN-RC designs against non-availability of all observations in block(s), row(s) and column(s) is studied in Chapter 5. It is observed that the problem of finding the eigen values of the information matrix of the residual design is specific to the design in use. Some general expressions of the relative efficiency are provided. The loss of block(s) in BN-RC designs, for v in the series 8t+i, i being in the range from 0 to 7, presented in Chapter 2 and Chapter 3, was studied and the relative efficiencies are presented in the form of a table, Table 1 of Appendix 2. It is observed that the loss of efficiency, in the residual design, after the loss of a block in these designs, is marginal for v more than 8. For a missing row, in BN-RC designs where each block is an LSD, the loss in efficiency is marginal for v more than 4 and listed in Table 2 of

Some well established results of matrix theory to find eigen values of residual designs for the investigation of robustness of BN-RC designs.

Result (R1): Let **A** and **B** are symmetric matrices. A necessary and sufficient condition for simultaneous diagonalisation of **A** and **B** by pre- and post- multiplication by $\mathbf{C}'$ and **C**, where **C** is an orthogonal matrix, is that **A** and **B** commute, i.e. $\mathbf{AB} = \mathbf{BA}$.

Result (R2) : Let **A** and **B** are two symmetric matrices of order n with eigen values λ_i and μ_i, $1 \le i \le n$, respectively and they commute (hence having the same set of eigen vectors). Then $A \pm B$ has the eigen values as $\lambda_i \pm \mu_i$, $1 \le i \le n$.

Result (R3) : Let A and D are symmetric matrices.

$$\text{Then det} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{A}).\det(\mathbf{D} - \mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B}), \text{ if } \mathbf{A} \text{ is non-singular}$$

$$= \det(\mathbf{D}).\det(\mathbf{A} - \mathbf{B}\,\mathbf{D}^{-1}\,\mathbf{A}), \text{if } \mathbf{D} \text{ is non-singular.}$$

Result (R4) : Given $\mathbf{A} = (a - b)\,\mathbf{I}_n + b\,\mathbf{J}_{n,n}$ where a and b are scalars, $\mathbf{I}_n$ an identity matrix order n and $\mathbf{J}_{n,n}$ is a matrix of order n x n with elements as unity. Then the eigen values of **A** are

$\theta_i = a - b$ with multiplicity n-1 and

$\theta_2 = a + (n - 1)\,b$ with multiplicity 1.

****A proof on the commutativity of two matrices occurring in C_{d*}, the information matrix of the residual design d*.**

Let $\mathbf{A} = \theta\,(\mathbf{I}_v - \frac{1}{v}\mathbf{J}_{v,v})$ where θ is a scalar quantity $\mathbf{I}_v$ an identity matrix of order v, $\mathbf{J}_{r,s}$ a matrix of order r x s with elements as unity. Then **A** can be presented in the form of a partion matrix

$$A = \theta \begin{bmatrix} A_{11} & A_{12} \\ A'_{12} & A_{22} \end{bmatrix} \quad \text{where}$$

$A_{11} = I_n - 1/v\, J_{n,n}$,

$A_{12} = -\,1/v\, J_{n,v-n}$,

$A_{22} = I_{v-n} - 1/v\, J_{v-n,v-n}$

Consider another matrix **B** is the following form $\mathbf{B} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where **F** is a **C**-matrix of order n. Then $\mathbf{F}.\mathbf{1}'_n = \mathbf{0}^{nx1}$ and $1'_n\, \mathbf{F} = \mathbf{0}'$.

Now **AB** $= \theta \begin{bmatrix} A_{11}\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \theta \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ since $A'_{12}\mathbf{F} = \mathbf{0}$

Similarly **BA** $= \theta \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{A}\,\mathbf{B}$.

i.e. **A** and **B** are commutative.

Then by result result R2 , A± B has the eigen values as $\lambda_i \pm \mu_i$, i = 1, 2, . . . , v, where corresponding to the eigen vector ξ_i's , λ_i's are the eigen values of **A** and μ_i's are that of **B**.

Table : 1: Efficiencies of BNRC designs with equal or unequal block sizes constructed in sections 2.5, 2.6 and 3.7after missing of disjoint blocks (s_1 blocks of size 2x3 , s_2 blocks of size 2x4 , s_3 blocks of size 2x5 , s_4 blocks of size 2x6).

v	μ	s_1	s_2	s_3	s_4	EX 100
4	2	0	1	0	0	64.28
5	2	0	1	0	0	77.41
	1	1	0	0	0	57.14
		0	1	0	0	42.86
6	4	0	1	0	0	92.90
	2	1	0	0	0	88.23
		0	1	0	0	84.74
7	4	0	1	0	0	94.93
	1	1	0	0	0	80.00
8	2	0	1	0	0	91.86
		1	0	0	0	93.81
		1	1	0	0	70.95
9	1	0	1	0	0	85.36
10	4	0	1	0	0	97.65
		0	2	0	0	95.41
	2	0	0	0	1	92.70
		1	0	0	0	96.22
11	4	0	1	0	0	98.08
		0	2	0	0	96.24
	1	0	0	1	0	88.75
12	2	0	1	0	0	96.64
		1	1	0	0	94.28
		0	2	0	0	93.51
13	2	0	1	0	0	97.18
		0	2	0	0	94.51
	1	1	0	0	0	95.24
		0	1	0	0	93.69
		1	1	0	0	89.50
		2	0	0	0	90.90
		0	2	0	0	88.13
		0	0	0	1	90.83
14	4	0	1	0	0	98.85
		0	2	0	0	97.73
	2	1	0	0	0	98.18
		0	1	0	0	97.59
		1	1	0	0	97.25
		0	2	0	0	95.30
		2	0	0	0	96.44

v	μ	s_1	s_2	s_3	s_4	EX 100
15	4	0	1	0	0	99.00
		0	2	0	0	98.03
	1	1	0	0	0	96.55
		0	1	0	0	95.42
		1	1	0	0	92.28
		0	2	0	0	91.25
		2	0	0	0	93.33
		1	2	0	0	88.37
		2	1	0	0	89.33
16	2	1	0	0	0	98.64
		2	0	0	0	97.31
		3	0	0	0	96.02
		0	1	0	0	98.19
		0	2	0	0	96.45
		1	1	0	0	96.88
		1	2	0	0	95.18
		2	1	0	0	95.60
17	1	0	1	0	0	96.53
		0	2	0	0	93.31
		0	3	0	0	90.28
18	4	0	1	0	0	99.32
		0	3	0	0	98.00
	2	0	0	0	1	97.90
		0	0	0	2	95.89
		1	2	0	0	96.22
		0	1	0	1	95.59
		4	0	0	0	95.89
19	4	0	1	0	0	99.39
		0	2	0	0	98.80
		0	3	0	0	98.20
	1	0	0	1	0	96.63
		0	0	2	0	93.48
		0	0	3	0	90.54
		0	1	1	0	94.09
		0	1	2	0	91.10
		0	2	1	0	91.69
20	2	0	1	0	0	98.87
		0	2	0	0	97.77
		1	1	0	0	98.05
		2	0	0	0	98.32
		2	2	0	0	96.17

v	μ	s_1	s_2	s_3	s_4	EX 100
21	2	0	1	0	0	98.98
		0	4	0	0	96.06
	1	1	0	0	0	98.36
		3	0	0	0	95.24
		4	0	0	0	93.75
		0	1	0	0	97.82
		1	1	0	0	96.25
		0	2	0	0	95.73
		2	1	0	0	94.73
		1	2	0	0	94.22
		0	4	0	0	91.81
		0	0	0	1	96.76
		0	1	0	1	94.72
		0	0	0	2	93.73
		0	2	0	1	92.76
		0	1	0	2	91.81
22	4	0	1	0	0	99.55
		0	4	0	0	98.24
	2	1	0	0	0	99.30
		5	0	0	0	96.63
		0	1	0	0	99.08
		0	2	0	0	98.17
		1	2	0	0	97.50
		0	4	0	0	96.41
		1	3	0	0	96.63
		2	3	0	0	95.98
23	4	1	0	0	0	99.69
		5	0	0	0	98.49
		0	1	0	0	99.59
		0	2	0	0	99.18
		0	4	0	0	98.39
		1	3	0	0	98.49
	1	1	0	0	0	98.65
		5	0	0	0	93.61
		0	1	0	0	98.20
		0	2	0	0	96.48
		0	4	0	0	93.20
		1	3	0	0	93.60
24	2	1	0	0	0	99.42
		6	0	0	0	96.64
		0	1	0	0	99.23
		1	4	0	0	96.45
25	1	0	1	0	0	98.50
		0	2	0	0	97.05
		0	3	0	0	95.64
		0	4	0	0	94.27
		0	5	0	0	92.94

Table: 2: Efficiencies ofBN-RC designs for a missing row where blocks are LSD of order q, and they are formed from blocks of a BIBD (v, k=q, λ)

v	q	λ	E x 100
4	3	2	86.76
5	3	3	94.74
	4	3	92.05
6	3	2	94.59
	4	6	97.74
	5	4	95.79
7	3	1	91.67
	4	2	95.00
	6	5	96.94
8	4	3	97.54
9	3	1	95.24
	4	3	98.09
	5	4	98.59
	6	5	98.31
10	3	2	98.23
	4	2	97.67
	5	4	98.59
	6	5	98.65
11	5	2	97.65
	6	3	98.15
12	3	2	98.80
	4	3	98.96
	6	5	99.08
13	4	1	97.22
	3	1	97.87
15	3	1	98.44
	7	3	98.87

REFERENCES

Agrawal, H.L. and Prasad, J. (1982). Some methods of construction of balanced incomplete block designs with nested rows and columns, *Biometrika*, **69**, 481 - 483.

Agrawal, H.L. and Prasad, J. (1983). On construction of balanced incomplete bolck designs with nested rows and columns. *Sankhya*, B **45**, 345 - 350.

Agrawal, N. (1995). Studies on block designs with nested rows and colums for test treatment control comparisons. *Unpublished Ph. D. thesis*, I.A.R.I., New Delhi.

Bagchi, S., Mukhopadhyay, A.C. and Sinha, B.K. (1990). A search for optimal nested row-column designs. *Sankhya, B* **52**, 93-104

Bagchi, S. and Shah, K.R. (1989). On the optimality of a class of row-column designs. *J. Statist. Plann. Infer.*, **23**, 397-402.

Baksalary, J.K. and Tabis, Z. (1987). Conditions of robustness of block designs against the unavailability of data. *J. Statist. Plann. Infer.*, **16**, 49-54.

Bhaumik, D.K. and Whittinghill III, D.C. (1991). Optimality and robustness to the unavailability of blocks in block designs. *J. Roy. Statist. Soc., **B*** **53**, 399 - 407.

Calinski, T. (1971). On some desirable patterns in block designs. *Biometrics*, **27**, 275 - 292.

Chandak, M.L. (1980). On the theory of resistant block designs. *Cal. Statist. Assoc. Bull.*, **29**, 27 - 34.

Chang, J.Y. and Notz, W.I. (1994). Some optimal nested row-colum designs. *Statist. Sinic.*, **4**, 249-263.

Chang, J.Y. and Notz, W.I. (1990). A method for constructing universaly optimal block designs with nested rows and columns. *Ulilit. Math.*, **38**, 263-276.

Cheng, C.S. (1981). Optimality and construction of Pseudo-Youden designs. *Ann. Statist.*, **9**, 200-205.

Cheng, C.S. (1986). A method for constructing balanced incomplete block designs with nested rows and columns. *Biometrika*, **73**, 695-700.

Cheng, C.S. (1978). optimality of certain asymmetrical experimental designs. *Ann. Statist.*, **6**, 1239 - 1261.

Chernoff, R. (1953). Locally optimum designs for estimating parameters. *Ann. Math. Statist.*, , **24**, 586 - 602.

Clatworthy, W.H. (1973). Tables of two associate-class partially balanced designs. Appl. Math Ser. 63, National Bureau of Standards, Washington.

Conniffe, D. and Stone, J. (1974). The efficiency factor of a class of incomplete block designs. *Biometrika,* **61**, 633 - 636.

Constantine, G.M. (1981). Some E-optimal block designs. *Ann. Statist.*, **9**, 886 - 892.

Das, M.N. and Giri, N.C. (1988). Design and Analysis of Experiments. John Wiley & Sons, New York.

Das, A. and Kageyama S. (1992). Robustness of BIB and extended BIB designs against the unavailability of any number of observations in a block. *Computational Statistics and Data Analysis*, **14**, 343 - 358.

Davis, P.J. (1979). Circulant matrices. J. Wiley & sons, New York.

Dey, A. (1986). Theory of block Designs. Wiley Eastern Ltd., New Delhi.

Dey, A. and Dhall, S.P. (1988). Robustness of augmented BIB designs. *Sankhya, B*. **50,** 376 - 381.

Dey, A. and Das, A. (1989). On some E-optimal block designs. *Metrika*, **36**, 269 - 278.

Dey, A. (1993). Robustness of block designs against missing data. *statist. Sinic,*, **3,** 219 - 231.

Dodge, Y. (1985) Analysis of Experiments with Missing Data. Wiley Eastern Ltd., London.

Duan, X. and Kageyama, S. (1995). Robustness of augmented BIB designs against unavailability of some observations. *Sankhya.*, **B 55,** 405 - 419.

Eeceleston, J.A. and Hedayat, A.S. (1974), On the theory of connected design : characterisation and optimality. *Ann. Statist.*, **2**, 1238 - 1255.

Eccleston, J.A. and John, J.A. (1988). Adjusted Orthogonal row-column designs. *Austral. J. Statist.,* **30,** 78 - 84.

Ehrenfeld, S. (1955). On the efficiency of experimental designs. *Ann. Math. Statist.*, **26**, 247 -255.

Elfving, G. (1952). Optimum allocation in linear regression theory. *Ann. Math. Statist.*,

23 , 255 - 262.

Ghosh, S. (1981). Robustness of three dimensional designs-1. Sankhya, **B 43**, 222-227.

Ghosh, S. (1982a). Robustness of designs against unavailability of data. *Sankhya, B* **44**, 50-62.

Ghosh, S. (1982b). Robustness of BIB designs against unavailability of data. *J. Statist. Plann. Infer.* , **6**, 29-32.

Ghosh, S. (1982c). Information in an observation in robust designs. *Comm. Statist. Theor. Meth., A* **11**, 355 - 364.

Ghosh, S., Rao, S.B. and Singhi, N.M. (1983). On a robust property of PBIB designs. *J. Statist.Plann. Inf.* , **8**, 355 - 364.

Gupta and Jones, B (1983). Equireplicate balanced block designs with unequal block sizes. *Biometrika* , **70**, 433 - 440.

Gupta, S. and Jones, B. (1983). Equireplicate balanced block designs with unequal block sizes. *Biometrika,* **70**, 433 - 440.

Gupta, V.K. and Singh, R.(1989). On E - optimal block designs. *Biometrika* , **76**, 184 - 188.

Gupta, V.K., Das, A. and Dey, A. (1991). Universal optimality of block designs with unequal block designs with unequal block sizes. *Statist . and Prob . Letters* , **11**, 177 - 180.

Gupta , S. (1992). Some optimal nested row - column designs. *Cal. Statist. Assoc. Bull.* , **42** , 261 - 265.

Gupta, V.K. (1993). Optimal nested block designs. *J. Ind. Soc. Ag. Statist.,* **45**, 187 - 194.

Gupta, V.K. and Srivastava, R. (1992). Investigations on robustness of block designs against missing observations. *Sankhya, B* **54**, 100 - 105.

Gupta V.K. (1995). Universally optimal block designs under a heteroscedastic model. *Sankhya , B* **57** , 420 - 427.

Hall, M. Jr. (1986). Combinatorial Theory. John Wiley & Sons, New York.

Hedayat, A. and John, P.W.M. (1974). Resistant and susceptible BIB designs. *Ann. Statist.*, **2**, 148 - 158.

Hedayat, A. and Federer, W.T. (1974). Pairwise and variance balanced incomplete block designs.

Ann. Inst. Statis. Math., **26**, 331 - 338.

Ipinyomi, R.A. and John, J.A. (1985). Nested generalised Cyclic row-column designs, *Biometrika*, **72**, 403 - 409.

Jacroux, M. (1986). Some E-optimal row-column designs. *Sankhya B* **48,** 31- 39.

Jacroux, M. (1986). Some E-optimal row -column designs *Sankhya.*, **B 48**, 31 - 39.

Jarrett, R.G. (1977). Bounds for the efficiency factor of block designs. *Biometrika*, **64,** 67 - 72.

Jarrett, R.G. and Hall, W.B.(1982). Some design considerations for varietal trials,. *Utilit. Math.*, .21, 153 - 168.

Jones, B. Sinha, K. and Kageyama, S. (1987). Further equireplicate variance balanced designs with unequal blocki sizes. *Utilit. Math.*, **32**, 5 - 10.

John, P.W.M. (1976). Robustness of incomplete block designs. *Ann. Statist.*, **4,** 960 - 962.

John, P.W.M. (1964) Balanced designs with unequal number of replications. *Ann. Math. Statist.*, **35,** 897 - 899.

John, J.A. (1987). Cyclic Designs. Monographs on Statistcs and Applied Probability. Chapman and Hall, London.

Kageyama, S. (1976). Construction of balanced block designs. *Utilit. Math.*, 9, 209 -229.

Kageyama, S. (1980). Robutness of incomplete block designs. *Ann. Inst. Statist. Math., A* **32,** 255 - 261.

Kageyama, S. and Tsuji, T. (1980). Characterization of equireplicated variance balanced block designs. *Ann. Inst. Statist. Math.*, **32,** 263-273.

Kageyama, S. (1982). The existence of locally resistant BIB designs of degree one. Tech. rept. No. 18/82, Stat. Math. Div., I.S.I., Calcutta.

Kiefer, J. (1958). On the non-randomised optimality and randomised non-optimality of symmetrical designs. *Ann. Math. Statist.*, **29**, 675 - 699.

Kleczkowski, A. (1960). Interpreting relationships between the concentrations of plant viruses and number of local lesions. *J. Gen. Microbiol.*, **4,** 532 -69.

Kulshreshtha, A.C., Dey, A. and Saha, G.M. (1972), Balanced designs with unequal replications and unequal block sizes. *Ann. Math. Statist.*, **43**, 1342 - 1345.

Lee, K.Y. and Jacroux, M. (1987a). Some sufficient condition for the E - and MV optimality of block designs having unequal block sizes. *Ann. Inst. Statist. Math.*, **39**, 385 - 397.

Lee, K.Y. and Jacroux, M. (1987b). On the construction of E- and MV- optimal group divisible designs with unequal block sizes. *J. Statist. Plann. Inf.*, **16**, 193 - 201.

Lee, K.Y. and Jacroux, M. (1987c). On the E- optimality of block designs having unequal block sizes. *Sankhya*, ***B*** **49**, 126 - 136.

Keifer, J. (1959). Optimum experimental designs. *J. Roy. Statist. Soc.*, ***B*** **21**, 272 - 313.

Kiefer, J. (1971). The role of symmetry and approximation in exact design optimality. In statistical decision theory and Related topics, Academic Press, New York, 109 - 118.

Kiefer. J. (1974). General equivalence theory for optimum designs (approximate theory). *Ann. Statist.*, **2**, 849 - 879.

Keifer, J. (1975). Construction and optimality of generalised Youden designs. A survey of Statistical Designs and Linear Models. *Ed. J.N. Srivasta, North Holland:Amsterdam*, 333 - 353.

Magda, C.G. (1979). On E-optimal block designs and schur-optimality. *Unpublished Ph.D. Thesis, Univ. of Illinois at chicago circle.*

Majumdar, D. (1986). Optimal incomplete block designs for comparing compring treatments with a control. *Ann. Statist.*, **11**, 258 - 266.

Mood, A.M. (1946). On Hotelling's weighing problem. *Ann. Math. Statist.*, **17**, 432 - 436.

Most, B.M. (1975). Resistance of balanced incomplete block desigtns. *Ann. Statist.*, **3**, 1149 - 1162.

Notz, W.I., Whittinghill, D.C. and Zhu, Y. (1994). Robustness to the unavailability of data in block designs. *Metrika.*, **41**, 263 -275.

Pal, S. (1980). A note on partially efficencey balanced designs. *Cal. Statist. Assoc. Bull.*, **29**, 185 - 190.

Pal, S. and Pal, S. (1988). Non - proper variance balanced designs and optimality. *Comm. Statist. Theo. Meth.*, **17**, 1697 - 1716.

Patterson, H.D. and Williams, E.R. (1976). A new clan of resolvable incomplete block designs. *Biometrika*, **63,** 83 - 92.

Pattersons, H.D., Williams, E.R. and Hunter, E.A. (1978). Block designs for varietal trials. *J. Agric. Sci. Comb.*, 90, 395 - 400.

Pearce, S.C. (1964). Expermenting with blocks of natural sizes. *Biometrics*, 20, 699 - 706.

Pearce, S.C. (1976). Concurrence and quasi-replication : An alternative approach to precision in designed experiments. *Biom. J.*, 18, 105-116.

Preece, D.A. (1967). Nested balanced incomplete block designs. *Biometrika,* **54**, 479 - 486.

Preece, D.A., Pearce, S.C. and Kerr, J.R. (1973). Orthogonal designs for three dimensional experiments. *Biometrika*, 60, 349-359.

Puri, P.D. and Nigam, A.K. (1977). Partially efficiency balanced designs, *Commun. Statist.*, A 6, 1171 - 1179.

Raghavarao, D. (1971). Construction and combinatorial Problems in Design of Experiments. John Wiley & Sons, New York.

Raghavarao, D. and Federer, W.T. (1975). On connectedness in two - way elimination of heterogeneity designs. *Ann. Statist.*, 3, 730 - 735.

Rao, M.V.R. Prasada and Das, A. (1988). A general method of obtaining eigen values of C-matrix of block designs. *Metron,* 46, 81 -98.

Saha, G.M. and Mitra, R.K. (1992). Some constructions of balanced incomplete block row - column designs, *ARS Combinatoria*, 34, 321 - 325.

Shah, K.R. (1960). Optimality criteria for incomplete block designs. *Ann. Math. Statist.*, 31, 791 - 794.

Singh, M. and Dey, A. (1979). Block designs with nested rows and columns. *Biometrika*, 66, 321 - 326.

Sinha, B.K. (1980). Optimal block Designs. *Unpublished Seminar Notes, Stat. Math. Div.*, ISI, Calcutta (Revised 1982).

Sinha, K. and Jones, B (1988). Further Equireplicate balanced designs with unequal block sizes . Statist and Prob. Letters , **6**, 229 -230.

Smith, K. (1918). On the standard deviations of adjusted and interpolated values of an observed polynomial function and its constants and the guidance they give towards a proper choice of the distribution of observations. *Biometrika* , **12**, 1 - 85.

Sreenath, P.R. (1989). Construction of some balnced incompleted block designs with nested rows and columns. *Biometrika* , **76** , 399 - 402.

Sreenath, P.R. (1991). Construction of balanced incomplete block designs with nested rows and columns through the method of differences. *Sankhya, B* **53,** 352 - 358.

Sreenath, P.R. (1996). Construction of block designs with nested rows and columns. Technical Report, I.A.S.R.I., New Delhi.

Srivastava, J.N. (1978) Statistical design of agricultural experiments. *J. Ind. Soc. Agri. Stat.,* **30,** 1- 10.

Srivastava, R. Gupta, V.K. and Dey, A. (1990). Robustness of some designs against missing observations. *Comm. Statist Theor. Method.* , **19** , 121 -126.

Tocher, K.D. (1952). The design and analysis of block experiments (with discussion). *J. R. Statist. Soc.* **B 14,** 45 - 100.

Tyagi B.N. (1979). On a class of variance balanced block designs. *J. Statist. Plann. Inf.* , **3** , 333 - 336.

Uddin, N. (1992). Construction for some balanced incomplete block designs with nested rows and columns. *J. Statist. Plann. Infer* ., **31**. 253 - 261.

Uddin, N. and Morgan, J.P. (1990). Some constructions for balanced incomplete block designs with nested rows and columns. *Biometrika* , **77**, 193 - 202.

Wald, A. (1943). On the efficient design of statistical investigations. *Ann. Math. Statist.* , **14** , 134 - 140.

Whittinghill, D.C. (1989). Balanced block designs robust against the loss of single observation *J. Statist. Plann. Infer.* , **22**, 71 - 80.

Williams, E.R. (1975). Efficiency balanced designs. *Biometrika*, **60**, 686-689.

Yates, F. (1936). Incomplete randomised blocks. *Ann. Eugen.*, **7** , 121-140.

Yates, F. (1940). Lattice squares. *J. Agric. Sci.*, **30**, 672 - 687.