Quaternionic Formalism of Curvature Space-Time and Einstein Field Equation

Thesis

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Pantnagar ***(Manisha Goutam)***
August, 2019 *Authoress*

C E R T I F I C A T E

This is to certify that the thesis entitled **"Quaternionic Formalism of Curvature Space-Time and Einstein Field Equation"** submitted in partial fulfilment of the requirements for the degree of **Master of Science** with major in **Physics** of the College of Post-Graduate, G.B. Pant University of Agriculture and Technology, Pantnagar, is a record of *bona-fide* research carried out by **Ms. Manisha Goutam Id. No. 52636**, under my supervision, and no part of the thesis has been submitted for any other degree or diploma.

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Pantnagar
August, 2019

(B. C. Chanyal)
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CERTIFICATE

We, the undersigned, members of the Advisory Committee of **Ms. Manisha Goutam, ID. No. 52636**, a candidate for the degree of **Master of Science** with major in **Physics**, agree that the thesis entitled **“Quaternionic Formalism of Curvature Space-Time and Einstein Field Equation”** may be submitted in partial fulfilment of the requirements for the degree.

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CONTENTS

LIST OF SYMBOLS

$\mathbb{R}$	:	Real Number
$\mathbb{C}$	:	Complex Number
$\mathbb{H}$	:	Quaternion
$S_{\mathbb{H}}$	:	Real quaternion
$V_{\mathbb{H}}$	:	Pure quaternion
$\mathbb{H}^*$	:	Quaternionic conjugate
$\lvert\mathbb{H}\rvert$	:	Norm of a quaternion
$T_{\mathbb{H}}$	:	Tensor of a quaternion
$U_{\mathbb{H}}$	:	Versor of a quaternion
Φ	:	Newtonian potential
ρ	:	Mass density
G	:	Gravitational constant
Γ^{i}_{jk}	:	Christoffel symbol
$[jk \quad i]$	:	Christoffel symbol of first kind
$\begin{Bmatrix} i \\ jk \end{Bmatrix}$	:	Christoffel symbol of second kind
∂_{μ}	:	Partial derivatives
D_{μ}	:	Covariant derivatives
H^{μ}	:	Tangential quaternion
$\eta_{\mu\vartheta}$	:	Minkowski metric
$g_{\mu\vartheta}$	:	Metric tensor
$R^{\alpha}_{\mu\beta\vartheta}$	:	Riemannian Christoffel curvature tensor
$R_{\mu\vartheta}$	:	Ricci tensor
R	:	Ricci scalar
$T_{\mu\vartheta}$	:	Energy-Momentum tensor
Λ	:	Cosmological constant

Introduction

The theory of the physical world has two fundamental components: Matter and its interactions. The fundamental forces or fundamental interactions define in which particles interact with one another and cannot be explained with respect to other communications. These interactions regulate molecular interactions as well as the destruction of macroscopic objects and particles. These fundamental interactions broke down into four fundamental forces of nature **(Zhang, T. 2010)**:

1. Gravitational force due to the attraction between massive bodies
2. Electromagnetic force due to the force between electrically charged particles
3. Weak force, the force responsible for the radioactive decay of atoms
4. Strong force, the force binding fundamental particles together.

Gravitational force is a weakest force experiences as attractive in nature. Because it is a very long-range force, the dominant force on the macroscopic scale is the reason for the formation, shape, and orbit of the astronomical bodies. This force is responsible for an object in space to move in circular orbit. In other words, we can say that every massive objects experience the gravitational force due to other massive object.

Albert Einstein developed the theory of relativity (in 1905, he published special theory and in 1916 general theory of relativity). The traditional Newtonian concept of absolute space and time has been replaced by the concept of space-time. Time is the fourth dimension used in space-time theory of special relativity and general relativity. The theory which explained the gravitation named as 'General Theory of Relativity' (GTR) **(Beiser, A. 2003)**. The special theory of relativity proves that no inertial reference system takes precedence over any other inertial reference frame, but takes precedence over the inertial reference frame of the non-inertial reference frame. General relativity eliminates the preference for an inertial frame of reference by indicating that there is no favoured frame of reference (inertial or non-inertial) for describing nature. In general relativity, gravity does not act as a force but significances space-time curvature where the source of the curved space-time is energy and matter. Einstein field equation is the second order non-linear partial differential equation that elaborates "mass and energy is responsible for curvature of space-time and geometry of space-time responsible for motion of matter". General relativity provides a unified description of a geometric property of space and time. Basically, the geometry of space-time in special relativity is Euclidean. The Euclidean geometry explains the Cartesian coordinate system in flat space-time. But the correct description of space-time geometry, according to Einstein, should be non-Euclidean geometry. So the space-time path becomes curved in presence of gravitational field. Thus, the motion of particles in such a field or in non-Euclidean space-time shows the curvilinear instead of being straight.

Objects in nature have basic properties that are independent of the manual selection of any specific coordinate system. The basic assumption of physics is that the nature's law are stated by equations, valid for all the coordinate systems. Vectors can be used to verify the covariance of these laws. But vector notation is not a general method to direct the relation in many physical problems. The theory of relativity required the 'covariant form' that are independent of coordinate system. Covariant form remains same in all co-ordinate system whether the reference frame will be changed or not. The laws of physics must be expressed by an equations that remains covariant for all coordinate system. The tensor has such properties that shows covariance form in all coordinate system. Covariant and contravariant terms have been introduced by **Sylvester, J. J. (1851)** in the general context of algebraic theories. The difference between covariance and contravariance is especially important for calculations with terms because tensors usually have mixed variance. The lowest index object (covectors) that usually have pull backs, which are contravariant, whereas the top index objects (vectors) have pushforwards, which are covariant. If all the equations, related to general theory of relativity, are written in the tensor form, then it will transform easily those equations from one reference frame to another. Therefore, in mathematics the tensor calculus is the best tool to explain theory of curvature as ordinary calculus.

For a flat space-time, the Pythagoras theorem is valid perfectly for calculating distance between two points. But for a curved space-time, this Pythagoras theorem no longer holds true. It is replaced by 'metric' to measure the distance. Metric is a mathematical tool which makes the correction to the Pythagoras theorem where the distance is determined between two points on a curved space-time instead of flat space. In general relativity, metric tensor behaves as the gravitational potential of Newtonian gravitation. The Christoffel symbol describes the connection of metric. The metric relationship is the specialization of affinity for surfaces with metrics or other collectors that allows you to measure distances on this surface. In general relativity, this connection plays the role of gravitational force related to gravitational potential which is a metric tensor. Riemannian Christoffel curvature tensor captures the notion of parallel transport. It gives the idea about the connections of the parallel transported vector on a curve space-time. The Riemann tensor is the derivatives of Christoffel symbols. Riemannian curvature is required for all the changes in the tangent vectors when we transport them around a curved space. The Riemann-Christoffel tensor is the only tensor that can be constructed from the second (or lower) derivative of the metric tensor and is linear in the second derivative **(Weinsberg, S. 1972)**. The contraction of Riemann tensor gives Ricci curvature and scalar curvature. Ricci curvature explains the concept of degree to which mass converge and diverge in time corresponding to the part of the curvature of space-time. Scalar curvature gives the single real number which represents the quantity of Riemannian manifold volume of geodesic ball differs from the normal ball in Euclidean space. 'Geodesic' generalized the conception of a straight line as a curved space-time, where the particles are moving freely always along a path of geodesic. After determining the geometry, the pathways of the particles and light rays are important to solve the geodetic equations in which the Christoffel symbols are explicitly displayed. The equation of motion of free particles, is recognised by geodesic equation followed the analogous to Newton's equation of motion which clarifies the equation for the acceleration of particles. The source of the curvature of space-time is generalised by energy and momentum of the field expressed as energy-momentum tensor. It will be a symmetric tensor and follow the conservation of energy and momentum. The general theory of relativity predicted that Universe must be either expanding or contracting.

Cosmology is the study of structures, matter, motions, origin and evolution of universe. The cosmological constant explains the notion of energy density that is responsible for universe expansion. It is the understanding of dark energy which is more general name given to the unknown reason of universe acceleration. In Einstein field equation, the cosmological constant, which is essential to keep the universe from becoming less dense with expanding volume, is equivalent to the existence of non-zero vacuum energy. Einstein originally presented a concept to unbalance the effect of gravity in 1917 and to achieve a stable universe, a perception that was an approved scene at that time. After the discovery of Hubble's expansion universe **(Rugh, S and Zinkernagel, H. 2002)**, Einstein abandoned this concept. With astonishing discovery in 1998 it changed that the expansion of the universe is increasing, thereby increasing the probability of a positive non-zero value for the cosmological constant.

The study of algebraic structure in curvature space-time has an important role to describe the natural world. In mathematical physics, algebra is neccessary to calculating space-time structures and distances, as the linear algebra for three-dimensions in which vector provides the directions and magnitude. Similarly, for understanding GTR it is essential to fully understand vectors, linear operators, tensors, and lots of differential equations as well. On the other hand, there are four types of division algebra given by **Dickson, L. E. (1919)**. These are: real numbers, complex algebra, quaternion algebra, and octonion algebra or Cayley algebra. Real numbers are the numbers, which are used normally without any imaginary number. The numbers written as the mixture of real number and imaginary number is renowned by complex numbers. Quaternions are the extension of complex number with non-commutative property, used to label the rotations in three-dimension. Cayley algebra is the extension of the quaternions with non-commutative and non-associative as well.

Hamilton, W. R. (1844) was the first who discovered the quaternions. It is the four-dimensional algebra, having some algebraic and geometric properties. Quaternions can be written as the combination of scalar part and vector part. It can easily handle 3-vectors. If scalar part vanished, it will be named as 'Right quaternions' **(Hardy, A. S. 1881)** or pure quaternions as it contains only vector part. The striking feature of two quaternions is that the product of two quaternions, always non-commutative. The distributive and associative properties on addition and multiplication are valid in quaternionic algebra. Initially, quaternions expressed the valuable ideas, e.g., the concept of scalar, notion of vectors, divergence, curl, gradient etc made it more general. They can be added, subtracted, multiplied and divided; there is nothing more general than this with four numbers. Many theories related to four-dimension can be coupled with the quaternions. Various classical and quantum theories are written in the form of quaternion such as Maxwell equations, Dirac equations, unification of gravity and electromagnetism etc. In general theory of relativity, space-time gravity can be formulate in the form of quaternions, it represents the new way to describe the field equations.

Review of Literature

Fundamental interactions are the interaction between objects or particles. Gravitational force is one of the fundamental force that exerts between two massive objects. Einstein developed the theory of gravitation called GTR which describes the concept of gravity originated from the distortion in space-time by massive objects.

Generally, the line element is the distance between two co-ordinate points of space-time. In general theory of relativity, it is determined by the metric tensor using Riemannian geometry. **Rosen, N. (1940)** introduced a notion where at each point of space-time, there is a corresponding Euclidean metric tensor in addition to the Riemannian metric tensor. By including the Euclidean metric tensor, he introduced new tensors and set up other field equation than original one. He discussed some more satisfactory explanations for describing the nature by adopting this different point of view.

Earman, J. and Friedman, M. (1973) discussed about the Newtonian physics in which they explained the law of inertia and the nature of gravitational forces. A four-dimensional approach to Newtonian physics was applied to differentiate between several different structure for Newtonian space-time and formulation of Newtonian gravitational theory. Their formalism explained the comparison between Newtonian and Einsteinian theory. It has been developed the combination of philosophy of space-time theories instead of philosophy of space and time.

Moffat, J. W. (1979) explained the geometry of space-time by using the concept of a non-symmetric field structure. It was a new theory of gravitation because the concept of space-time has no meaning at the singularities of collapsed stars and cosmology. For analysing this limitation, he used the field equations which was derived from the Lagrangian action principle. It has been explained that the space-time geometry should be non-Riemannian and notion of a non-symmetric tensor field.

In Einstein field equation of GTR, the energy-momentum is the cause of space-time curvature structures. **Babak, S. V. and Grishchuk, L. P. (1999)** derived the energy-momentum tensor for the gravitational field with metrical and canonical routes which satisfy six conditions i.e., the routes must be symmetric, conserved, free from second derivative of variables, should not contain field variables, follow rule of coordinate transformation. This formulation of gravitational energy-momentum tensor is very useful in practical difficulties.

Baskaran, D. and Grishchuk, L. P. (2004) studied the components of gravitational force in the field of a gravitational wave. They described that there existed an analogy between the motion of free masses in the field of gravitational wave and the motion of free charges in the field of electromagnetic wave. They introduced the analogous of electric and magnetic components of the gravitational force to describe the linear gravitational field equations.

Boi, L. (2006) discussed the application of Riemannian geometry to Einstein's General theory of Relativity in terms of space-time geometry. The evolution of the interaction between physics and geometry was discussed. He discussed the geometry of space-time and explained the Einstein field equation for gravitation. He also discussed the unification of electromagnetic and general theory of relativity by using the Kaluza-Klein principle of 5-dimension.

Kumar, A. (2010) explained the concept of covariance and contravariance of vectors. According to him, vectors can be expanded in terms of the orthogonal basis. In many different area of physics, non-orthogonal basis (or oblique co-ordinate system) can be chosen. The notion of contravariant and covariant vector made the non-orthogonal basis behave like orthonormal basis. The same concepts are also used in quantum mechanics, Kets are analogous to contravariant vectors and Bras are analogous to covariant vectors. This notion was utilised in differential geometry which takes into account the non-orthogonality of basis to make it similar to orthogonal basis. Tensors are the arrangements of numbers in multilinear algebra having aspects of covariance and contravariance. In tensor analysis, the covariance and contravariance illustrates how the change in the image of some geometric or physical creatures has changed the basis.

The General relativity (GR) has the restriction that affine connection be symmetric, then the space-time is said to be torsion less. But space-time torsion as a possible remedy to major problems in gravity and cosmology explained by **Poplawski, N. J. (2011)**. The Einstein-Cartan-Sciama-Kibble theory of gravity in the presence of torsion extended the general relativity to explain the natural spin of matter but with this it also gave the answer to major problems occur in gravity and cosmology like beginning of universe, existence of black holes, presence of matter and anti-matter symmetry in universe and the nature of dark matter. The space-time torsion removed the limitation on GR that affine connection be symmetric, infact with the torsion it should be antisymmetric.

Correspondingly, tensors are used for the formalism of curvature space-like equations, as it has some properties which fulfil the basic requirements of the equations like covariant property, systematic nature, generality, and even elegance. Equation of motion of a classical fluid in an external gravitational field which transforms one coordinate system to another coordinate system, has been written in the tensor form by **Charron, M. et al., (2013)**. In their formalism, the set of equations in three-dimensional are curvilinear but by adding vectors with additional terms to the equations, it became time varying also.

On the basis of GTR, the existence of gravitational waves was predicted by Albert Einstein. There are many observations which were done, indirect evidences were offered by scholars to prove the existence of gravitational waves. **Grote, H. and Reitze, D. H. (2011)** gave the idea of first-generation interferometric detectors for gravitational waves. Here, first-generation demonstrates the time at which instruments were made and operated. There are five observatories for gravitational waves: two branches of LIGO in United States, Virgo in Italy, GEO site 600 in Europe and TAMA 300 in Japan. Second-generation interferometric detectors for gravitational waves were explained by **Acernese, F. et al., (2014)** which was known as Advanced Virgo. It is the upgraded version of Virgo interferometric detector of gravitational waves. The main aim of this version was to increase the rate of detection or to increase the number of galaxies which was observed. In detection of gravitational waves, Advanced Virgo became a part of other observatories which was operated in first-generation and gave a new path of observation on the universe. First-generation established an infrastructure which was also utilised for second-generation and gave very good achievements.

Einstein gave the theory of gravitation on a large scale explaining gravity as a warping of space-time and developed an Einstein field equation. The theory explained that the geometry of space-time is equivalent to matter and energy. **Walters, S. (2016)** derived the Einstein field equation using differential geometric background related to curvilinear coordinates and tensor to see how Einstein used this geometry to give the concept of gravity. At the end of the equation he also explained some applications or predictions of the theory.

On the other hand, the gravitational waves are the waves that are generated by acceleration of heavy objects in the curvature of space-time. These waves are produced in curvature space-time which propagates outward from their source. **Abbott, B. P. et al., (2016)** discussed about the observation of gravitational waves by the combination of two Binary Black holes. The two Laser-Interferometer Gravitational waves observatory (LIGO) detectors observed simultaneously a temporary gravitational wave signals. These observed waveform were found to match with the gravitational waves which were predicted by general relativity.

GTR is the extended theory of special theory of relativity. With the development of GTR, similar effect of time dilation was found to occur in the presence of high gravity. This effect is known as Gravitational time dilation which was first confirmed by **Pound, R. V. and Rebka, G. A. (1959)**. The test showed a general relativity prediction that the clock should work at different speeds in different places of the gravitational field. A stronger gravitational attraction is observed if an object is closer with another object results in more dragging of time.

In classical mechanics and geometry, the rotational group $SO(3)$ is a group of all 3-D rotations relative to the origin of the three-dimensional Euclidean space R^3 during the operation of the composition. As such, the rotation group $SO(4)$ in a four-dimensional Euclidean space can be use to study the four-dimensional GTR. **Myszkowski, M. (2019)** recently discussed the general four-dimensional rotation matrix by replacing three-dimension rotations with spatio-temporal rotations. This four-dimensional rotation matrix was found to be similar with special orthogonal group of order 4 $i.e., SO(4)$. General four-dimensional matrix was composed by multiplication of six matrices (each matrix describes the different rotation) which was given by **Zhelezov, O. I. (2018)** who presented the N-dimensional Rotation Matrix Generation algorithm (NRMG). It was explained for N-dimensional vector that every one rotation can be constructed by $2(N-1)$ two-dimensional rotations. He considered simple rotations of plane that was the combination of two axis along which rest of the two axis rotate and the angle that helps in constructing six matrices.

A four-dimensional division algebra was developed by **Hamilton, W. R. (1844)** in the form of quaternion. He explained quaternion as the extension of complex number dealing with the algebraic and geometric properties of object which consisting a scalar and a vector. He also gave another form of a quaternion in terms of tensor and versor instead of a combination of a scalar and a vector. It is also described as the quotient of division of two vectors.

For the unification of quantum and gravity theory **Edmonds, J. D. (1974)** derived quaternion wave equation in curved space-time. The extension in the quaternion formulation of relativistic quantum theory by including curvilinear coordinates are discussed. Quaternion quantum theory was explained in curvature form which worked as a bridge to fill the gap between quantum and gravity theory. Further, the second problem was to quantized the gravity which was not done but these steps towards this unification theory provides a natural framework to solve this problem.

Rawat, A. S. and Negi, O. P. S. (2011) developed the quaternionic formalism for gravi-electromagnetism. They explained charge, potential, current and fields as complex quantities where gravitation was represented by real part and electromagnetism was represented by imaginary part. The equation of motion, continuity equation, and the corresponding field equation derived in covariant form by them. The concept of Lagrangian density of gravity and electromagnetic charges for generalised fields has been explained to obtain these equations in consistent manner. They also discussed that the unified field equations remain same under quaternion Lorentz transformation and duality transformation as well.

Arbab, A. I. (2011) derived the many problems related quantum mechanics using the quaternion algebra. He described the wave function, energy, and momentum of the particle in terms of quaternion. The momentum eigen value problems explained in terms of quaternion that gave the scalar and vector part which described a new wave function in a single equation. He also derived the energy momentum relation of Einstein in terms of quaternion. Analogous to this, **Chanyal, B. C. (2018)** discussed the quaternion and derived the quantised Proca-Maxwell's equations for dyons. Quaternionic quantum wave equation for massive dyons was obtained by the quaternionic momentum eigen value equation. The quaternionic quantization of electromagnetic field of dyons had been formulated for massive particles and classical behaviour of field equations in free space was also shown. He expressed the quantized continuity equation, Klein-Gordan equation and relativistic Dirac wave equation which generates the new concept of antiparticle of dyons known as anti-dyons.

Weng, Z. H. (2016) studied the gravitational properties of complex curved space using by quaternions and octonion. He studied the quaternionic metric, covariant derivative and parallel translation in reimannian of complex curved space of quaternion. The octonion form of metric, covariant derivative and parallel translation was obtained in curved space. The gravitational field potential, field strength, field source, linear and angular momentum, and force was deduced with the help of quaternion and octonion metric, orthogonality, and covariant derivative.

Jafari, M. (2016) explained algebra of quaternions and its applications. He discussed the special cases of quaternions: real quaternions, split quaternions, semi-quaternions, split semi-quaternions and quasi-quaternions (or 1/4 quaternions). These cases were defined with different properties and study of these basic algebraic properties of different algebras was done in his investigation. Split quaternions or co-quaternions were used in study of differential geometry and superstring theory. Semi-quaternions were used to examine De Moivre's formula.

Chanyal, B. C. and Pathak, M. (2018) derived the dual fields equations for dyonic cold plasma by using the quaternion algebra. By using quaternionic dual velocity and dual enthalpy of dyonic cold plasma, they formulated the quaternionic hydroelectric and hydromagnetic field equation. Further, quaternionic Dirac-Maxwell equations for dyonic cold plasma were derived.

An application point of view, the quaternions are used in fourier transformation as Quaternion Fourier Transform (QFT) **Bahri, M. et al., (2018)** in the process of colour image and analysis of signal. Further, QFT was extended to QLCT (Quaternion Linear Canonical Transform) by **Bahri, M. and Musdalifah, S. (2018)** which is used in signal processing. This approach established uncertainity principle for QLCT.

From the study of above literature we may describe the geometry of curvature space-time in quaternionic algebra. The quaternionic theory has been used to describe different theories in 3-D such as gravi-electromagnetism, wave equation in curved space-time, Proca-Maxwell's equations for dyons, etc. In this work we apply the quaternionic algebra for describing the four-dimensional quaternionic theory of curvature space-time and apply it to construct the Einstein field equation in curvature space-time.

Materials and Methods

3.1 Tensor algebra

Tensors are defined as simply the arrangements of numbers, or functions, that transform according to certain rules under a change of coordinates. We also can say that tensor is a generalised form of vector **(De, U. C. 2012)**. In curved space-time (non-euclidean geometry) we use tensor analysis as it transforms the equation covariantly under the change of coordinate system.

3.1.1 Notations:

Many expression need to be written in compact form for which we use the concept of notations that is the collection of indices, see **(Brand, L. 1947).** When indices are used as a superscript for denoting the components of a tensor it will be a contravariant form. When indices are used as a subscript for denoting the components of a tensor it will be a covariant form. Suppose, T is any tensor then the components of this tensor can be written in following forms like

T_{ij} is a covariant components of a tensor

T^{ij} is a contravariant components of a tensor

T^i_j is the components of a mixed tensor.

where, i and j are the indices $1, 2, 3, ..., n$.

3.1.2 Summation convention:

According to Einstein summation convention, if any same index appear twice in a term then that index stands for the sum of all terms at the complete range of values. For example, we can write the two forms of vector as **(Heinbockel, J. H. 1996)**

$$\mathbf{u} = u^i e_i, \quad \mathbf{u} = u_j e^j \qquad (\forall,\ i,\ j = 1, 2, 3)\,, \tag{1}$$

where, i and j are the repeated indices or summation indices. $\mathbf{u}$ is any vector and $\left(u^i, u_j\right)$ are the components in contravariant and covariant form respectively. It can also written in other form as

$$\mathbf{u} = u^r e_r = u^1 e_1 + u^2 e_2 + u^3 e_3 \,. \tag{2}$$

Here, r is the repeated indices in place of i and j, these indices are also known as dummy indices. If any term having indices which are not repeated in a term then these indices are called a real or free indices. For example, $a_i^{\mu} x_i$ here μ is a real indices. According to **(Kelly, P. 2013)**, some generalised tensors can be written as

$$\begin{array}{lll}
\alpha, \beta, \gamma, \phi .... 0th - \text{order tensors} & (\text{"scalars"}) \\
\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} .... 1st - \text{order tensors} & (\text{"vectors"}) \\
\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D} .... 2nd - \text{order tensors} & (\text{"dyadics"}) \\
\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} .... 3rd - \text{order tensors} & (\text{"triadics"}) \\
A, B, C, D .... 4th - \text{order tensors} & (\text{"tetradics"}) \,.
\end{array}$$

These generalised tensor also recognised by indices where scalar have no indices, vectors have one indices, dyadics have two indices, triadics have three indices and tetradics have four indices. Scalar product of two vectors can be represented in terms of indices **(Ferkinghoff-Borg, J. 2007)** and the components can be written as

$$\mathbf{a} \cdot \mathbf{b} = \left(\sum_i a_i e_i \right) \cdot \left(\sum_j b_j e_j \right) = \sum_{ij} a_i b_j \left(e_i \cdot e_j \right) = \sum_{ij} a_i b_j \delta_{ij} = \sum_i a_i b_i \,, \tag{3}$$

where, δ_{ij} is the kroneckar delta symbol defined as $\delta_{ij} = 1,$ for $(i = j)$ and $\delta_{ij} = 0,$ for $(i \neq j)$. Similarly, we can represent the cross product as

$$[\mathbf{a} \times \mathbf{b}]_i = \sum_{jk} \varepsilon_{ijk} a_j b_k \,, \tag{4}$$

where, ε_{ijk} is the Levi-civita symbol with three indices having value $\varepsilon_{ijk} = +1$ for cyclic permutation, $\varepsilon_{ijk} = -1$ for non-cyclic permutation, and $\varepsilon_{ijk} = 0$ for any two repeated indices.

3.1.3 Addition and Multiplication of tensors:

We have two tensors of same rank by adding them it gives the tensor of same order, i.e.,

$$\mathbf{A}^r_{st} = \mathbf{B}^r_{st} + \mathbf{C}^r_{st}\,, \tag{5}$$

where, $\mathbf{A}^r_{st}, \mathbf{B}^r_{st}$ and $\mathbf{C}^r_{st}$ are the tensors of same rank, *i.e.*, three-rank tensor. They are also known as mixed tensor because they are having contravariant and covariant indices both. The product of two tensors are given as

$$\mathsf{B}^r_s \mathbf{C}^t_{uv} = \mathsf{A}^{rt}_{suv}\,, \tag{6}$$

here, B^r_s is tensor of rank two having indices r and s, $\mathbf{C}^t_{uv}$ is a tensor of rank three having indices t, u and v, and A^{rt}_{suv} is a tensor of rank five. This type of product is known as direct product of two tensor which gives the tensor whose rank is equal to the sum of the rank of both the tensors multiplied in left hand side. We can also defined the contraction product as

$$\mathbf{b}_t = \mathbf{A}^r_{rt}\,, \tag{7}$$

where, $\mathbf{A}^r_{rt}$ is a three-rank tensor having two same indices which contact and gives the components of vector of order one.

3.1.4 Basic Transformations:

Any scalar quantity (ϕ) transform from one reference frame to another reference frame, it remains invariant, such that **(Hay, G. E. 1953)**

$$\phi^{'} = \phi\,. \tag{8}$$

This invariant quantity may also be known as contravariant tensor of rank zero, or covariant tensor of rank zero. A vector components can be transforms as **(Gron, O. and Naess, A. 2002)**

$$v^{'r} = \frac{\partial x^{'r}}{\partial x^s} v^s\,. \tag{9}$$

The above equation signifies that components of vector v^s transformed to components of vector v'^r when the coordinate x^s transformed to x'^r. We can also define the contravariant tensor components of rank two represented as **(Narlikar, J. V. 1978)**

$$A'^{ru} = \frac{\partial x'^r}{\partial x^s}\frac{\partial x'^u}{\partial x^t}A^{st}\,. \tag{10}$$

This is the transformation of tensor components of rank two with a change of coordinates. It is the dyadic product of two vectors transformation which is also known as dyads (**Mitiguy, P. 2009**). Similarly, we can defined analogously the contavariant tensors of higher rank.

3.2 Quaternionic algebra

Over real algebra $(\mathbb{R})$, a complex number system can use the imaginary number i to define an algebraic extension of a normal real number. The complex number $\mathbb{C}$ will be expressed in the form of basis $(1,i)$,

$$Z = \xi_1 + i\xi_2, \qquad \forall\,(\xi_1,\,\xi_2) \in \mathbb{R} \text{ and } Z \in\, \mathbb{C}\,, \tag{11}$$

where ξ_1 is known as the real and ξ_2 is known as the imaginary part of the complex number Z. If $Re(Z) = 0$ then the complex number is called as purely imaginary. On the other hand, a quaternion is a number system that expands the complex number and form an algebra having some properties over addition and multiplcation. **Hamilton, W. R. (1853)** extended the complex number to explain the quaternionic algebra. Quaternions are the four-dimensional norm-division algebra over real algebra $\mathbb{R}$. In normed division algebra, there is a concept of "size" (given by a norm) and each non-zero element has a left and right multiplication inverse. It can be written as the combination of scalar quantity (represents by one-dimension) and vector quantity (represents by three-dimension). Quaternions have four unit basis (e_0,e_1,e_2,e_3) in which e_0 is the scalar unit and e_1,e_2,e_3 are the imaginary units. A quaternion $\mathbb{H}$ can be expressed as

$$\mathbb{H} = e_0w + e_1x + e_2y + e_3z = \; e_0w + \sum_{j=1}^{3} e_j r_j\,, \tag{12}$$

where, w,x,y,z are the real numbers. The quaternion may also be compose as

$$\mathbb{H} = S_{\mathbb{H}} + V_{\mathbb{H}} \,. \tag{13}$$

Here, $e_0 w$ is the scalar part of quaternion denoted by $S_{\mathbb{H}}$ and $(e_1 x + e_2 y + e_3 z)$ is the vector part of quaternion denoted by $V_{\mathbb{H}}$. If scalar part is zero then

$$\mathbb{H} \rightarrow \; V_{\mathbb{H}} = e_1 x + e_2 y + e_3 z \,, \tag{14}$$

it is known as right quaternion or pure quaternion. Although any quaternion can be seen as a vector in a four-dimensional vector space, it is usual to refer to the pure quaternion as a vector in a three-dimensional space. Quaternionic vector part can be understood as coordinate vector at $\mathbb{R}^3$ therefore, the quaternion algebraic operations represent the geometry of $\mathbb{R}^3$. Similarly, if vector part is zero then

$$\mathbb{H} \rightarrow \; S_{\mathbb{H}} = e_0 w \,, \tag{15}$$

it is simply a real number known as scalar quaternion.

3.2.1 Properties of quaternion for addition:

Consider two quaternions $\mathbb{A}$ and $\mathbb{B}$ can be written as

$$\mathbb{A} = e_0 A_0 + e_1 A_1 + e_2 A_2 + e_3 A_3 \,, \tag{16}$$

$$\mathbb{B} = e_0 B_0 + e_1 B_1 + e_2 B_2 + e_3 B_3 \,. \tag{17}$$

Addition of two quaternions will be

$$\begin{aligned} \mathbb{A} + \mathbb{B} = & \; (e_0 A_0 + e_1 A_1 + e_2 A_2 + e_3 A_3) + (e_0 B_0 + e_1 B_1 + e_2 B_2 + e_3 B_3) \\ & = e_0 (A_0 + B_0) + e_1 (A_1 + B_1) + e_2 (A_2 + B_2) + e_3 (A_3 + B_3) \,, \\ & = e_0 (H_0) + e_j \left(H_j\right) = \mathbb{H}, \;\; \forall \; (j = 1,2,3) \,, \end{aligned} \tag{18}$$

where H_0 is the scalar part of quaternion and H_j is the vector part of quaternion. Since, equation (18) shows that addition of two quaternions gives a new quaternion which satisfies the closure property with respect to addition. The quaternionic algebra also satisfy the associative property with respect to addition i.e.,

$$(\mathbb{A}+\mathbb{B})+\mathbb{C}=\mathbb{A}+(\mathbb{B}+\mathbb{C}),\;\;\forall\;(\mathbb{A},\mathbb{B},\mathbb{C})\in\mathbb{H}\,. \tag{19}$$

The additive identity of quaternion can be written as

$$0=e_0 0+e_1 0+e_2 0+e_3 0\,, \tag{20}$$

where, 0 is the identity element. The additive inverse of quaternion can also be written as

$$-\mathbb{A}=e_0\left(-A_0\right)+e_1\left(-A_1\right)+e_2\left(-A_2\right)+e_3\left(-A_3\right)\,, \tag{21}$$

here, -1 is the additive inverse element. The commutative property is defined by

$$\begin{aligned}\mathbb{A}+\mathbb{B} &= e_0\left(A_0+B_0\right)+e_1\left(A_1+B_1\right)+e_2\left(A_2+B_2\right)+e_3\left(A_3+B_3\right)\\ &= e_0\left(B_0+A_0\right)+e_1\left(B_1+A_1\right)+e_2\left(B_2+A_2\right)+e_3\left(B_3+A_3\right)\\ &= \mathbb{B}+\mathbb{A}\,. \end{aligned} \tag{22}$$

These properties shows that it is an abelian group of quaternion with respect to addition.

3.2.2 Properties of quaternion for multiplication:

Multiplication properties of quaternion are similar to the addition properties of quaternion but it does not satisfy the commutative property. Now, the quaternionic product of equation (16) and equation (17) becomes,

$$\begin{aligned}\mathbb{A}\circ\mathbb{B} &= \left(e_0A_0+e_1A_1+e_2A_2+e_3A_3\right)\circ\left(e_0B_0+e_1B_1+e_2B_2+e_3B_3\right)\\ &= e_0P_0+e_1P_1+e_2P_2+e_3P_3\;=\;\mathbb{P}\,, \end{aligned} \tag{23}$$

where $'\circ'$ is the symbol used for the quaternionic multiplication. Since, above equation shows that multiplication of two quaternions gives again a quaternion which proves the closure property with respect to multiplication of quaternions. The quaternionic components of $\mathbb{P}$ in equation (23) are written by

$$
\begin{aligned}
P_0 &= (A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3) \quad \text{(scalar coefficient)} \\
P_1 &= (A_0B_1 + A_1B_0 + A_2B_3 - A_3B_2) \quad (e_1 \text{ coefficient}) \\
P_2 &= (A_0B_2 + A_2B_0 + A_3B_1 - A_1B_3) \quad (e_2 \text{ coefficient}) \\
P_3 &= (A_0B_3 + A_3B_0 + A_1B_2 - A_2B_1) \quad (e_3 \text{ coefficient}).
\end{aligned} \tag{24}
$$

Here, the quaternionic multipication follows the given relations of quaternionic unit basis (e_0, e_1, e_2, e_3), *i.e.*,

$$
\begin{aligned}
e_0^2 &= 1, \\
e_i^2 &= -1, \\
e_0 e_i &= e_i e_0 = e_i, \\
e_i e_j &= -\delta_{ij} e_0 + \varepsilon_{ijk} e_k, \ \forall \ (i, j, k = 1, 2, 3),
\end{aligned} \tag{25}
$$

where δ_{ij} is the kroneckar delta symbol defined by $\delta_{ij} = 1$ $(\forall\, i = j)$ and $\delta_{ij} = 0$ $(\forall\, i \neq j)$ and ε_{ijk} is the Levi-civita symbol having value $\varepsilon_{ijk} = +1$ for cyclic permutation, $\varepsilon_{ijk} = -1$ for non-cyclic permutation, and $\varepsilon_{ijk} = 0$ for any two repeated indices. All indices in ε_{ijk} is antisymmetric and it satisfy $e_i \times e_j = \sum_k \varepsilon_{ijk} e_k$. Thus, the equation (23) can be written in compact form in terms of ordinary dot and cross product as,

$$
\mathbb{A} \circ \mathbb{B} = \left(A_0 B_0 - \overrightarrow{A} \cdot \overrightarrow{B},\, A_0 \overrightarrow{B} + B_0 \overrightarrow{A} + \left(\overrightarrow{A} \times \overrightarrow{B} \right) \right). \tag{26}
$$

Here, $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors and, the scalar and vector product of vectors are denoted by $'\cdot'$ and $'\times'$ symbols. The product of quaternion with scalar quantity ϕ is given by

$$
\phi \circ \mathbb{A} = e_0 (\phi A_0) + e_1 (\phi A_1) + e_2 (\phi A_2) + e_3 (\phi A_3). \tag{27}
$$

Scalar product of two quaternions **(Rastall, P. 1964)** is given by

$$
\begin{aligned}
\mathbb{A} \cdot \mathbb{B} &= -\frac{1}{2} \left(\mathbb{A} \circ \mathbb{B}^* + \mathbb{B} \circ \mathbb{A}^* \right) \\
&= -\frac{1}{2} \left(\mathbb{A}^* \circ \mathbb{B} + \mathbb{B}^* \circ \mathbb{A} \right).
\end{aligned} \tag{28}
$$

The scalar product of two quaternions is same as scalar product of two vectors the only difference will be of sign. The triple product of quaternions are given by

$$
\begin{aligned}
(\mathbb{A}\circ\mathbb{B})\circ\mathbb{C} = & \,(e_0P_0+e_1P_1+e_2P_2+e_3P_3)\circ(e_0C_0+e_1C_1+e_2C_2+e_3C_3) \\
& = e_0\left(P_0C_0-P_1C_1-P_2C_2-P_3C_3\right) \\
& + e_1\left(P_0C_1+P_1C_0+P_2C_3-P_3C_2\right) \\
& + e_2\left(P_0C_2+P_2C_0+P_3C_1-P_1C_3\right) \\
& + e_3\left(P_0C_3+P_3C_0+P_1C_2-P_2C_1\right) .
\end{aligned} \tag{29}
$$

Similarly, the quaternionic multiplication of $\mathbb{A}\circ(\mathbb{B}\circ\mathbb{C})$ gives

$$
\begin{aligned}
\mathbb{A}\circ(\mathbb{B}\circ\mathbb{C}) = & \,(e_0A_0+e_1A_1+e_2A_2+e_3A_3)\circ(e_0Q_0+e_1Q_1+e_2Q_2+e_3Q_3) \\
& = e_0\left(A_0Q_0-A_1Q_1-A_2Q_2-A_3Q_3\right) \\
& + e_1\left(A_0Q_1+A_1Q_0+A_2Q_3-A_3Q_2\right) \\
& + e_2\left(A_0Q_2+A_2Q_0+A_3Q_1-A_1Q_3\right) \\
& + e_3\left(A_0Q_3+A_3Q_0+A_1Q_2-A_2Q_1\right) .
\end{aligned} \tag{30}
$$

Here, equations (29) and (30) satisfy the associative property with respect to multiplication, *i.e.*, $(\mathbb{A}\circ\mathbb{B})\circ\mathbb{C} = \mathbb{A}\circ(\mathbb{B}\circ\mathbb{C})$. The multiplicative identity can be written as

$$
1 = 1e_0+0e_1+0e_2+0e_3 \,, \tag{31}
$$

where $'1'$ is the identity element. In order to check the non-commutative property, we may write

$$
\begin{aligned}
\mathbb{B}\circ\mathbb{A} = & \,(e_0B_0+e_1B_1+e_2B_2+e_3B_3)\circ(e_0A_0+e_1A_1+e_2A_2+e_3A_3) \\
& = e_0\left(B_0A_0-B_1A_1-B_2A_2-B_3A_3\right)+e_1\left(B_0A_1+B_1A_0+B_2A_3-B_3A_2\right) \\
& + e_2\left(B_0A_2+B_2A_0+B_3A_1-B_1A_3\right)+e_3\left(B_0A_3+B_3A_0+B_1A_2-B_2A_1\right) \\
& = e_0\left(B_0A_0-\overrightarrow{A}\cdot\overrightarrow{B}\right)+e_j\left(A_0\overrightarrow{B}+B_0\overrightarrow{\mathrm{A}}+\left(\overrightarrow{A}\times\overrightarrow{B}\right)_j\right) \\
& \neq \mathbb{A}\circ\mathbb{B} .
\end{aligned} \tag{32}
$$

From the above expression, it is clear that product of two quaternions is non-commutative because product of two vectors are always non-commutative *i.e.*, $\overrightarrow{A}\times\overrightarrow{B}=-\overrightarrow{B}\times\overrightarrow{A}$. Every quaternion has its conjugate, then the quaternionic conjugate of equation (12) will be

$$\mathbb{H}^{*}=e_{0}w-(e_{1}x+e_{2}y+e_{3}z)\,. \tag{33}$$

From the above definitions, we can easily define the following property

$$\begin{aligned}(\mathbb{H}^{*})^{*}&=e_{0}w-(-e_{1}x-e_{2}y-e_{3})\\&=\mathbb{H}.\end{aligned} \tag{34}$$

Equation (34) signifies that the conjugate of a quaternionic conjugate will always a quaternion. The addition of quaternion and its quaternionic conjugate will be

$$\begin{aligned}\mathbb{H}+\mathbb{H}^{*}&=(e_{0}w+e_{1}x+e_{2}y+e_{3}z)+(e_{0}w-e_{1}x-e_{2}y-e_{3}z)\\&=2e_{0}w.\end{aligned} \tag{35}$$

It is clear that if we add a quaternion with its quaternionic conjugate it gives twice times of its scalar part. Now, similarly the multiplication of quaternion with its quaternionic conjugate will be

$$\begin{aligned}\mathbb{H}\circ\mathbb{H}^{*}&=(e_{0}w+e_{1}x+e_{2}y+e_{3}z)\circ(e_{0}w-e_{1}x-e_{2}y-e_{3}z)\\&=e_{0}^{2}w^{2}-e_{1}^{2}x^{2}-e_{2}^{2}y^{2}-e_{3}^{2}z^{2}\\&=w^{2}+x^{2}+y^{2}+z^{2}\\&=\mathbb{H}^{*}\circ\mathbb{H}\,.\end{aligned} \tag{36}$$

Above equation signifies that the quaternionic multiplication of a quaternion and its quaternionic conjugate is always commutative. Similarly, we can product the quaternion $\mathbb{B}^{*}$ with quaternion $\mathbb{A}^{*}$, and get the following property

$$\mathbb{B}^{*}\circ\mathbb{A}^{*}=(e_{0}B_{0}-e_{1}B_{1}-e_{2}B_{2}-e_{3}B_{3})\circ(e_{0}A_{0}-e_{1}A_{1}-e_{2}A_{2}-e_{3}A_{3})$$

$$
\begin{aligned}
&= e_0\left(A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3\right)\\
&- e_1\left(A_0B_1 + A_1B_0 + A_2B_3 - A_3B_2\right)\\
&- e_2\left(A_0B_2 + A_2B_0 + A_3B_1 - A_1B_3\right)\\
&- e_3\left(A_0B_3 + A_3B_0 + A_1B_2 - A_2B_1\right)\\
&= e_0P_0 - e_1P_1 - e_2P_2 - e_3P_3\\
&= \left(\mathbb{A}\circ\mathbb{B}\right)^* . \qquad (37)
\end{aligned}
$$

The norm of a quaternion $\mathbb{H}$ denoted by $|\mathbb{H}|$ will be represented as

$$
\begin{aligned}
|\mathbb{H}| &= \sqrt{\mathbb{H}\circ\mathbb{H}^*}\\
&= \sqrt{w^2 + x^2 + y^2 + z^2} . \qquad (38)
\end{aligned}
$$

We can also define the norm of the product of two quaternions

$$
\begin{aligned}
|\mathbb{A}\mathbb{B}|^2 &= (\mathbb{A}\mathbb{B})\circ(\mathbb{A}\mathbb{B})^* = \mathbb{A}\left(\mathbb{B}\circ\mathbb{B}^*\right)\mathbb{A}^* = \left(\mathbb{A}\circ\mathbb{A}^*\right)|\mathbb{B}|^2 = |\mathbb{A}|^2\circ|\mathbb{B}|^2\\
&= |\mathbb{A}|^2\circ|\mathbb{B}|^2\\
|\mathbb{A}\mathbb{B}| &= |\mathbb{A}|\circ|\mathbb{B}| . \qquad (39)
\end{aligned}
$$

From the above expression it is clear that the norm of the product of two quaternion will be equal to the product of individual norms of quaternions $\mathbb{A}$ and $\mathbb{B}$. Now, the inverse of a quaternion will be defined as

$$
\begin{aligned}
\mathbb{H}^{-1} &= \frac{\mathbb{H}^*}{|\mathbb{H}|^2}\\
&= \frac{e_0w - e_1x - e_2y - e_3z}{w^2 + x^2 + y^2 + z^2} . \qquad (40)
\end{aligned}
$$

We can also verify that

$$
\mathbb{H}^{-1}\mathbb{H} = \mathbb{H}\mathbb{H}^{-1} = 1 . \qquad (41)
$$

This above expression shows that the multiplication of quaternion and inverse of a quaternion is equal to the unity.

3.2.3 Another general decomposition of quaternion:

According to **Hamilton, W. R. (1853)**, the quotient of the two vectors is known as quaternion which will be represented as

$$\mathbb{H} = \frac{\alpha}{\beta}, \tag{42}$$

where, $\mathbb{H}$ is a quaternion, α and β are the two vectors. The Hamilton tensor is, in fact the absolute value of the quaternion algebra, which makes it a normalised vector space. Hamilton defines the tensor as a positive quantity, or rather an unsigned number **(Tait, P. G. 1890)**. Tensor of a quaternion $(\mathbf{T}_{\mathbb{H}})$ can be written as

$$\begin{aligned} (\mathbf{T}_{\mathbb{H}})^2 &= \mathbb{H}\mathbb{H}^* \\ &= w^2 + x^2 + y^2 + z^2 \\ \mathbf{T}_{\mathbb{H}} &= \sqrt{(w^2 + x^2 + y^2 + z^2)}. \end{aligned} \tag{43}$$

Tensor of a quaternion is same as the norm of a quaternion, it will be always positive or real number **(Weinsberg, P. 2014)**. If scalar part is zero then

$$\mathbf{T}(\mathbf{V}_{\mathbb{H}}) = \sqrt{(x^2 + y^2 + z^2)}, \tag{44}$$

and if vector part is zero then

$$\mathbf{T}(\mathbf{S}_{\mathbb{H}}) = w, \tag{45}$$

where, $\mathbf{T}(\mathbf{V}_{\mathbb{H}})$ is known as tensor of a vector quaternion and $\mathbf{T}(\mathbf{S}_{\mathbb{H}})$ is known as tensor of a scalar quaternion. The 'versor' of a quaternion which indicates its direction, it is a unit quaternion and also known as normalised quaternion. It is denoted by $\mathbf{U}_{\mathbb{H}}$ as **(Hamilton, W. E. 1866)**

$$\begin{aligned} \mathbf{U}_{\mathbb{H}} &= \frac{\mathbb{H}}{|\mathbb{H}|} \\ &= \frac{e_0 w + e_1 x + e_2 y + e_3 z}{\sqrt{(w^2 + x^2 + y^2 + z^2)}}. \end{aligned} \tag{46}$$

Scalar and vector of a quaternion versor can be expressed as

$$\mathbf{S}\left(\mathbf{U}_{\mathbb{H}}\right)=\frac{w}{\sqrt{\left(w^{2}+x^{2}+y^{2}+z^{2}\right)}}\,, \tag{47}$$

$$\mathbf{V}\left(\mathbf{U}_{\mathbb{H}}\right)=\frac{e_{1}x+e_{2}y+e_{3}z}{\sqrt{\left(w^{2}+x^{2}+y^{2}+z^{2}\right)}}\,. \tag{48}$$

Quaternion can also be written in terms of tensor of a quaternion and versor of a quaternion as **(Hamilton, W. E. 1866)**

$$\begin{aligned}\mathbb{H}=\mathbf{T}_{\mathbb{H}}\mathbf{U}_{\mathbb{H}}= & \sqrt{\left(w^{2}+x^{2}+y^{2}+z^{2}\right)}\frac{\left(e_{0}w+e_{1}x+e_{2}y+e_{3}z\right)}{\sqrt{\left(w^{2}+x^{2}+y^{2}+z^{2}\right)}}\\ = & \,e_{0}w+e_{1}x+e_{2}y+e_{3}z\,.\end{aligned} \tag{49}$$

3.3 Quaternionic basis transformations

When any system of coordinates displaced from its original frame of reference then there will be change in unit vector with respect to original unit vector. This change of unit vectors represent the transformation equations. Basically, the transformation equations are those equations which tells how the basis transform from one frame to another frame. Starting with the transformation equations for 2-dimensional system in which the equations are used for the transformation of coordinates from one plane to another plane **(Goldstein, H. 1950)**, *i.e.*,

$$x^{'}=\,x\cos\phi+y\sin\phi\,,\,y^{'}=\,-x\sin\phi+y\cos\phi\,, \tag{50}$$

which gives

$$\begin{aligned}\frac{\partial x^{'}}{\partial x}=\cos\phi, & \quad\frac{\partial x^{'}}{\partial y}=\,\sin\phi\\ \frac{\partial y^{'}}{\partial x}=-\sin\phi, & \quad\frac{\partial y^{'}}{\partial y}=\,\cos\phi\,.\end{aligned}$$

Similarly, in the same way the transformation of basis can be written as in matrix form such that

$$\left(\begin{array}{c} e_{1}^{'}\\ e_{2}^{'}\end{array}\right)=\left(\begin{array}{cc} \cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{array}\right)\left(\begin{array}{c} e_{1}\\ e_{2}\end{array}\right), \tag{51}$$

which leads to

$$e_1^{'} = \frac{\partial x^{'}}{\partial x} e_1 + \frac{\partial x^{'}}{\partial y} e_2,$$
$$e_2^{'} = \frac{\partial y^{'}}{\partial x} e_1 + \frac{\partial y^{'}}{\partial y} e_2, \tag{52}$$

where e_1 and e_2 are the unit vectors corresponding to x and $y-$ axis in $XY-$ plane, while $e_1^{'}$ and $e_2^{'}$ are the unit vectors corresponding to $x^{'}$ and $y^{'}-$ axis in $X^{'}Y^{'}-$ plane, respectively. ϕ is the angle between these two planes.

Like 2-D transformation, we define 3-D transformation of basis which derived from rotating frame in three-dimensional Euler angles. Thus, the 3-D transformation matrix (D) can be written by **(Goldstein, H. 1950),**

$$D = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta & -\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & -\cos\psi\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}, \tag{53}$$

where ψ, θ, ϕ are the three independent parameters known as Euler angles which represents the rotation of axis in new frame of reference with respect to original axis. Let (e_1, e_2, e_3) are unit vectors in (x, y, z) axis denoted by $S-$ frame and $\left(e_1^{'}, e_2^{'}, e_3^{'}\right)$ are unit vectors in $\left(x^{'}, y^{'}, z^{'}\right)$ axis for $S^{'}-$ frame. Using equation (53) we may write the following $3D-$transformation as **(Brand, L. 1947)**

$$e_1^{'} = \frac{\partial x^{'}}{\partial x} e_1 + \frac{\partial x^{'}}{\partial y} e_2 + \frac{\partial x^{'}}{\partial z} e_3,$$
$$e_2^{'} = \frac{\partial y^{'}}{\partial x} e_1 + \frac{\partial y^{'}}{\partial y} e_2 + \frac{\partial y^{'}}{\partial z} e_3,$$
$$e_3^{'} = \frac{\partial z^{'}}{\partial x} e_1 + \frac{\partial z^{'}}{\partial y} e_2 + \frac{\partial z^{'}}{\partial z} e_3, \tag{54}$$

where

$$\frac{\partial x^{'}}{\partial x} = \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, \qquad \frac{\partial x^{'}}{\partial y} = -\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi,$$

$$\frac{\partial x'}{\partial z}=\sin\psi\sin\theta,\qquad \frac{\partial y'}{\partial x}=\ \sin\psi\cos\phi+\cos\psi\sin\phi\cos\theta\,,$$
$$\frac{\partial y'}{\partial y}=-\sin\psi\sin\phi+\cos\psi\cos\theta\cos\phi,\qquad \frac{\partial y'}{\partial z}=\ -\cos\psi\sin\theta\,,$$
$$\frac{\partial z'}{\partial x}=\sin\theta\sin\phi,\qquad \frac{\partial z'}{\partial y}=\ \sin\theta\cos\phi,\qquad \frac{\partial z'}{\partial z}=\ \cos\theta\,.$$

Interestingly, these transformation equations satisfy the properties of vector algebra. The cross product of two vectors is always a non-commutative, this property is also shown by pure quaternion. Quaternions have some different properties also form vector algebra with respect to multiplication. So, when these unit basis follows the properties of quaternion shown in equation (25) they are called the unit basis of quaternions. As 4-D quaternion, having four unit basis (e_0,e_1,e_2,e_3) in which e_0 represents a scalar unit and (e_1,e_2,e_3) represents the imaginary units corresponding to vector unit basis. If we vanish the scalar part then the remaining part will show only vector part which is known as pure quaternion. We take 3-D transformations for pure quaternion because of the same behaviour of pure quaternionic algebra to the vector algebra in 3-D. The properties of pure quaternionic unit elements (e_1,e_2,e_3) corresponding to vector algebra basis can be represented as

$$\begin{aligned}
e_1'\circ e_1' &= \left(\frac{\partial x'}{\partial x}e_1+\frac{\partial x'}{\partial y}e_2+\frac{\partial x'}{\partial z}e_3\right)\circ\left(\frac{\partial x'}{\partial x}e_1+\frac{\partial x'}{\partial y}e_2+\frac{\partial x'}{\partial z}e_3\right)\\
e_1'\circ e_1' &= -\left(\frac{\partial x'}{\partial x}\right)^2-\left(\frac{\partial x'}{\partial y}\right)^2-\left(\frac{\partial x'}{\partial z}\right)^2\\
&= -\left(\cos\psi\cos\phi-\sin\psi\sin\phi\cos\theta\right)^2-\left(-\cos\psi\sin\phi-\sin\psi\cos\theta\cos\phi\right)^2\\
&- \ (\sin\psi\sin\theta)^2=\ -1\,.
\end{aligned}\tag{55}$$

Similarly, we have

$$e_2'\circ e_2'=\ -1,\quad e_3'\circ e_3'=\ -1\,.\tag{56}$$

On the other hand,

$$e_1'\circ e_2'=\left(\frac{\partial x'}{\partial x}e_1+\frac{\partial x'}{\partial y}e_2+\frac{\partial x'}{\partial z}e_3\right)\circ\left(\frac{\partial y'}{\partial x}e_1+\frac{\partial y'}{\partial y}e_2+\frac{\partial y'}{\partial z}e_3\right)$$

$$
\begin{aligned}
&= -\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial z}\frac{\partial y'}{\partial z} - \left(\frac{\partial x'}{\partial z}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial z}\right) e_1 \\
&- \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial z} - \frac{\partial x'}{\partial z}\frac{\partial y'}{\partial x}\right) e_2 - \left(\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x} - \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}\right) e_3 \\
&= -(\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta) \\
&- (-\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi)(-\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi) \\
&- (\sin\psi\sin\theta)(-\cos\psi\sin\theta) - e_1[(\sin\psi\sin\theta)(-\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi) \\
&+ (-\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi)(-\cos\psi\sin\theta)] \\
&- e_2[(\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)(-\cos\psi\sin\theta) \\
&+ (\sin\psi\sin\theta)(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta)] \\
&- e_3[(-\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi)(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta) \\
&+ (\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)(-\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi)] \\
&= e_3', \qquad (57)
\end{aligned}
$$

and,

$$
\begin{aligned}
e_2' \circ e_1' &= \left(\frac{\partial y'}{\partial x}e_1 + \frac{\partial y'}{\partial y}e_2 + \frac{\partial y'}{\partial z}e_3\right) \circ \left(\frac{\partial x'}{\partial x}e_1 + \frac{\partial x'}{\partial y}e_2 + \frac{\partial x'}{\partial z}e_3\right) \\
&= -\frac{\partial y'}{\partial x}\frac{\partial x'}{\partial x} - \frac{\partial y'}{\partial y}\frac{\partial x'}{\partial y} - \frac{\partial y'}{\partial z}\frac{\partial x'}{\partial z} + \left(\frac{\partial y'}{\partial y}\frac{\partial x'}{\partial z} - \frac{\partial y'}{\partial z}\frac{\partial x'}{\partial y}\right) e_1 \\
&+ \left(\frac{\partial y'}{\partial z}\frac{\partial x'}{\partial x} - \frac{\partial y'}{\partial x}\frac{\partial x'}{\partial z}\right) e_2 + \left(\frac{\partial y'}{\partial x}\frac{\partial x'}{\partial y} - \frac{\partial y'}{\partial y}\frac{\partial x'}{\partial x}\right) e_3 \\
&= -(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta)(\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta) \\
&- (-\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi)(-\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi) \\
&- (-\cos\psi\sin\theta)(\sin\psi\sin\theta) + e_1[(\sin\psi\sin\theta)(-\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi) \\
&- (-\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi)(-\cos\psi\sin\theta)] \\
&+ e_2[(\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)(-\cos\psi\sin\theta) \\
&- (\sin\psi\sin\theta)(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta)] \\
&+ e_3[(-\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi)(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta) \\
&- (\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)(-\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi)] = -e_3'. \qquad (58)
\end{aligned}
$$

Similarly, we have

$$e_2^{'} \circ e_3^{'} = e_1^{'}\,,\;\; e_3^{'} \circ e_2^{'} = \; -e_1^{'}\,,$$
$$e_3^{'} \circ e_1^{'} = e_2^{'}\,,\;\; e_1^{'} \circ e_3^{'} = \; -e_2^{'}\,. \tag{59}$$

Now, we can extent the pure quaternion to quaternion (4-D rotation) by adding the scalar (e_0) unit. **Myszkowski, M. (2019)** have give the idea of 4-D rotation similar to the 3-D rotation. In 4-D quaternionic rotation, we assume that the transformations can be written as

$$\begin{aligned} e_0^{'} &= e_0\,, \\ e_1^{'} &= \frac{\partial x^{'}}{\partial t}e_0 + \frac{\partial x^{'}}{\partial x}e_1 + \frac{\partial x^{'}}{\partial y}e_2 + \frac{\partial x^{'}}{\partial z}e_3\,, \\ e_2^{'} &= \frac{\partial y^{'}}{\partial t}e_0 + \frac{\partial y^{'}}{\partial x}e_1 + \frac{\partial y^{'}}{\partial y}e_2 + \frac{\partial y^{'}}{\partial z}e_3\,, \\ e_3^{'} &= \frac{\partial z^{'}}{\partial t}e_0 + \frac{\partial z^{'}}{\partial x}e_1 + \frac{\partial z^{'}}{\partial y}e_2 + \frac{\partial z^{'}}{\partial z}e_3\,. \end{aligned} \tag{60}$$

These are the required quaternionic basis transformations which will be use to formulate the Einstein field equation. In the above transformation equations, imaginary basis of quaternion transforms with respect to other four basis of different frame of reference with there components but scalar unit basis will not transform. According to quaternionic property in equation (25) scalar unit basis (e_0) having the unit value which cannot be transformed from one to another frame of reference.

3.4 Quaternionic formulation of Riemannian geometry

3.4.1 Quaternion transformation:

In this geometry, the motion of object takes place in curvature space-time in which 4-dimensional frame of reference rotate along the curve. The structure of four space-time coordinates can be written as $P^{\mu} = \left(P^{\xi}, P^1, P^2, P^3\right)$ where, $\mu = (\xi, j)$. The time coordinate may refer corresponding to e_0 e.g. P^{ξ},and the spatial coordinates may refer corresponding to e_j e.g. $\left(P^1, P^2, P^3\right)$ where, $j = 1, 2, 3$. The basis transformation of quaternion can be written as

$$e_0^{'} = e_0,$$
$$e_i^{'} = \frac{\partial P^{\mu}}{\partial P^{'i}} e_{\mu}, \qquad (\forall \;\; i = \; 1,2,3)\,, \tag{61}$$

where $P^{'\mu} = \left(P^{'\xi}, P^{'1}, P^{'2}, P^{'3}\right)$ represented four spatial-temporal structure for other coordinate system. Now, quaternionic transformation may derive with the help of basis transformation from one reference frame to another reference frame similar to the vector transformation **(Heinbockel, J. H. 1996; Gron, O. and Hervik, S. 2007)** *i.e.*,

$$\mathbb{H} = e_{\mu} H^{\mu} = \; e_{\mu}^{'} H^{'\mu}, \qquad (\forall \;\; \mu = \; \xi, 1,2,3)\,. \tag{62}$$

Equation (62) implies that the flat quaternionic space-time equation transform linearly with $e_{\mu} = \; e_{\mu}^{'}$ *i.e.*, similar to vector transform in flat space **(Bertschinger, C. E. 1999)**.

$$\mathbb{H} = \; e_0 H^{\xi} + e_i H^{i} \rightarrow \; e_0^{'} H^{'\xi} + e_i^{'} H^{'i}, \tag{63}$$

where $\left(H^{\xi}, H^{i}\right)$ are quaternionic scalar and vector components in $S-$ frame while $\left(H^{'\xi}, H^{'i}\right)$ are quaternionic scalar and vector components in $S^{'}-$ frame. Correspondingly, in curved space-time the quaternion may transform by using equation (61) we get

$$\begin{aligned} e_0 H^{\xi} + e_i H^{i} &= e_0 H^{'\xi} + \left(\frac{\partial P^{\mu}}{\partial P^{'i}} e_{\mu}\right) H^{'i} = \; e_0 H^{'\xi} + \frac{\partial P^{\xi}}{\partial P^{'i}} e_0 H^{'i} + \frac{\partial P^{j}}{\partial P^{'i}} e_j H^{'i} \\ &= e_0 \left(H^{'\xi} + \frac{\partial P^{\xi}}{\partial P^{'i}} H^{'i}\right) + e_j \left(\frac{\partial P^{j}}{\partial P^{'i}} H^{'i}\right)\,. \end{aligned} \tag{64}$$

On comparing coefficient of quaternionic basis elements given in equation (64), we obtain

$$H^{\xi} = H^{'\xi} + \frac{\partial P^{\xi}}{\partial P^{'i}} H^{'i} \qquad \text{(coefficient of } e_0) \tag{65}$$
$$H^{j} = \frac{\partial P^{j}}{\partial P^{'i}} H^{'i} \qquad \text{(coefficient of } e_j)\,. \tag{66}$$

Equations (65) and (66) show the transformation of scalar and vector components of a quaternion from $S^{'}$ to S frame. Thus,

$$\mathbb{H} \longmapsto \mathbb{H}^{'} = \left(H^{'\xi} + \frac{\partial P^{\xi}}{\partial P^{'i}} H^{'i}, \ \frac{\partial P^{j}}{\partial P^{'i}} H^{'i}\right). \tag{67}$$

Here, in the above transformation of scalar component the effect of $H^{'\xi}$ component on H^{ξ} is negligible because it shows the simple transformation on flat space-time. So, we can neglect the $H^{'\xi}$ component to get the quaternionic transformation in curved space-time only. Therefore,

$$\mathbb{H} \longmapsto \mathbb{H}^{'} = \left(\frac{\partial P^{\xi}}{\partial P^{'i}} H^{'i}, \ \frac{\partial P^{j}}{\partial P^{'i}} H^{'i}\right). \tag{68}$$

Equation (68) represents transformation in contravariant form, we can also write the above transformation in quaternionic covariant form as **(Kumar, A. 2010),**

$$H_{\xi} = \frac{\partial P^{'i}}{\partial P^{\xi}} H_{i}^{'} \qquad \text{(coefficient of } e_0\text{)} \tag{69}$$

$$H_{j} = \frac{\partial P^{'i}}{\partial P^{j}} H_{i}^{'} \qquad \text{(coefficient of } e_j\text{)}. \tag{70}$$

Now, we can write the quaternionic differential operator $\Box$ as **(Gutierrez, S. E. A. 2016)**,

$$\Box = \left(e_0 \partial_{\phi} + e_k \partial_k\right) \rightarrow \left(e_0^{'} \partial_{\phi}^{'} + e_k^{'} \partial_k^{'}\right), \tag{71}$$

where $k = 1,2,3;$ $\left(\partial_{\phi} = \frac{\partial}{\partial P^{\phi}}, \partial_k = \frac{\partial}{\partial P^{k}}\right)$ are quaternionic scalar and vector partial derivatives in $S-$ frame while $\left(\partial_{\phi}^{'} = \frac{\partial}{\partial P^{'\phi}}, \partial_k^{'} = \frac{\partial}{\partial P^{'k}}\right)$ are quaternionic scalar and vector partial derivatives in $S^{'}-$ frame. By using transformation of quaternionic basis given in equation (61), we get,

$$\begin{aligned} \Box &= e_0 \partial_{\phi}^{'} + \frac{\partial P^{\lambda}}{\partial P^{'k}} e_{\lambda} \partial_k^{'} = e_0 \partial_{\phi}^{'} + \frac{\partial P^{\phi}}{\partial P^{'k}} e_0 \partial_k^{'} + \frac{\partial P^{l}}{\partial P^{'k}} e_l \partial_k^{'} \\ &= e_0 \left(\partial_{\phi}^{'} + \frac{\partial P^{\phi}}{\partial P^{'k}} \partial_k^{'}\right) + e_l \left(\frac{\partial P^{l}}{\partial P^{'k}} \partial_k^{'}\right). \end{aligned} \tag{72}$$

On comparing quaternionic scalar and vector parts in given equation (72), we obtain

$$\partial_{\phi} = \frac{\partial P^{\phi}}{\partial P^{'k}} \partial_k^{'}, \tag{73}$$

$$\partial_l = \frac{\partial P^l}{\partial P'^k}\partial_k^{'}. \tag{74}$$

Here, we have neglected the effect of $\partial_\phi^{'}$ for curved space-time transformation. Since, components of quaternion can also represented in form of coordinates, therefore, we can write the transformation of quaternionic coordinates **(Edmonds, J. D. 1974)**, as

$$dP^\xi = \frac{\partial P^\xi}{\partial P^{'i}}dP^{'i}, \tag{75}$$

$$dP^j = \frac{\partial P^j}{\partial P^{'i}}dP^{'i}, \quad \forall\,(i, j = 1,2,3)\,. \tag{76}$$

According to Einstein's principle of general covariance **(Pathria, R. K. 1974),** all the required laws of physics can be expressed in terms of covariant form using tensors. Similarly, product of two scalar component of quaternionic tensor $\left(T_{\xi\xi}\right)$ will be the transformation as

$$T_{\xi\xi} = H_\xi H_\xi = \left(\frac{\partial P^{'i}}{\partial P^\xi}H_i^{'}\right)\left(\frac{\partial P^{'i}}{\partial P^\xi}H_i^{'}\right) = \frac{\partial P^{'i}}{\partial P^\xi}\frac{\partial P^{'i}}{\partial P^\xi}H_i^{'}H_i^{'} = \frac{\partial P^{'i}}{\partial P^\xi}\frac{\partial P^{'i}}{\partial P^\xi}T_{ii}^{'}. \tag{77}$$

The product of one scalar and vector component of quaternionic tensor $\left(T_{\xi n}\right)$ will be, transform as

$$T_{\xi n} = H_\xi H_n = \left(\frac{\partial P^{'i}}{\partial P^\xi}H_i^{'}\right)\left(\frac{\partial P^{'m}}{\partial P^n}H_m^{'}\right) = \frac{\partial P^{'i}}{\partial P^\xi}\frac{\partial P^{'m}}{\partial P^n}H_i^{'}H_m^{'} = \frac{\partial P^{'i}}{\partial P^\xi}\frac{\partial P^{'m}}{\partial P^n}T_{im}^{'}. \tag{78}$$

The product of one vector and scalar component of quaternionic tensor $\left(T_{j\xi}\right)$ will be, transform as

$$T_{j\xi} = H_j H_\xi = \left(\frac{\partial P^{'i}}{\partial P^j}H_i^{'}\right)\left(\frac{\partial P^{'i}}{\partial P^\xi}H_i^{'}\right) = \frac{\partial P^{'i}}{\partial P^j}\frac{\partial P^{'i}}{\partial P^\xi}H_i^{'}H_i^{'} = \frac{\partial P^{'i}}{\partial P^j}\frac{\partial P^{'i}}{\partial P^\xi}T_{ii}^{'}. \tag{79}$$

Similarly, product of two vector components of quaternionic tensor $\left(T_{jn}\right)$ gives the following transformation,

$$T_{jn} = H_j H_n = \left(\frac{\partial P^{'i}}{\partial P^j}H_i^{'}\right)\left(\frac{\partial P^{'m}}{\partial P^n}H_m^{'}\right) = \frac{\partial P^{'i}}{\partial P^j}\frac{\partial P^{'m}}{\partial P^n}H_i^{'}H_m^{'} = \frac{\partial P^{'i}}{\partial P^j}\frac{\partial P^{'m}}{\partial P^n}T_{im}^{'}. \tag{80}$$

Equations (77)-(80) are the required transformation components of the quaternionic tensors of rank-2. It is known as dyadic product of components of two quaternions **(Kelly, P. 2013).**

3.4.2 Quaternionic covariant derivative:

In curved space-time partial derivative extended to the covariant derivative with some additional term. On transporting different points in curved space-time covariant derivative shows the change in field. In space-time curvature structure the quaternionic derivative can also shows the curvature form. Now, applying $\partial_{\phi}$from equation (73) to quaternionic scalar transformation given in equation (68) as

$$\partial_{\phi}H^{\xi}=\frac{\partial P'^{l}}{\partial P^{\phi}}\partial_{l}^{'}\left(\frac{\partial P^{\xi}}{\partial P'^{i}}H'^{i}\right)=\frac{\partial P'^{l}}{\partial P^{\phi}}\frac{\partial P^{\xi}}{\partial P'^{i}}\partial_{l}^{'}H'^{i}+\frac{\partial P'^{l}}{\partial P^{\phi}}\partial_{l}^{'}\left(\frac{\partial P^{\xi}}{\partial P'^{i}}\right)H'^{i}$$
$$=\frac{\partial P'^{l}}{\partial P^{\phi}}\frac{\partial P^{\xi}}{\partial P'^{i}}\partial_{l}^{'}H'^{i}+\frac{\partial^{2}P^{\xi}}{\partial P^{\phi}\partial P^{\phi}}H^{\phi}\,. \tag{81}$$

In equation (81), we see that the quaternionic transformation of $\partial_{\phi}H^{\xi}$ is not transform according to quaternionic condition (77). The extra term $\frac{\partial}{\partial P^{\phi}}\left(\frac{\partial P^{\xi}}{\partial P^{\phi}}\right)$ shows the change in one tangent plane to another tangent plane in curvature space-time. Therefore,

$$D_{\phi}H^{\xi}=\partial_{\phi}H^{\xi}+\Gamma_{\phi\phi}^{\xi}H^{\phi}\,, \tag{82}$$

where D_{ϕ} represents the quaternionic covariant derivative, ∂_{ϕ} is the ordinary quaternionic partial derivative, and $\Gamma_{\phi\phi}^{\xi}$ represents the three-index symbol known as christoffel symbols. Thus,

$$\Gamma_{\phi\phi}^{\xi}=\frac{\partial}{\partial P^{\phi}}\left(\frac{\partial P^{\xi}}{\partial P^{\phi}}\right)\,. \tag{83}$$

This covariant derivative is also known as dyads of rank-2. If the extra term $\Gamma_{\phi\phi}^{\xi}\rightarrow0$, then the quaternionic covariant derivative $\left(D_{\phi}\right)$ transforms into the quaternionic partial derivative $\left(\partial_{\phi}\right)$. The quaternionic covariant derivative $D_{\phi}H^{\xi}$ is often denoted by $H_{;\phi}^{\xi}$, with a semicolon which is placed in front of the indices connected to the direction along which it will differentiates. In the same way, the partial derivative $\partial_{\phi}H^{\xi}$ is denoted by $H_{,\phi}^{\xi}$**.** Equation (82) can be written as **(Das, A. and DeBenedictis, A. 2012)**

$$H_{;\phi}^{\xi}=H_{,\phi}^{\xi}+\left\{ \begin{array}{c} \xi\\ \phi\phi \end{array}\right\} H^{\phi}\,, \tag{84}$$

where this covariant derivatives involve the christoffel symbol $\Gamma^{\xi}_{\phi\phi}$ is also denoted by $\begin{Bmatrix} \xi \\ \phi\phi \end{Bmatrix}$.

The quaternionic covariant derivative of a scalar field can be expressed as

$$H_{\xi;\phi}=H_{\xi,\phi}-\begin{Bmatrix} \phi \\ \phi\xi \end{Bmatrix}H_{\phi}\,. \tag{85}$$

Now, applying quaternionic partial derivative ∂_{ϕ} from equation (73) into vector component of quaternion H^{j} from equation (68) as

$$D_{\phi}H^{j}=\frac{\partial P^{'l}}{\partial P^{\phi}}\partial_{l}^{'}\left(\frac{\partial P^{j}}{\partial P^{'i}}H^{'i}\right)=\frac{\partial P^{'l}}{\partial P^{\phi}}\frac{\partial P^{j}}{\partial P^{'i}}\partial_{l}^{'}H^{'i}+\frac{\partial P^{'l}}{\partial P^{\phi}}\partial_{l}^{'}\left(\frac{\partial P^{j}}{\partial P^{'i}}\right)H^{'i}, \tag{86}$$

which may expressed as

$$H^{j}_{;\phi}=H^{j}_{,\phi}+\begin{Bmatrix} j \\ \phi\phi \end{Bmatrix}H^{\phi}\,. \tag{87}$$

Equation (87) shows the contravariant quaternionic vector transformation. In covariant form, the quaternionic rank -2 transformation can be written as

$$H_{j;\phi}=H_{j,\phi}-\begin{Bmatrix} \phi \\ \phi j \end{Bmatrix}H_{\phi}\,. \tag{88}$$

Similarly, we can show the change in the contravariant scalar and vector field with respect to the vector component of quaternionic covariant derivative as

$$H^{\xi}_{;k}=H^{\xi}_{,k}+\begin{Bmatrix} \xi \\ km \end{Bmatrix}H^{m}, \tag{89}$$

$$H^{j}_{;k}=H^{j}_{,k}+\begin{Bmatrix} j \\ km \end{Bmatrix}H^{m}\,. \tag{90}$$

Equation (89) signifies that when the vector operator as a differential applied to scalar functions (scalar fields) it is a vector field **(Kelly, P. 2013)**. In general, the components of $H^{\xi}_{;k}$ in any direction is the rate of change of scalar function in that direction. Equation (90) is the quaternionic covariant derivative of a quaternionic vector field H_j. We can also write the equation (89) and (90) in covariant form as

$$H_{\xi;k}=H_{\xi,k}-\left\{\begin{array}{c}m\\ k\xi\end{array}\right\}H^{m}\,, \tag{91}$$

$$H_{j;k}=H_{j,k}-\left\{\begin{array}{c}m\\ kj\end{array}\right\}H^{m}\,. \tag{92}$$

The main advantage of the quaternionic covariant derivative is that it gives enlargements to rank of quaternionic tensors because here the quaternionic covariant derivative increases the order (rank) of quaternionic tensors as it gives the information about the change with respect to another coordinate in possibly all space-time directions.. Using ∂_ϕ given in equation (73) on quaternionic scalar rank -2 tensor $\left(T_{\xi\xi}\right)$ as

$$\partial_{\phi}\left(T_{\xi\xi}\right)=\frac{\partial P^{\phi}}{\partial P^{\prime l}}\partial_{l}^{\prime}\left(\frac{\partial P^{\prime i}}{\partial P^{\xi}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}T_{ii}^{\prime}\right)=\frac{\partial P^{\phi}}{\partial P^{\prime l}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}\frac{\partial T_{ii}^{\prime}}{\partial P^{\prime l}}-\frac{\partial}{\partial P^{\phi}}\left(\frac{\partial P^{\eta}}{\partial P^{\xi}}\right)\frac{\partial P^{\prime i}}{\partial P^{\eta}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}T_{ii}^{\prime}$$
$$-\frac{\partial}{\partial P^{\phi}}\left(\frac{\partial P^{\eta}}{\partial P^{\xi}}\right)\frac{\partial P^{\prime i}}{\partial P^{\eta}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}T_{ii}^{\prime}, \tag{93}$$

which gives

$$T_{\xi\xi;\phi}=T_{\xi\xi,\phi}-\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}T_{\xi\eta}-\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}T_{\eta\xi}\,. \tag{94}$$

Equation (94) is simply an extension of quaternionic rank -2 tensor which gives rank -3 tensor, where last two terms show scalar variable (ϕ) change is being considered in tangent planes with respect to scalar variables (ξ) from curvature space-time. Now, using ∂_k given in equation (74) on quaternionic scalar rank -2 tensor $\left(T_{\xi\zeta}\right)$ as

$$\partial_{k}\left(T_{\xi\xi}\right)=\frac{\partial P^{k}}{\partial P^{\prime l}}\partial_{l}^{\prime}\left(\frac{\partial P^{\prime i}}{\partial P^{\xi}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}T_{ii}^{\prime}\right)=\frac{\partial P^{\phi}}{\partial P^{\prime l}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}\frac{\partial T_{ii}^{\prime}}{\partial P^{\prime l}}-\frac{\partial}{\partial P^{k}}\left(\frac{\partial P^{l}}{\partial P^{\xi}}\right)\frac{\partial P^{\prime i}}{\partial P^{l}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}T_{ii}^{\prime}$$
$$-\frac{\partial}{\partial P^{k}}\left(\frac{\partial P^{l}}{\partial P^{\xi}}\right)\frac{\partial P^{\prime i}}{\partial P^{l}}\frac{\partial P^{\prime i}}{\partial P^{\xi}}T_{ii}^{\prime}, \tag{95}$$

which gives

$$T_{\xi\xi;k} = T_{\xi\xi,k} - \left\{ \begin{array}{c} l \\ k\xi \end{array} \right\} T_{\xi l} - \left\{ \begin{array}{c} l \\ k\xi \end{array} \right\} T_{l\xi} \,. \tag{96}$$

In equation (96) last two terms are showing an extra terms in vector variable (k), which may considered to changes in tangent planes with respect to scalar variables (ξ) in curvature space-time. Similarly, we can analysis to change in quaternionic rank -2 tensors given by equation (78)-(80) with respect to scalar and vector components of quaternionic covariant derivative as

$$T_{\xi n;\phi} = T_{\xi n,\phi} - \left\{ \begin{array}{c} \eta \\ \phi n \end{array} \right\} T_{\xi\eta} - \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} T_{\eta n} \,, \tag{97}$$

$$T_{\xi n;k} = T_{\xi n,k} - \left\{ \begin{array}{c} l \\ kn \end{array} \right\} T_{\xi l} - \left\{ \begin{array}{c} l \\ k\xi \end{array} \right\} T_{ln} \,, \tag{98}$$

$$T_{j\xi;\phi} = T_{j\xi,\phi} - \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} T_{j\eta} - \left\{ \begin{array}{c} \eta \\ \phi j \end{array} \right\} T_{\eta\xi} \,, \tag{99}$$

$$T_{j\xi;k} = T_{j\xi,k} - \left\{ \begin{array}{c} l \\ k\xi \end{array} \right\} T_{jl} - \left\{ \begin{array}{c} l \\ kj \end{array} \right\} T_{l\xi} \,, \tag{100}$$

$$T_{jn;\phi} = T_{jn,\phi} - \left\{ \begin{array}{c} \eta \\ \phi n \end{array} \right\} T_{j\eta} - \left\{ \begin{array}{c} \eta \\ \phi j \end{array} \right\} T_{\eta n} \,, \tag{101}$$

$$T_{jn;k} = T_{jn,k} - \left\{ \begin{array}{c} l \\ kn \end{array} \right\} T_{jl} - \left\{ \begin{array}{c} l \\ kj \end{array} \right\} T_{ln} \,. \tag{102}$$

Equation (97)-(102), the first term shows the tensorial transformation of derivative of quaternionic tensor and the last two terms show the christoffel symbols which tell us how the geodesic path changes from point to point.

3.4.3 Quaternionic metric tensor:

We can write the distance from one point to another point in quaternionic curved space-time as similar to Minkowski 4-D space-time. So, the line element for flat space-time will be

(Kilmister, C. W. 1973)

$$\begin{aligned} ds^2 &= -dP^{\xi}dP^{\xi} + dP^1dP^1 + dP^2dP^2 + dP^3dP^3 \\ &= \eta_{\mu\nu}\,dP^{\mu}\,dP^{\nu} = -d\tau^2, \qquad (\forall\ \mu = (\xi,\, j),\ \nu = (\zeta,\, n))\,, \end{aligned} \tag{103}$$

where $\eta_{\mu\nu} = +1\ \forall\,(\mu,\,\nu = 1,2,3)$ for $\mu = \nu$, $\eta_{\mu\nu} = -1$ for $(\mu = \nu = \xi)$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. Here $\eta_{\mu\nu}$ is the Minkowski metric used in special relativity and $d\tau^2$is the proper time. The essential property of this proper time is that it is invariant in all coordinates systems i.e.,

$$d\tau^2 = d\tau'^2\,. \tag{104}$$

In matrix form, Minkowski metric will be **(Walters, S. 2016)**

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{105}$$

This is known as semi-riemannian or Lorentzian metric tensors (Minkowski metric). Analogous to this, we can define the line element in quaternionic curved space-time by using equation (75) and (76) as

$$\begin{aligned} ds^2 &= \eta_{\xi\zeta}\left(\frac{\partial P^{\xi}}{\partial P'^i}dP'^i\right)\left(\frac{\partial P^{\zeta}}{\partial P'^m}dP'^m\right) + \eta_{jn}\left(\frac{\partial P^{j}}{\partial P'^i}dP'^i\right)\left(\frac{\partial P^{n}}{\partial P'^m}dP'^m\right) \\ ds^2 &= \left(\eta_{\xi\zeta}\frac{\partial P^{\xi}}{\partial P'^i}\frac{\partial P^{\zeta}}{\partial P'^m}\right)dP'^i dP'^m + \left(\eta_{jn}\frac{\partial P^{j}}{\partial P'^i}\frac{\partial P^{n}}{\partial P'^m}\right)dP'^i dP'^m\,. \end{aligned} \tag{106}$$

It is quaternionic line element in curved space-time where

$$g_1 = g'_{im} = \frac{\partial P^{\xi}}{\partial P'^i}\,\frac{\partial P^{\zeta}}{\partial P'^m}\,\eta_{\xi\zeta}\,, \tag{107}$$

$$g_2 = g'_{im} = \frac{\partial P^{j}}{\partial P'^i}\,\frac{\partial P^{n}}{\partial P'^m}\,\eta_{jn}\,. \tag{108}$$

Here, we obtain $3D$ metric tensors (g_1, g_2) in terms of scalar and vector Minkowski metric form. **(Ganihar, S. A. et al., 2015)** have showed the $3D$ metric tensor that consists six independent components. Now, combining equation (107) and (108) we get

$$g'_{im} = \frac{\partial P^{\mu}}{\partial P'^{i}} \frac{\partial P^{\nu}}{\partial P'^{m}} \eta_{\mu\nu} \,. \tag{109}$$

Therefore, we can express the $3D$ metric tensor for $3D$ pure quaternionic space as **(Ganihar, S. A. et al., 2015)**

$$g'_{im} = \begin{pmatrix} \frac{\partial P^{\mu}}{\partial P'^{1}} \frac{\partial P^{\nu}}{\partial P'^{1}} \eta_{\mu\nu} & \frac{\partial P^{\mu}}{\partial P'^{1}} \frac{\partial P^{\nu}}{\partial P'^{2}} \eta_{\mu\nu} & \frac{\partial P^{\mu}}{\partial P'^{1}} \frac{\partial P^{\nu}}{\partial P'^{3}} \eta_{\mu\nu} \\ \frac{\partial P^{\mu}}{\partial P'^{2}} \frac{\partial P^{\nu}}{\partial P'^{1}} \eta_{\mu\nu} & \frac{\partial P^{\mu}}{\partial P'^{2}} \frac{\partial P^{\nu}}{\partial P'^{2}} \eta_{\mu\nu} & \frac{\partial P^{\mu}}{\partial P'^{2}} \frac{\partial P^{\nu}}{\partial P'^{3}} \eta_{\mu\nu} \\ \frac{\partial P^{\mu}}{\partial P'^{3}} \frac{\partial P^{\nu}}{\partial P'^{1}} \eta_{\mu\nu} & \frac{\partial P^{\mu}}{\partial P'^{3}} \frac{\partial P^{\nu}}{\partial P'^{2}} \eta_{\mu\nu} & \frac{\partial P^{\mu}}{\partial P'^{3}} \frac{\partial P^{\nu}}{\partial P'^{3}} \eta_{\mu\nu} \end{pmatrix} .$$

The line element in matrix form can be represented as,

$$ds^2 = \begin{pmatrix} -\frac{\partial P^{\xi}}{\partial P'^{i}} \frac{\partial P^{\zeta}}{\partial P'^{m}} & 0 & 0 & 0 \\ 0 & \frac{\partial P^{1}}{\partial P'^{i}} \frac{\partial P^{1}}{\partial P'^{m}} & 0 & 0 \\ 0 & 0 & \frac{\partial P^{2}}{\partial P'^{i}} \frac{\partial P^{2}}{\partial P'^{m}} & 0 \\ 0 & 0 & 0 & \frac{\partial P^{3}}{\partial P'^{i}} \frac{\partial P^{3}}{\partial P'^{m}} \end{pmatrix} dP'^{i} dP'^{m} . \tag{110}$$

The structure of above matrix shows a $'-, +, +, +'$ which is similar to Euclidean space-time. For flat space-time metric tensor, all the diagonal terms in ds^2 matrix are equal to unity, i.e., no variation in $S-$frame coordinate with respect to $S'-$ frame coordinate.

3.4.4 Quaternionic geodesic equation:

In GTR, geodesic gives the replacement of linear space-time to curved space-time. In other words, a falling or freely moving particle in a curvature space-time structure is always follows the path of geodesic. The geodesic path in curvature space-time structure can be describe in terms of four-vector form. Thus, the tangent of a four-vector (U^{μ}) is expressed as

$$U^{\mu} = \frac{dx^{\mu}}{ds} \,, \tag{111}$$

where the tangent of this four vector shows the direction of the motion of a particle. But for a rectilinear motion,

$$dU^{\mu} = 0. \tag{112}$$

The four-vector (U^{μ}) can also take as quaternionic form, i.e.,

$$dH^{\mu} = 0, \tag{113}$$

where $H^{\mu} = \frac{dP^{\mu}}{ds}$. The analogous motion in a quaternionic curved space-time is not explained by equation (113). In curvature space-time, after the transformation of quaternionic tangent from equation (68), the equation (113) becomes **(Weinberg, S. 1972)**

$$\begin{aligned} H^{\xi}_{;\phi} = H^{\xi}_{,\phi} + \left\{ \begin{array}{c} \xi \\ \phi\phi \end{array} \right\} H^{\phi} = 0 \\ \Rightarrow \frac{dH^{\xi}}{dP^{\phi}} + \left\{ \begin{array}{c} \xi \\ \phi\phi \end{array} \right\} H^{\phi} = 0 \\ \Rightarrow dH^{\xi} + \left\{ \begin{array}{c} \xi \\ \phi\phi \end{array} \right\} dP^{\phi} H^{\phi} = 0. \end{aligned} \tag{114}$$

Now, dividing equation (114) by ds and substituting the value $H^{\xi} = \frac{dP^{\xi}}{ds}$, $H^{\phi} = \frac{dP^{\phi}}{ds}$ we get

$$\frac{d^2P^{\xi}}{ds^2} + \left\{ \begin{array}{c} \xi \\ \phi\phi \end{array} \right\} \frac{dP^{\phi}}{ds}\frac{dP^{\phi}}{ds} = 0. \tag{115}$$

Similarly, equations (87), (89) and (90) give the following relations,

$$\frac{d^2P^{j}}{ds^2} + \left\{ \begin{array}{c} j \\ \phi\phi \end{array} \right\} \frac{dP^{\phi}}{ds}\frac{dP^{\phi}}{ds} = 0, \tag{116}$$

$$\frac{d^2P^{\xi}}{ds^2} + \left\{ \begin{array}{c} \xi \\ km \end{array} \right\} \frac{dP^{k}}{ds}\frac{dP^{m}}{ds} = 0, \tag{117}$$

$$\frac{d^2P^j}{ds^2}+\left\{\begin{array}{c} j \\ km \end{array}\right\}\frac{dP^k}{ds}\frac{dP^m}{ds}=0\,. \tag{118}$$

These are the quaternionic geodesic equations which show the path of curved space-time travelled by freely moving particles. We can combine the quaternionic vector components (116) and (118) in a compact manner as

$$\frac{d^2P^j}{ds^2}+\left\{\begin{array}{c} j \\ \mu\nu \end{array}\right\}\frac{dP^\mu}{ds}\frac{dP^\nu}{ds}=0 \qquad (\text{Coefficient of } e_j)\,, \tag{119}$$

where $\mu=\,(k,\,\phi)\;\nu=\,(m,\,\phi)$ whereas the quaternionic scalar components (115) and (117) provide the following compact form as

$$\frac{d^2P^\xi}{ds^2}+\left\{\begin{array}{c} \xi \\ \mu\nu \end{array}\right\}\frac{dP^\mu}{ds}\frac{dP^\nu}{ds}=0 \qquad (\text{Coefficient of } e_0)\,, \tag{120}$$

where $\mu=\,(\phi,\,k)$ and $\nu=\,(\zeta,\,m)$ Equations (119) is showing the compact form of quaternionic geodesic equation of motion which are similar to the differential form of Classical Newton's equation of motion. These equations show the non-linear equations arises due to the effect of Christoffel symbols. In quaternionic formulation we also can conclude that the space-time curvature path not only followed by vector component but also followed by the scalar component of a quaternion variable it shows the change of scalar component with respect to time in equation (120). If christoffel symbol is zero, then the quaternionic geodesic equations of motion will become

$$\frac{d^2P^j}{ds^2}=0\,,$$
$$\frac{d^2P^\xi}{ds^2}=0\,.$$

This implies that the acceleration of quaternionic vector will be zero or uniform velocity which show the particles moving in a straight line (i.e., flat space-time) and the change of scalar component will be zero which shows in flat space-time there will be no change in scalar component with respect to time. For expressing quaternionic metric tensor in terms of christoffel symbol, the covariant derivative of quaternionic metric tensor is used. Since, quaternionic metric tensor of rank-2 transformed from one frame to another similar to the transformation of quaternionic tensor. Then the covariant derivative of quaternionic tensor from equation (94) can be expressed as

$$g_{\xi\xi;\phi} = g_{\xi\xi,\phi} - \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} g_{\xi\eta} - \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} g_{\eta\xi} \,. \tag{121}$$

Under the covariant differentiation the quaternionic metric tensor is constant **(Pathria, R. K 1974)** i.e.,

$$\begin{aligned} 0 &= g_{\xi\xi,\phi} - \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} g_{\xi\eta} - \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} g_{\eta\xi} \\ g_{\xi\xi,\phi} &= \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} g_{\xi\eta} + \left\{ \begin{array}{c} \eta \\ \phi\xi \end{array} \right\} g_{\eta\xi} \,. \end{aligned} \tag{122}$$

We also may use the general properties of christoffel symbol as, **(Narlikar, J. V. 1978)**

$$\left\{ \begin{array}{c} m \\ il \end{array} \right\} g_{mk} = \Gamma_{k,il} = \left[\begin{array}{cc} il, & k \end{array} \right], \tag{123}$$

$$\left\{ \begin{array}{c} i \\ kl \end{array} \right\} = \left\{ \begin{array}{c} i \\ lk \end{array} \right\}. \tag{124}$$

Equation (124) shows the symmetrical form of christoffel symbol with respect to last two indices. The christoffel symbol $\Gamma_{k,il}$ can be represented as $\left[\begin{array}{cc} il, & k \end{array} \right]$ **(Struik, D. J. 1961)**. The inverse of the metric tensor can be written in the superscript form such that

$$g^{ik} = \frac{1}{g_{ik}} \,. \tag{125}$$

However, the metric tensor are used in increasing and decreasing of indices on tensors, e.g.,

$$A^{i} g_{ik} = A_{k}, \qquad g_{ik} R^{i}_{jlm} = R_{kjlm} \,.$$

Thus, using the general properties of equation (123) and (124) in equation (122), we obtain

$$g_{\xi\xi,\phi} = \left[\ \phi\xi,\ \ \xi\ \right] + \left[\ \phi\xi,\ \ \xi\ \right]. \tag{126}$$

By cyclic interchange of indices, we get the following relations

$$g_{\phi\xi,\xi} = \left[\ \xi\xi,\ \ \phi\ \right] + \left[\ \xi\phi,\ \ \xi\ \right], \tag{127}$$

$$g_{\xi\phi,\xi} = \left[\ \xi\phi,\ \ \xi\ \right] + \left[\ \xi\xi,\ \ \phi\ \right]. \tag{128}$$

Now, adding equations (127) and (128), with subtracting from equation (126) from this we obtain

$$\left[\ \xi\xi,\ \ \phi\ \right] = \frac{1}{2}\left(g_{\phi\xi,\xi} + g_{\xi\phi,\xi} - g_{\xi\xi,\phi}\right). \tag{129}$$

Now, using equation (123) we get the value of scalar christoffel symbol corresponding to quaternionic scalar component as,

$$\left\{ \begin{array}{c} \eta \\ \xi\xi \end{array} \right\} g_{\eta\phi} = \frac{1}{2}\left(g_{\phi\xi,\xi} + g_{\xi\phi,\xi} - g_{\xi\xi,\phi}\right)$$

$$\Rightarrow \left\{ \begin{array}{c} \eta \\ \xi\xi \end{array} \right\} = \frac{1}{2} g^{\eta\phi}\left(g_{\phi\xi,\xi} + g_{\xi\phi,\xi} - g_{\xi\xi,\phi}\right), \tag{130}$$

where $g^{\eta\phi}$ is the inverse matrix of $g_{\eta\phi}$. Similarly, from equation (96)-(102) we get

$$\left\{ \begin{array}{c} l \\ \xi\xi \end{array} \right\} = \frac{1}{2} g^{lk}\left(g_{k\xi,\xi} + g_{\xi k,\xi} - g_{\xi\xi,k}\right), \tag{131}$$

$$\left\{ \begin{array}{c} \eta \\ \xi n \end{array} \right\} = \frac{1}{2} g^{\eta\phi}\left(g_{\phi\xi,n} + g_{n\phi,\xi} - g_{\xi n,\phi}\right), \tag{132}$$

$$\left\{ \begin{array}{c} l \\ \xi n \end{array} \right\} = \frac{1}{2} g^{lk}\left(g_{k\xi,n} + g_{nk,\xi} - g_{\xi n,k}\right), \tag{133}$$

$$\left\{\begin{array}{c}\eta\\ jn\end{array}\right\}=\frac{1}{2}g^{\eta\phi}\left(g_{\phi j,n}+g_{n\phi,j}-g_{jn,\phi}\right), \tag{134}$$

$$\left\{\begin{array}{c}l\\ jn\end{array}\right\}=\frac{1}{2}g^{lk}\left(g_{kj,n}+g_{nk,j}-g_{jn,k}\right), \tag{135}$$

where equations (131)-(134) show the mixed form of the value of quaternionic christoffel symbol while equation (135) shows the pure vector form of the quaternionic christoffel symbol in terms of metric tensor. Here, form equation (130)-(135) the partial derivative of metric tensor shows the quaternionic gravitational potential **(Walters, S. 2016)**. It is also known as quaternionic tensor potential which may reflect the contribution of mass, energy, and pressure in terms to gravity.

3.4.5 Quaternionic Riemannian christoffel curvature tensor:

The Riemannian christoffel curvature tensor is a four-indices tensor that eleborating the curvature of Riemannian manifolds. The non-Euclidean geometry of curvature space-time was described by the help of Riemannian tensor. It is obtained by subtraction of quaternionic covariant derivative of quaternionic tensor from the another in which indices are interchanged, such that equation (94) becomes

$$T_{\xi\phi;\xi}=T_{\xi\phi,\xi}-\left\{\begin{array}{c}\eta\\ \xi\phi\end{array}\right\}T_{\xi\eta}-\left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}T_{\eta\phi}. \tag{136}$$

Now, subtracting equations (94) and (136) we obtain

$$\begin{aligned}T_{\xi\xi;\phi}-T_{\xi\phi;\xi}=&\left(T_{\xi\xi,\phi}-\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}T_{\xi\eta}-\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}T_{\eta\xi}\right)\\ &-\left(T_{\xi\phi,\xi}-\left\{\begin{array}{c}\eta\\ \xi\phi\end{array}\right\}T_{\xi\eta}-\left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}T_{\eta\phi}\right).\end{aligned}$$

Using equation (85) it gives

$$T_{\xi\xi;\phi} - T_{\xi\phi;\xi} = \partial_{\phi}\left(H_{\xi,\xi} - \left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}H_{\eta}\right) - \left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\left(H_{\xi,\eta} - \left\{\begin{array}{c}\varphi\\ \eta\xi\end{array}\right\}H_{\varphi}\right)$$
$$- \left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\left(H_{\eta,\xi} - \left\{\begin{array}{c}\varphi\\ \eta\xi\end{array}\right\}H_{\varphi}\right) - \partial_{\xi}\left(H_{\xi,\phi} - \left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}H_{\eta}\right)$$
$$+ \left\{\begin{array}{c}\eta\\ \xi\phi\end{array}\right\}\left(H_{\xi,\eta} - \left\{\begin{array}{c}\varphi\\ \eta\xi\end{array}\right\}H_{\varphi}\right) + \left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}\left(H_{\eta,\phi} - \left\{\begin{array}{c}\varphi\\ \eta\phi\end{array}\right\}H_{\varphi}\right)$$
$$= \partial_{\xi}\left(\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\right)H_{\eta} - \partial_{\phi}\left(\left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}\right)H_{\eta} + \left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\left\{\begin{array}{c}\varphi\\ \eta\xi\end{array}\right\}H_{\varphi}$$
$$- \left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}\left\{\begin{array}{c}\varphi\\ \eta\phi\end{array}\right\}H_{\varphi}$$
$$= \left(R^{\eta}_{\xi\xi\phi}\right)H_{\eta}\,, \tag{137}$$

where $R^{\eta}_{\xi\xi\phi} = \partial_{\xi}\left(\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\right) - \partial_{\phi}\left(\left\{\begin{array}{c}\eta\\ \xi\xi\end{array}\right\}\right) + \left\{\begin{array}{c}\varphi\\ \phi\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi\xi\end{array}\right\} - \left\{\begin{array}{c}\varphi\\ \xi\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi\phi\end{array}\right\}$ is the purely quaternionic scalar component of Riemannian chirstoffel curvature tensor. It is a four-indices tensor in curved space-time which described the curvature of manifolds. The left hand part of equation (137) is the difference of two quaternionic scalar tensor component, each having rank -3. In right hand side of equation (137) H_{η} is a covariant quaternionic scalar component and it follows the quotient law with Riemannian christoffel curvature tensor $\left(R^{\eta}_{\xi\xi\phi}\right)$ of rank-4. Similarly, we can obtain the Riemannian christoffel curvature tensor with respect to other quaternionic components given in equation (96)-(102), i.e.,

$$T_{\xi\xi;k} - T_{\xi k;\xi} = \left(R^{l}_{\xi\xi k}\right)H_{l}\,, \tag{138}$$
$$T_{\xi n;\phi} - T_{\xi\phi;n} = \left(R^{\eta}_{\xi n\phi}\right)H_{\eta}\,, \tag{139}$$
$$T_{\xi n;k} - T_{\xi k;n} = \left(R^{l}_{\xi nk}\right)H_{l}\,, \tag{140}$$
$$T_{j\xi;\phi} - T_{j\phi;\xi} = \left(R^{\eta}_{j\xi\phi}\right)H_{\eta}\,, \tag{141}$$
$$T_{j\xi;k} - T_{jk;\xi} = \left(R^{l}_{j\xi k}\right)H_{l}\,, \tag{142}$$

$$T_{jn;\phi} - T_{j\phi;n} = \left(R^{\eta}_{jn\phi}\right) H_{\eta}\,, \tag{143}$$

$$T_{jn;k} - T_{jk;n} = \left(R^{l}_{jnk}\right) H_{l}\,, \tag{144}$$

where the Riemannian christoffel curvature tensor leads to the following way as

$$R^{l}_{\xi\xi k} = \partial_{\xi}\left(\begin{Bmatrix} l \\ k\xi \end{Bmatrix}\right) - \partial_{k}\left(\begin{Bmatrix} l \\ \xi\xi \end{Bmatrix}\right) + \begin{Bmatrix} i \\ k\xi \end{Bmatrix}\begin{Bmatrix} l \\ i\xi \end{Bmatrix} - \begin{Bmatrix} i \\ \xi\xi \end{Bmatrix}\begin{Bmatrix} l \\ ik \end{Bmatrix}, \tag{145}$$

$$R^{\eta}_{\xi n\phi} = \partial_{n}\left(\begin{Bmatrix} \eta \\ \phi\xi \end{Bmatrix}\right) - \partial_{\phi}\left(\begin{Bmatrix} \eta \\ \xi n \end{Bmatrix}\right) + \begin{Bmatrix} \varphi \\ \phi\xi \end{Bmatrix}\begin{Bmatrix} \eta \\ \varphi n \end{Bmatrix} - \begin{Bmatrix} \varphi \\ n\xi \end{Bmatrix}\begin{Bmatrix} \eta \\ \varphi\phi \end{Bmatrix}, \tag{146}$$

$$R^{l}_{\xi nk} = \partial_{n}\left(\begin{Bmatrix} l \\ k\xi \end{Bmatrix}\right) - \partial_{k}\left(\begin{Bmatrix} l \\ \xi n \end{Bmatrix}\right) + \begin{Bmatrix} i \\ k\xi \end{Bmatrix}\begin{Bmatrix} l \\ in \end{Bmatrix} - \begin{Bmatrix} i \\ n\xi \end{Bmatrix}\begin{Bmatrix} l \\ ik \end{Bmatrix}, \tag{147}$$

$$R^{\eta}_{j\xi\phi} = \partial_{\xi}\left(\begin{Bmatrix} \eta \\ \phi j \end{Bmatrix}\right) - \partial_{\phi}\left(\begin{Bmatrix} \eta \\ j\xi \end{Bmatrix}\right) + \begin{Bmatrix} \varphi \\ \phi j \end{Bmatrix}\begin{Bmatrix} \eta \\ \varphi\xi \end{Bmatrix} - \begin{Bmatrix} \varphi \\ \xi j \end{Bmatrix}\begin{Bmatrix} \eta \\ \varphi\phi \end{Bmatrix}, \tag{148}$$

$$R^{l}_{j\xi k} = \partial_{\xi}\left(\begin{Bmatrix} l \\ kj \end{Bmatrix}\right) - \partial_{k}\left(\begin{Bmatrix} l \\ j\xi \end{Bmatrix}\right) + \begin{Bmatrix} i \\ kj \end{Bmatrix}\begin{Bmatrix} l \\ i\xi \end{Bmatrix} - \begin{Bmatrix} i \\ \xi j \end{Bmatrix}\begin{Bmatrix} l \\ ik \end{Bmatrix}, \tag{149}$$

$$R^{\eta}_{jn\phi} = \partial_{n}\left(\begin{Bmatrix} \eta \\ \phi j \end{Bmatrix}\right) - \partial_{\phi}\left(\begin{Bmatrix} \eta \\ jn \end{Bmatrix}\right) + \begin{Bmatrix} \varphi \\ \phi j \end{Bmatrix}\begin{Bmatrix} \eta \\ \varphi n \end{Bmatrix} - \begin{Bmatrix} \varphi \\ nj \end{Bmatrix}\begin{Bmatrix} \eta \\ \varphi\phi \end{Bmatrix}, \tag{150}$$

$$R^{l}_{jnk} = \partial_{n}\left(\begin{Bmatrix} l \\ kj \end{Bmatrix}\right) - \partial_{k}\left(\begin{Bmatrix} l \\ jn \end{Bmatrix}\right) + \begin{Bmatrix} i \\ kj \end{Bmatrix}\begin{Bmatrix} l \\ in \end{Bmatrix} - \begin{Bmatrix} i \\ nj \end{Bmatrix}\begin{Bmatrix} l \\ ik \end{Bmatrix}. \tag{151}$$

Here, quaternionic Riemannian tensor keeps the track that how much a scalar and vector component of quaternion change when we parallel propagate along a small parallelogram. If the value of quaternionic Riemannian christoffel curvature is zero then our quaternionic curved space-time converted into flat space-time. Other tensor can also formed from the contraction of quaternionic Riemannian christoffel curvature tensor with metric tensor. The quaternionic Ricci tensor is an important contraction of quaternionic Riemannian christoffel tensor which explains the volume changes when object parallel transport along a geodesic. Starting with equation (146) as

$$\begin{aligned} g_{h\eta}R^{\eta}_{\xi n\phi} &= R_{h\xi n\phi} \\ R_{h\xi n\phi} &= g_{h\eta}\left[\partial_n\left(\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\right)-\partial_\phi\left(\left\{\begin{array}{c}\eta\\ \xi n\end{array}\right\}\right)+\left\{\begin{array}{c}\varphi\\ \phi\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi n\end{array}\right\}-\left\{\begin{array}{c}\varphi\\ n\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi\phi\end{array}\right\}\right] \\ R_{h\xi n\phi} &= \partial_n\left(g_{h\eta}\left\{\begin{array}{c}\eta\\ \phi\xi\end{array}\right\}\right)-\partial_\phi\left(g_{h\eta}\left\{\begin{array}{c}\eta\\ \xi n\end{array}\right\}\right)+g_{h\eta}\left\{\begin{array}{c}\varphi\\ \phi\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi n\end{array}\right\} \\ &\quad - g_{h\eta}\left\{\begin{array}{c}\varphi\\ n\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi\phi\end{array}\right\}. \end{aligned} \tag{152}$$

Using equations (123) and (129), we get following relation from equation (152) as

$$\begin{aligned} R_{h\xi n\phi} &= \partial_n\left(\frac{1}{2}\left(g_{h\phi,\xi}+g_{\xi h,\phi}-g_{\phi\xi,h}\right)\right)-\partial_\phi\left(\frac{1}{2}\left(g_{h\xi,n}+g_{nh,\xi}-g_{\xi n,h}\right)\right) \\ &\quad +\left\{\begin{array}{c}\varphi\\ \phi\xi\end{array}\right\}\left[\begin{array}{cc}\varphi n, & h\end{array}\right]-\left\{\begin{array}{c}\varphi\\ n\xi\end{array}\right\}\left[\begin{array}{cc}\varphi\phi, & h\end{array}\right] \\ &= \frac{1}{2}\left[\frac{\partial^2 g_{h\phi}}{\partial P^n\partial P^\xi}+\frac{\partial^2 g_{\xi n}}{\partial P^h\partial P^\phi}-\frac{\partial^2 g_{\xi\phi}}{\partial P^n\partial P^h}-\frac{\partial^2 g_{nh}}{\partial P^\phi\partial P^\xi}\right] \\ &\quad + g_{\eta h}\left\{\begin{array}{c}\varphi\\ \phi\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \phi n\end{array}\right\}-g_{\eta h}\left\{\begin{array}{c}\varphi\\ n\xi\end{array}\right\}\left\{\begin{array}{c}\eta\\ \varphi\phi\end{array}\right\}. \end{aligned} \tag{153}$$

This is the Riemannian christoffel curvature tensor in terms of metric tensor. Now, contracting equation (152) with the metric tensor having indices in superscript such that

$$\begin{aligned} g^{hn}R_{h\xi n\phi} &= \partial_n\left(g^{hn}\left[\begin{array}{cc}\phi\xi, & h\end{array}\right]\right)-\partial_\phi\left(g^{hn}\left[\begin{array}{cc}\xi n, & h\end{array}\right]\right) \\ &\quad +g^{hn}\left\{\begin{array}{c}\varphi\\ \phi\xi\end{array}\right\}\left[\begin{array}{cc}\varphi n, & h\end{array}\right]-g^{hn}\left\{\begin{array}{c}\varphi\\ n\xi\end{array}\right\}\left[\begin{array}{cc}\varphi\phi, & h\end{array}\right] \end{aligned}$$

$$R_{\xi\phi} = \partial_n \left(\left\{ \begin{array}{c} n \\ \phi\xi \end{array} \right\} \right) - \partial_\phi \left(\left\{ \begin{array}{c} n \\ \xi n \end{array} \right\} \right) + \left\{ \begin{array}{c} \varphi \\ \phi\xi \end{array} \right\} \left\{ \begin{array}{c} n \\ \varphi n \end{array} \right\} - \left\{ \begin{array}{c} \varphi \\ n\xi \end{array} \right\} \left\{ \begin{array}{c} n \\ \varphi\phi \end{array} \right\}. \tag{154}$$

Similarly, from equation (145)-(151) we get other possible components of quaternionic ricci tensor i.e.,

$$R_{\xi k} = \partial_n \left(\left\{ \begin{array}{c} n \\ k\xi \end{array} \right\} \right) - \partial_k \left(\left\{ \begin{array}{c} n \\ \xi n \end{array} \right\} \right) + \left\{ \begin{array}{c} i \\ k\xi \end{array} \right\} \left\{ \begin{array}{c} n \\ in \end{array} \right\} - \left\{ \begin{array}{c} i \\ n\xi \end{array} \right\} \left\{ \begin{array}{c} n \\ ik \end{array} \right\}, \tag{155}$$

$$R_{j\phi} = \partial_n \left(\left\{ \begin{array}{c} n \\ \phi j \end{array} \right\} \right) - \partial_\phi \left(\left\{ \begin{array}{c} n \\ jn \end{array} \right\} \right) + \left\{ \begin{array}{c} \varphi \\ \phi j \end{array} \right\} \left\{ \begin{array}{c} n \\ \varphi n \end{array} \right\} - \left\{ \begin{array}{c} \varphi \\ nj \end{array} \right\} \left\{ \begin{array}{c} n \\ \varphi\phi \end{array} \right\}, \tag{156}$$

$$R_{jk} = \partial_n \left(\left\{ \begin{array}{c} n \\ kj \end{array} \right\} \right) - \partial_k \left(\left\{ \begin{array}{c} n \\ jn \end{array} \right\} \right) + \left\{ \begin{array}{c} i \\ kj \end{array} \right\} \left\{ \begin{array}{c} n \\ in \end{array} \right\} - \left\{ \begin{array}{c} i \\ nj \end{array} \right\} \left\{ \begin{array}{c} n \\ ik \end{array} \right\}. \tag{157}$$

The evolution of a quaternion is governed by Riemannian tensor similar to the evolution of volume of objects governed by quaternionic Ricci tensor. We should notice that the algebraic properties of Riemannian tensor is symmetric with respect to first with third indices and second with fourth indices **(Weinberg, S. 1972)**. Therefore, the Ricci tensor is also symmetric. The quaternionic Ricci tensor can further be contract with metric tensor and obtain scalar curvature tensor, such that

$$g^{\xi\phi} R_{\xi\phi} = R. \tag{158}$$

The scalar curvature (R) of rank-0 tells about the amount of change in the volume of a small geodesic ball in a riemannian manifold. It is also known as Ricci scalar.

Here, the Riemannian geometry studied in quaternionic form. By using the concept of quaternions we study that 3-D space can also transformed into 4-D space-time. Since, 3-D can also expressed by quaternions according to **Goldman, R. (2010)** quaternions can be used to rotate 3-D vector. It is also used to visualised rotations, projections and reflections on point and vector in 3-D.

3.5 Quaternionic Energy-Momentum tensor

Generally, the energy-momentum tensor describes the density and flux in space-time. It contains the information about the stuff in space, the mass, the energy and the pressure. The two things are important in constructing energy-momentum tensor that it should be symmetric and contain energy and momentum. We use the concept of special relativity mainly energy, momentum and unit vector or tangent vector. Thus, the energy-momentum tensor can be written as

$$e_0 E = e_0 mc^2 \gamma \qquad \text{(energy)}\,, \tag{159}$$

$$e_j P^j = e_j m v^j \gamma \qquad \text{(momentum)}\,, \tag{160}$$

where, e_0 and e_j are the quaternionic unit basis, E is the energy corresponding to scalar component of a quaternion and P^j is the momentum corresponding to vector component of a quaternion. $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, $j = 1,2,3$, m is the mass of the object and v^j is the velocity of the object. Since, we have quaternionic tangent such that

$$e_\mu H^\mu = e_\mu \frac{dP^\mu}{ds}\,, \tag{161}$$

where the scalar and vector components of quaternionic tangent may be expressed as,

$$H^\xi = \frac{dP^\xi}{ds} = \gamma\ \text{(say)}\,, \quad H^j = \frac{dP^j}{ds}\,. \tag{162}$$

Here $\left(H^\xi, H^j\right)$ are the quaternionic tangential scalar and vector component, ds is the proper time and dP^ξ is also the component of time in moving frame. In equation (162), we assume the time in moving frame is equal to the γ times of proper time **(Beiser, A. 2003)**. Then, the components of quaternionic energy momentum tensor can be written as

$$\left(E, P^j\right) = \left(mH^\xi, mH^1, mH^2, mH^3\right) = m\left(H^\xi, H^j\right)\,. \tag{163}$$

When we choose arbitrary volume of a particle in space then the number density in rest frame is represented as n_0. If particle are in motion then the number density observe by stationary observer increase by factor γ and energy density of the system after volume contraction will be

$$e_0 E = e_0 n_0 m \gamma^2 = n_0 m H^{\xi} H^{\xi} . \tag{164}$$

In per unit time the amount of stuff flows across the surface may express the current density as,

$$e_j J^j = e_j \rho v^j , \tag{165}$$

where J^j is the current density, ρ is the density of stuff and v^j is the velocity of stuff. Further,

$$e_j \sigma^j = e_j E v^j , \tag{166}$$

where σ^j is the energy current, E is the energy density of the stuff. From equation (164), we get

$$e_j \sigma^j = e_j n_0 m H^{\xi} H^j . \tag{167}$$

Here, the equation of continuity may written as

$$\frac{\partial E}{\partial P^{\phi}} + \overrightarrow{\nabla} \cdot \overrightarrow{\sigma} = 0 . \tag{168}$$

Now, we can write equation (168) in terms of quaternionic four-dimensional form as

$$\frac{\partial E^{\mu}}{\partial P^{\lambda}} = 0 , \tag{169}$$

where $\frac{\partial}{\partial P^{\lambda}} = e_0 \frac{\partial}{\partial P^{\phi}} + e_k \frac{\partial}{\partial P^k}$ is quaternionic nabla operator and $E^{\mu} = \left(E, \sigma^1, \sigma^2, \sigma^3, \right)$. Similarly, for momentum space we may write,

$$e_j P^j = n_0 m e_j H^{\xi} H^j \qquad (\text{momentum density}) , \tag{170}$$

$$e_i \circ e_j P^{ij} = n_0 m \left(e_i \circ e_j\right) H^i H^j \qquad (\text{momentum current}) , \tag{171}$$

$$\frac{\partial P^{ij}}{\partial P^{\lambda}}=0 \qquad\qquad \text{(equation of continuity)}. \tag{172}$$

The continuity equation can be written in matrix form as

$$\begin{pmatrix} \frac{\partial}{\partial P^{\phi}} & \frac{\partial}{\partial P^{1}} & \frac{\partial}{\partial P^{2}} & \frac{\partial}{\partial P^{3}} \end{pmatrix} \begin{bmatrix} E & P^1 & P^2 & P^3 \\ \sigma^1 & P^{11} & P^{12} & P^{13} \\ \sigma^2 & P^{21} & P^{22} & P^{23} \\ \sigma^3 & P^{31} & P^{32} & P^{33} \end{bmatrix} = 0. \tag{173}$$

The terms of quaternionic energy-momentum tensor can be written as

$$E = n_0 m H^{\xi} H^{\xi}, \tag{174}$$

$$\sigma^i = n_0 m H^i H^{\xi}, \tag{175}$$

$$P^j = n_0 m H^{\xi} H^j, \tag{176}$$

$$P^{ij} = n_0 m H^i H^j. \tag{177}$$

In matrix contravariant form, generally the energy-momentum tensor written as

$$T^{\mu\nu} = n_0 m \begin{bmatrix} H^{\xi}H^{\xi} & H^{\xi}H^1 & H^{\xi}H^2 & H^{\xi}H^3 \\ H^1H^{\xi} & H^1H^1 & H^1H^2 & H^1H^3 \\ H^2H^{\xi} & H^2H^1 & H^2H^2 & H^2H^3 \\ H^3H^{\xi} & H^3H^1 & H^3H^2 & H^3H^3 \end{bmatrix}. \tag{178}$$

Similarly, in covariant form $T_{\mu\nu}$ in quaternionic form can be determined as similar to the energy-momentum tensor in **(Pathria, R. K. 1974)** written as

$$T_{\mu\nu} = n_0 m \left[e_0 \left(H_{\xi}H_{\xi} - \overrightarrow{H_i}\cdot\overrightarrow{H_j} \right) + e_k \left(H_{\xi}\overrightarrow{H_i} + \overrightarrow{H_j}H_{\xi} + \left(\overrightarrow{H_i}\times\overrightarrow{H_j} \right)_k \right) \right], \tag{179}$$

where $T_{\xi\xi}$ represents the energy density, $T_{\xi i}$ represents the momentum density, T_{ii} represents the pressure and T_{ij} represents the shear stress. In other form,

$$T_{\mu\nu} = n_0 m \left[e_0 \left(H_{\xi}H_{\xi} - H_iH_i \right) + e_k \left(H_{\xi}\overrightarrow{H_i} + \overrightarrow{H_j}H_{\xi} + \sum_{ij} \left(\varepsilon_{ijk} H_i H_j \right) \right) \right]. \tag{180}$$

This is the required symmetric quaternionic energy-momentum tensor.

3.6 Quaternionic study of Einstein field equation

For deriving Einstein's field equation we have to supersedes the poisson equation for Newton's law of gravitation i.e.,

$$\nabla^2\Phi = 4\pi G\rho\,, \tag{181}$$

where Φ is the Newtonian potential, G is gravitational constant and ρ is the mass density. In equation (181), the left hand side involves the second derivative of Newtonian gravitational potential and the right hand side shows the mass distribution. Let consider weak static fields in which particles are moving slowly then the equation of motion, i.e., from equation (119), we have

$$\frac{d^2P^j}{ds^2} + \left\{\begin{array}{c} j \\ \mu\nu \end{array}\right\} \frac{dP^\mu}{ds}\frac{dP^\nu}{ds} = 0\,.$$

Because the particles are moving slowly, the spatial components of second term in equation (119) will dwarfed by the terms consisting the time component. Therefore, we can assume the following approximation

$$\frac{d^2P^j}{ds^2} + \left\{\begin{array}{c} j \\ \xi\xi \end{array}\right\} \frac{dP^\xi}{ds}\frac{dP^\xi}{ds} = 0\,. \qquad \text{(corresponding to scalar function)} \tag{182}$$

From equation (131) we get

$$\left\{\begin{array}{c} j \\ \xi\xi \end{array}\right\} = \frac{1}{2}\eta^{jk}\left(g_{k\xi,\xi} + g_{\xi k,\xi} - g_{\xi\xi,k}\right)\,,$$

here we considering the static field so time derivative of $g_{k\xi,\xi}$ is zero. Therefore,

$$\left\{\begin{array}{c} j \\ \xi\xi \end{array}\right\} = -\frac{1}{2}\eta^{jk}g_{\xi\xi,k}\,. \tag{183}$$

The elements of metric tensor will be slighty different from the original one when the weak field conditions are implies **(Pe'er, A. 2014)**

$$g_{\xi\xi} = \eta_{\xi\xi} + h_{\xi\xi} = -1 + h_{\xi\xi}\,, \tag{184}$$

then equation (183) becomes

$$\begin{Bmatrix} j \\ \xi\xi \end{Bmatrix} = -\frac{1}{2}\eta^{jk}\left[\frac{\partial}{\partial P^k}\left(-1+h_{\xi\xi}\right)\right]$$
$$= -\frac{1}{2}\frac{\partial h_{\xi\xi}}{\partial P^k}\,. \tag{185}$$

Now, using equation (185) in equation (182), we get

$$\frac{d^2P^j}{ds^2} = -\begin{Bmatrix} j \\ \xi\xi \end{Bmatrix}\left(\frac{dP^\xi}{ds}\right)^2$$
$$= -\left[-\frac{1}{2}\frac{\partial h_{\xi\xi}}{\partial P^k}\right]\left(\frac{dP^\xi}{ds}\right)^2\,, \tag{186}$$

it reduces to

$$\frac{d^2P^j}{d\left(P^\xi\right)^2} = \frac{1}{2}\frac{\partial h_{\xi\xi}}{\partial P^k}\,. \tag{187}$$

According to Newtonian result **(Gron, O. and Hervik, S. 2007),**

$$\frac{d^2P^j}{d\left(P^\xi\right)^2} = -\nabla\Phi\,. \tag{188}$$

On comparing equation (187) and (188) we get

$$h_{\xi\xi} = -2\Phi\,.$$

From equation (184), the metric tensor in scalar form becomes

$$g_{\xi\xi} = -\left(1+2\Phi\right)\,. \tag{189}$$

On double differentiating equation (189) we get

$$\nabla^2 g_{\xi\xi} = -2\nabla^2\Phi\,,$$
$$\Rightarrow \nabla^2\Phi = -\frac{1}{2}\nabla^2 g_{\xi\xi}\,. \tag{190}$$

On putting equation (190) in equation (181), we get

$$\nabla^2 g_{\xi\xi} = -8\pi G\rho\,. \tag{191}$$

Therefore, the left hand side of equation (191) is the correct term for the replacement of Newtonian potential with tensor potential i.e., quaternionic form of scalar metric tensor. Accordingly, the right hand side of equation (191) can replaced with quaternionic four energy-momentum tensor given in equation (180). Thus, the scalar form of quaternionic energy density becomes

$$T_{\xi\xi} \simeq \rho\,. \tag{192}$$

Then, equation (191) becomes

$$\nabla^2 g_{\xi\xi} = -8\pi G T_{\xi\xi} \qquad \text{(quaternionic scalar component)}. \tag{193}$$

The quaternionic Ricci tensor can also be written in the form of double derivative of quaternionic metric tensor. In equation (154), time derivative $\left(\text{i.e.,}\,\partial_\phi\right)$ will be zero for static fields so the second term will be vanished. For the non-double derivative form of metric tensor, third and fourth term must be negligible. So, we get the derivative of order-2 for left hand side of equation (193) becomes

$$R_{\xi\xi} = -8\pi G T_{\xi\xi}\,. \tag{194}$$

Similarly, we can write rest of the components of quaternionic $R_{\mu\nu}$ as

$$\partial_n\left(\begin{Bmatrix} n \\ 1\xi \end{Bmatrix}\right) - \begin{Bmatrix} i \\ n\xi \end{Bmatrix}\begin{Bmatrix} n \\ i1 \end{Bmatrix} = -8\pi G\left(H_\xi H_1\right)$$

$$\Rightarrow R_{\xi 1} = -8\pi G T_{\xi 1}, \tag{195}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 2\xi \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n\xi \end{matrix} \right\} \left\{ \begin{matrix} n \\ i2 \end{matrix} \right\} = -8\pi G \left(H_\xi H_2 \right)$$

$$\Rightarrow R_{\xi 2} = -8\pi G T_{\xi 2}, \tag{196}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 3\xi \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n\xi \end{matrix} \right\} \left\{ \begin{matrix} n \\ i3 \end{matrix} \right\} = -8\pi G \left(H_\xi H_3 \right)$$

$$\Rightarrow R_{\xi 3} = -8\pi G T_{\xi 3}, \tag{197}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 11 \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n1 \end{matrix} \right\} \left\{ \begin{matrix} n \\ i1 \end{matrix} \right\} = -8\pi G \left(H_1 H_1 \right)$$

$$\Rightarrow R_{11} = -8\pi G T_{11}, \tag{198}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 22 \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n2 \end{matrix} \right\} \left\{ \begin{matrix} n \\ i2 \end{matrix} \right\} = -8\pi G \left(H_2 H_2 \right)$$

$$\Rightarrow R_{22} = -8\pi G T_{22}, \tag{199}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 33 \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n3 \end{matrix} \right\} \left\{ \begin{matrix} n \\ i3 \end{matrix} \right\} = -8\pi G \left(H_3 H_3 \right)$$

$$\Rightarrow R_{33} = -8\pi G T_{33}, \tag{200}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 21 \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n1 \end{matrix} \right\} \left\{ \begin{matrix} n \\ i2 \end{matrix} \right\} = -8\pi G \left(H_1 H_2 \right)$$

$$\Rightarrow R_{12} = -8\pi G T_{12}, \tag{201}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 32 \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n2 \end{matrix} \right\} \left\{ \begin{matrix} n \\ i3 \end{matrix} \right\} = -8\pi G \left(H_2 H_3 \right)$$

$$\Rightarrow R_{23} = -8\pi G T_{23}, \tag{202}$$

$$\partial_n \left(\left\{ \begin{matrix} n \\ 13 \end{matrix} \right\} \right) - \left\{ \begin{matrix} i \\ n3 \end{matrix} \right\} \left\{ \begin{matrix} n \\ i1 \end{matrix} \right\} = -8\pi G \left(H_3 H_1 \right)$$

$$\Rightarrow R_{31} = -8\pi G T_{31}, \tag{203}$$

where other components i.e., $R_{1\xi}, R_{2\xi}, R_{3\xi}, R_{21}, R_{32}, R_{13}$ are symmetrical to $R_{\xi 1}, R_{2\xi}, R_{3\xi}$,

R_{12}, R_{23}, R_{31} respectively. It is reasonable to consider that the equation of general relativity in generalised form can be written as

$$R_{\mu\nu} \propto T_{\mu\nu} \,. \tag{204}$$

Since, the quaternionic energy-momentum tensor is conserved quantity but quaternionic ricci tensor is not so, we have to prove that it should be divergence-free. Now by using Bianchi identity **(Patharia, R. K. 1974)** i.e.,

$$R_{\alpha\mu\beta\gamma;\sigma} + R_{\alpha\mu\gamma\sigma;\beta} + R_{\alpha\mu\sigma\beta;\gamma} = 0 \,, \tag{205}$$

taking the inverse metric tensor on the both sides of equation (205) as,

$$\begin{aligned} g^{\alpha\nu}\left(R_{\alpha\mu\beta\gamma;\sigma} + R_{\alpha\mu\gamma\sigma;\beta} + R_{\alpha\mu\sigma\beta;\gamma}\right) &= 0, \\ R^{\nu}_{\mu\beta\gamma;\sigma} + R^{\nu}_{\mu\gamma\sigma;\beta} + R^{\nu}_{\mu\sigma\beta;\gamma} &= 0, \\ R^{\nu}_{\mu\nu\gamma;\sigma} + R^{\nu}_{\mu\gamma\sigma;\nu} + R^{\nu}_{\mu\sigma\nu;\gamma} &= 0, \qquad (\beta = \nu) \\ R_{\mu\gamma;\sigma} + R^{\nu}_{\mu\gamma\sigma;\nu} - R_{\mu\sigma;\gamma} &= 0. \end{aligned} \tag{206}$$

Again, we take the inverse metric tensor on the both sides of equation (206), i.e.,

$$\begin{aligned} g^{\mu\delta}\left(R_{\mu\gamma;\sigma} + R^{\nu}_{\mu\gamma\sigma;\nu} - R_{\mu\sigma;\gamma}\right) &= 0 \,, \\ R^{\delta}_{\gamma;\sigma} + R^{\nu\delta}_{\gamma\sigma;\nu} - R^{\delta}_{\sigma;\gamma} &= 0 \,, \\ R^{\delta}_{\delta;\sigma} - R^{\nu\delta}_{\sigma\delta;\nu} - R^{\delta}_{\sigma;\delta} &= 0 \,, \qquad (\gamma = \delta) \\ R_{;\sigma} - R^{\nu}_{\sigma;\nu} - R^{\delta}_{\sigma;\delta} &= 0 \,, \\ R_{;\sigma} - R^{\nu}_{\sigma;\nu} - R^{\nu}_{\sigma;\nu} &= 0 \,, \\ 2R^{\nu}_{\sigma;\nu} - R_{;\sigma} &= 0 \,, \\ R^{\nu}_{\sigma;\nu} - \frac{1}{2}R_{;\sigma} &= 0 \,, \\ \left(R^{\nu}_{\sigma} - \frac{1}{2}\delta^{\nu}_{\sigma}R\right)_{;\nu} &= 0 \,, \\ g^{\mu\sigma}\left(R^{\nu}_{\sigma} - \frac{1}{2}\delta^{\nu}_{\sigma}R\right)_{;\nu} &= 0 \,, \end{aligned}$$

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\nu}=0\,,$$
$$\Rightarrow\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)_{;\nu}=0\,. \tag{207}$$

This shows the quaternionic Einstein tensor which fulfill the required properties to explain the geometric part of Einstein field equation.

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\propto T_{\mu\nu}\,. \tag{208}$$

Equation (208) gives the Einstein field equation, where the right hand side shows quaternionic matter and energy while the left hand side shows geometry of space-time in quaternionic form. Here, the Einstein field equation stated that motion of matter tells how quaternionic space-time is curved and vive-versa. To analysis the unstability of universe the cosmological constant (Λ) can be introduced, then equation (208) becomes

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\propto T_{\mu\nu}$$
$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=8\pi G T_{\mu\nu}\,. \tag{209}$$

The equation (209) can be used for the case if the universe is expanding. Cosmological constant also represents the concept of non-zero vacuum energy.

Results and Discussion

In present thesis work, we have used the four-dimensional division algebra to formulate the space-time expression in curvature form. This thesis contains five chapters. **Chapter-1** discussed about the basic introduction of fundamental interactions. We have introduced special theory of relativity and general theory of relativity. In order to describe the Einstein field equation, we introduced about non-Euclidean geometry of four space-time. Also, we have introduced the concept of four-dimensional quaternionic algebra to examine the geometry of curvature space-time. In **chapter-2**, we have deliberated about literatures with the application of quaternions. We have studied about theories like Riemannian geometry, formulation of Newtonian gravitation, Einstein field equation, etc. We have reviewed the experimental results of GTR such as gravitational waves and gravitational time-dilation. We also have discussed the literatures based on the role of quaternionic four-dimensional algebra in various theories such as, to analysed the wave equation in curved space-time, to obtain quantum wave equation for massive dyons, to study the field equations for dyonic cold plasma, etc.

Chapter-3 described the materials and methods of our quaternionic analysis. This chapter includes six sections. In **section-3.1**, we have discussed about the tensor algebra which is used to describe the equations in covariant form under the change of coordinate system. This section is further divided into four subsections. **Subsection-3.1.1** explained the concept of notations used to write the expressions in compact form. **Subsection-3.1.2** discussed about the Einstein summation convention. The scalar product and vector product of two vectors can be represented in compact form which has been shown in equation (3) and (4) respectively. **Subsection-3.1.3** shows that the addition of two tensors of same rank always yields the tensor of same rank and multiplication of tensors increases the rank of tensor represented in equation (6). In **subsection-3.1.4**, we have discussed about the basis transformations. Equation (8) expressed that when any scalar quantity of rank -0 transformed from one reference frame to another it always remains unchanged. Similarly, transformation of vector component of rank -1 has been represented by equation (9).

Section-3.2 discussed the four-dimensional algebra of quaternion. We have described the various properties of quaternion in terms of their basis elements (e_0, e_1, e_2, e_3). Equation (14) showed the pure vector part only when scalar part is zero and this is known as pure quaternion while equation (15) showed the pure scalar part if vector part is vanished and known as scalar quaternion. This section is divided into three subsection. **Subsection-3.2.1** expressed the quaternionic additive property while **subsection-3.2.2** discussed the properties for quaternionic multiplication. We have expressed that the quaternion is associative but non-commutative with respect to multiplication of quaternions. **Subsection-3.2.3** discussed the alternative general disintegration of quaternions. Equation (42) represented that the quotient of two vectors is also a quaternion. The tensor of a quaternion is similar to the norm of a quaternion which has been showed in equation (43).

In **section-3.3**, we have discussed the basis transformations of quaternion. We have started with the transformation of basis from one plane to another plane in 2-D system as given in equation (52). Then, we have described the 3-D transformation of basis elements of vector and the analogous to 3-D rotation we created the idea to extent vector transformation to quaternionic transformation. Here, in our study we introduced the quaternionic basis transformation given in equation (60), in which the imaginary basis element (e_1, e_2, e_3) transform from one frame to another frame but scalar basis (e_0) remain unchanged.

Section-3.4, we have constructed the Riemannian geometry in terms of quaternionic form. In **subsection-3.4.1**, we started with the basis transformation of quaternion and expressed the transformation for quaternionic space-time in equation (61). The linear transformation of quaternion in flat space-time also has been represented. After applying the quaternionic basis transformation, we have extended the quaternionic transformation in curved space-time. The transformation of scalar and vector components of quaternion have been separately expressed in equation (65) and (66) for $S^{'}$ to S frame. We have also represented the transformation of quaternionic component in covariant form given in equations (69) and (70). The transformation of quaternionic differential operator has been written in equations (71) and (72). Further, for describing the change from one point to another point in curvature space-time we have used the quaternionic tensors. The product of the components of quaternion yields the quaternionic tensor of rank-2 which has been expressed by equations (77)-(80). **Subsection-3.4.2** described the quaternionic covariant derivative in curved space-time. Equation (81) showed the change in contravariant scalar component with respect to partial derivatives of scalar component. Due to the presence of christoffel term, it does not transformed similar to the equation (77). Thus, the christoffel term arise due to effect of curvature from one tangent plane to another tangent plane. Also, the change in contravariant vector field with respect to scalar component of quaternionic is expressed by equation

(87). Similarly, the extension of rank-2 to rank-3 tensor has been represented by equations (94)-(102) by using the quaternionic covariant derivative. This showed the tensorial transformation along with the two terms of christoffel symbol for the change in geodesic path from one point to another point. **Subsection-3.4.3** discussed about the quaternionic metric tensor. For describing the distance from one point to another point in quaternionic curved space-time, we have started with the line element in Minkowski flat space-time. where the line element in flat space-time has been converted to line element in quaternionic curved space-time. In **subsection-3.4.4** we have derived the equation of motion in quaternionic curved space-time which is also known as qauternionic geodesic equation. The freely moving particle in quaternionic curved space-time follows the geodesic path. The direction of motion of a particle shows by tangent of the four vector given in equation (111). Equation (119) represented the actual three-dimensional path in terms of quaternionic basis e_j. The first term in equation (119) shows the acceleration of quaternionic vector and the curvature path arises from the second term which represented as force. If the second term of the equation (119) tends to zero it shows the straight track of the freely moving particles, so that quaternionic geodesic equation reduced to equation of motion in straight path. The change in scalar component of a quaternion variable with respect to time has been showed by equation (120). Like the transformation of quaternionic tensor, quaternionic metric tensor also transformed from one frame to another frame. The covariant derivative of scalar component of quaternionic metric tensor has been represented by equation (121). Equation (130)-(135) have been expressed the value of christoffel symbols in terms of quaternionic metric tensor whereas the metric tensor represented as a quaternionic tensorial potential. **Subsection-3.4.5** discussed the Riemannian christoffel curvature tensor in quaternionic form. The tensor which takes as an input three tangent vector and gives as an output one tangent vector is shown by quaternionic Riemannian christoffel curvature tensor. It is a four-indices tensor that shows the change in vector and scalar component when it transport parallely around any small parallelogram, which represented that the quaternionic space-time is curved see equations (137)-(151). Riemannian christoffel curvature tensor represents all the aspects of curvature, it is also written in terms of christoffel symbol. If the value of quaternionic Riemannian christoffel curvature tensor becomes zero then it converted into flat space-time. One of the important contraction of quaternionic Riemannian tensor is the quaternionic Ricci tensor which represents the change in volume when object parallel transport along a geodesic path as expressed in equation (154)-(157). Another contraction of quaternionic Ricci tensor gives the Ricci scalar shown in equation (158), which tells the amount of change in volume of Riemannian manifolds. Therefore, this section includes the geometric representation of quaternionic curved space-time in which 3-D space is converted into 4-D space-time from one frame to another frame.

In **section-3.5**, we discussed the quaternionic energy-momentum tensor which gives the information about the mass, the energy, and the pressure. The required expression for energy and momentum has been written in equations (159) and (160). The vector component of quaternionic tangent represents the momentum and the scalar component of quaternionic tangent represents the energy. The conservation of quaternionic energy-momentum tensor shown by continuity equation (173). The quaternionic energy-momentum tensor in matrix form has been represented by equation (178) where first term is the energy density, the other diagonal terms represents the pressure, first column excluding first term shows how much energy and mass is moving through the volume at a given instant. Similarly, first row excluding first term tells how much energy and mass is moving through the volume over time. And rest off diagonal terms shows the shear stress. The unified energy-momentum tensor has been written in the quaternionic form given in equations (179) and (180).

In **section-3.6**, we studied the Einstein field equation with quaternionic components. First, we have motivated the Einstein field equations in an attempt to replace Newton law of gravity with covariant tensor equation. We began with the local form of Newton's law of gravity represented by equation (181). The right hand side of this equation contained the material density, we replaced this aspect with quaternionic energy-momentum tensor of rank-2 and the left hand side involves the gravitational potential. The gravitational potential therefore, must be represented in some way by the quaternionic metric tensor. Since, the left hand side of the Poisson equation includes second derivative potential, we have replaced the left hand side of the poisson equation with a tensor involving second derivatives of the quaternionic metric tensor. Here, acceleration of gravitational field is showed by the christoffel symbol in equation (182). The first derivative of quaternionic metric tensor is christoffel symbol and first derivative of christoffel symbol is quaternionic Riemannian curvature tensor. We have also consider the weak field, the metric tensor is slighlty change from original expression showed by equation (184). The components of quaternionic Ricci tensor in terms of energy-momentum form has been showed by equation (194)-(203). But quaternionic Ricci tensor is not divergence free, therefore after using the bianchi identity from equation (205) we obtained equation (207) that represented the Einstein curvature tensor which shows geometric part. To describe the accelerating universe, cosmological constant was introduced in Einstein field equation that represent the expansion of universe showed by equation (209).

Summary and Conclusion

Chapter 5 SUMMARY AND CONCLUSIONS

We need a mathematical algebra to describe the behaviour of any physical theory. For studying the four-dimensional theory we have four-dimensional division algebra called Hamilton algebra. The Hamilton (quaternion) algebra is a linear combination of scalar and vector fields. It is an extension of complex algebra. In present days, quaternion are widely used in modern high energy physics. Therefore, in this thesis, we have applied the notion of quaternion and discussed its importance in the curvature space-time geometry and Einstein field equation. We have used the 3-D transformation of vectors along with their unit elements by rotating frame. Accordingly, we have introduced the tranformation rule for four dimensional quaternionic basis to study the four-dimensional structure of space-time geometry. We have shown that how scalar and vector components of a quaternion transformed from one reference frame to another. The transformation of scalar, vector and their mixed components are also written in terms of quaternionic tensor. We have also expressed the christoffel symbols from the covariant derivative in the form of quaternionic tensor. The effect of curvature space-time arises due to quaternionic christoffel symbol. The quaternionic line element which representing the distance between two points is also described in curved space-time. The quaternionic line element shows the extension of Minkowski metric to quaternionic 4-D metric tensor. It is concluded that the curved path of freely moving particles is effectively explained by the quaternionic geodesic equation. This equation also represents the change in scalar component with respect to time and interprets the christoffel symbol as force. The christoffel symbol is well established in terms of derivative of quaternionic metric tensor while the gravitational potential in tensorial form represented by the metric tensor. It has been confirmed that in present work, using quaternionic Riemannian christoffel curvature tensor for a curvature space-time, the vector and scalar change during its transformation along a curve. The volume of the object obeying a geodesic path changes due to the contraction of quaternionic Riemannian tensor while the contraction of quaternionic Ricci tensor gives a scalar quantity representing the magnitude of change in volume of the object. Further, we discussed about the general energy-momentum tensor which is responsible for the curvature of space-time. The components of quaternionic energy-momentum tensor indicates the energy density, momentum density, pressure and the shear stress. Finally, we have obtained quaternionic Einstein field equation describing the field gravitation as a result of space-time being curved by mass and energy. Also, the relation of the space-time curvature with the energy and momentum is also explained by Einstein field equation.

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VITA

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ABSTRACT

In present work, the four-dimensional quaternionic algebra has been used to describing the space-time geometry in curvature form. The properties of pure quaternion are expressed using the transformation of vector basis from one frame to another. Further, we have expressed the transformation of quaternionic variable with the help of basis transformation of quaternion. The transformation of quaternionic scalar and vector derivatives has been shown. We deduced the quaternionic covariant derivative that explains the change in quaternionic components with respect to scalar and vector components. We have also derived covariant derivative for quaternionic tensor of rank-2. An additional term appeared in the transformation known as quaternionic Christoffel symbol which explain the change from one tangent plane to another. The quaternionic metric tensor has been discussed to describing the line element in quaternionic curved space-time. We also expressed the quaternionic geodesic equation for the curved space-time. We deduced the expression for Christoffel symbol and the Riemannian Christoffel curvature tensor in terms of quaternionic metric tensor. We have also described the energy-momentum tensor in terms of quaternion. Finally, form these expressions we have proposed the quaternionic form of Einstein field equation.

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शोध का शीर्षक : **''वक्रता दिक्–काल का क्वाटरनियोनिक निरूपण एवं आइन्सटीन क्षेत्र समीकरण''**

सलाहकार : डॉ0 बी0 सी0 चन्याल

सारांश

वर्तमान कार्य में, दिक्–काल ज्यामिती के वक्रता रूप का वर्णन करने के लिये चार–विमीय क्वाटरनियोनिक बीजगणित का उपयोग किया गया है। एक तंत्र से दूसरे तंत्र में सदिश आधार के रूपान्तरण का उपयोग करके शुद्ध क्वाटरनियोन के गुणों को व्यक्त किया गया है। इसके अलावा हमने क्वाटरनियोन के आधार रूपान्तरण की सहायता से क्वाटरनियोनिक चर के रूपान्तरण को व्यक्त किया है। क्वाटरनियोनिक सदिश एवं अदिश अवकलजों का रूपान्तरण किया गया है। हमने क्वाटरनियोनिक सहसंयोजक अवकलज को व्युत्पन्न किया जो सदिश एवं अदिश घटकों के सम्बन्ध में क्वाटरनियोनिक घटकों में परिवर्तन को दर्शाता है। हमने रैंक–2 के क्वाटरनियोनिक प्रदिश के लिए सहसंयोजक अवकलज को भी व्युत्पन्न किया। रूपान्तरण में एक अतिरिक्त पद प्रकट हुआ जो क्वाटरनियोनिक क्रिस्टोफल प्रतीक के रूप में जाना जाता है, तथा यह प्रतीक एक स्पर्शरेखा तल से दूसरे स्पर्शरेखा तल में परिवर्तन की व्याख्या करता है। क्वाटरनियोनिक वक्रित दिक्–काल में रेखा तत्व का वर्णन करने के लिये क्वाटरनियोनिक आव्यूह प्रदिश की भी व्याख्या की। हमने वक्रित दिक्–काल के लिए क्वाटरनियोनिक जियोडेसिक समीकरण को भी व्यक्त किया। हमनें क्वाटरनियोनिक आव्यूह प्रदिश के पदों में, क्रिस्टोफल प्रतीक एवं रीमानियन क्रिस्टोफल वक्रता प्रदिश के व्यंजक को भी व्युत्पन्न किया। हमने क्वाटरनियोन के पदों में ऊर्जा–संवेग प्रदिश की भी व्याख्या की। अंततः इन सभी व्यंजकों की सहायता से हमनें आइन्सटीन क्षेत्र समीकरण के क्वाटरनियोनिक रूप को प्रस्तावित किया।

(बी0 सी0 चन्याल)
सलाहकार

(मनीषा गौतम)
लेखिका