Generalized Row-Column Designs for Single and Multi-Factor Experiments

एकल एवं बहु कारक प्रयोगों के लिये सामान्यीकृत पंक्ति-स्तम्भ अभिकल्पनायें

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Generalized Row-Column Designs for Single and Multi-Factor Experiments

By

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CERTIFICATE

This is to certify that the thesis entitled "Generalized Row-Column Designs for Single and Multi-Factor Experiments" submitted to the Faculty of the Post-Graduate School, Indian Agricultural Research Institute, New Delhi, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Agricultural Statistics, embodies the results of bona fide research work carried out by Ms. Anindita Datta under my guidance and supervision and that no part of this thesis has been submitted for any other degree or diploma.

It is further certified that any assistance and help availed during the course of investigation as well as source of information have been duly acknowledged by her.

Place: New Delhi
Date: 25-01-2016

(Seema Jaggi)
Chairperson
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Dedicated to Maa, Baba, Bhai and my Chairperson

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CONTENTS

CHAPTER I

INTRODUCTION

Designing an experiment and drawing valid conclusions based on the experiment by using appropriate statistical tools are two important aspects of any scientific investigation. In many experimental situations, in addition to the treatments there may be some other factors which potentially contribute to the variability in the observed response. The technique of blocking is used to control the experimental error variation due to these types of extraneous sources.

1.1 Row-Column Designs

When the heterogeneity present in the experimental material is from two sources, then two-dimensional blocking or double blocking of the experimental units is recommended for control or reduction of experimental error. The two blocking systems are referred to generally as row blocking and column blocking and the resulting designs are termed as Row-Column (RC) designs. These designs are used to control variability in field and animal experiments. For example, in a greenhouse experiment on tobacco mosaic virus, the experimental unit is a single leaf. The plant and the position of the leaf on the plant may affect the number of lesions produced per leaf by rubbing the leaf with a solution, which contain the virus. Thus, here individual plant is one source of variability and represents rows and the position of the leaf from top to bottom on each plant represent columns. Another situation is in case of a laboratory trial to compare the percentage of protein in various grains, rows may be the different analysts and columns may be the occasions. Further, in an irrigation experiment in horticultural research, rows may be represented by channels and columns by the positions along the channels.

Latin square design is the simplest row-column design. In a Latin square design, v treatments are arranged in v rows and v columns in such a way that each treatment occurs once in each row and once in each column e.g. an animal experiment is conducted to compare the effects of four feeds eliminating the variation due to four breeds and four age groups of calves. Data is on growth rate of calves during a certain period. Here, rows

represent age groups and columns represent different breeds. Following is the arrangement of a Latin square design for this situation with rows and columns complete.

Rows (Age Groups)	Columns (Breeds)			
	I	II	III	IV
I	1	2	3	4
II	2	3	4	1
III	3	4	1	2
IV	4	1	2	3

Latin squares have the restriction that the rows and columns must be equal in number to one another and to the number of treatments. In practical experiments Latin squares are very useful where the number of treatments is small. The available range of designs is generally restricted to sizes from about 4 × 4 to about 7 × 7. The upper end of the size range can be extended by using incomplete Latin squares of size (n-1) × n or size n × (n-1) obtained by deleting one complete row or one complete column from a Latin square of size n × n (Yates, 1936) whereas the lower end of the scale can be extended by using augmented Latin squares of size (n + 1) × n or of size n × (n + 1) obtained by repeating a complete row or a complete column of a Latin square of size n × n (Pearce, 1952). So when experimental units are in rectangular array then these designs are useful. Various types of row-column designs and their properties are discussed in Hinkelmann and Kempthorne (2005).

1.2 Generalized Row-Column (GRC) Designs

Most of the row-column designs developed in the literature have one unit corresponding to the intersection of row and column. However, there may be instances when the number of treatments is substantially large with limited number of replicates. A more general class of row-column designs is required where there are more than one unit in each row-column intersection. These designs may be called as Generalized Row-Column (GRC) designs. GRC design is an arrangement of v treatments in p rows and q columns such that the intersection of each row and column consist of more than one unit. Following are some examples:

- To compare a number of dietary treatments on mice, the different breeds and different age groups constitute the two sources of variability. The cages available

with the experimenter have two partitions accommodating two mice of same parity, one in each partition. Hence, corresponding to each breed-age combination there are two mice, each receiving one treatment.

- An experiment is taken to compare twelve pest control treatments on apple trees. For cultural reasons, four long replicate rows, each one tree wide, are used with twelve plots per row. Each row is subdivided into four blocks or cells of three plots and the four adjacent blocks at any one position along the four rows formed a replicate column of twelve plots.

These designs are studied in the literature in different names such as Semi-Latin square [$(n \times n)/k$ i.e. there are n rows and n columns and intersection of each row and column contains a cell of k units], Trojan square (based on the existence of mutually orthogonal Latin squares), Semi-Latin rectangles, generalized incomplete Trojan-type designs, Row-column designs with multiple units per cell. These designs can be randomized by rows, by columns and by units within cells. Following is a semi-Latin square with n = 3, k = 2 and number of treatments v = 6:

Rows	Columns		
	I	II	III
I	1 2	3 4	5 6
II	3 6	5 2	1 4
III	5 4	1 6	3 2

The design is a doubly resolvable incomplete block design if rows and columns are ignored. When experimental material having two orthogonal nuisance factors Semi-Latin squares are appropriate designs. Semi-Latin squares are also sometimes called modified Latin squares or pseudo-Latin squares. A semi-Latin square possesses two features, it controls heterogeneity in two directions as does the ordinary Latin square and does not require a large number of replications.

Following is a semi-Latin rectangle for 4 treatments in 2 rows, 4 columns with k = 2 units per cell:

Rows	Columns			
	I	II	III	IV
I	1 2	2 3	3 4	4 1
II	3 4	4 1	1 2	2 3

Here each treatment appears in each row twice and in each column once.

1.3 Experimental Situations

Some of the experimental situations are described below along with designs appropriate for such situations.

Situation 1.3.1 (Bailey and Monod, 2001): An experiment was conducted on tobacco plants at Rothamsted Experimental Station to check whether a mechanism to inhibit tobacco mosaic virus had been carried over to following generations. Each treatment was a solution made from an extract of one of the offspring plants. The solution was rubbed onto several half-leaves of normal tobacco plants. The number of lesions per half leaf was measured and the logarithm of this number analyzed by ANOVA. There are eight plants and pair of half leaves at four heights. A row-column design which has less number of rows than columns is useful in such situations as the number of plants available for the experiment is typically more than the number of usable leaves and their positions per plant. The experimenter is interested to compare more than two treatments in c plants each with leaves at r heights, where typically $r < c$. Generally, the two half leaves of each of the rc leaves form the plots. So here leaf heights represent rows and the plants as columns and two plots in the intersection of each row and column. For such situations semi-Latin rectangles are useful.

Rows (Heights)	Columns (Plants)							
	I	II	III	IV	V	VI	VII	VIII
I	4 5	5 6	6 7	7 0	0 1	1 2	2 3	3 4
II	1 7	2 0	3 1	4 2	5 3	6 4	7 5	0 6
III	0 3	1 4	2 5	3 6	4 7	5 0	6 1	7 2
IV	2 6	3 7	4 0	5 1	6 2	7 3	0 4	1 5

Situation 1.3.2 (Bailey, 1992): Consider a food sensory experiment conducted in 3 sessions where 6 food items are to be compared. There are 6 panelists. So each of them will taste 2 food items at each session. In this case, a GRC design with 3 rows, 6 columns with each row-column intersection having cell of size 2 can be used. Following is the arrangement of such a design:

Rows (Sessions)	Columns (Panelists)					
	I	II	III	IV	V	VI
I	1 4	2 6	2 5	3 5	6 3	4 1
II	2 3	1 5	4 6	6 1	4 5	3 2
III	6 5	4 3	3 1	2 4	1 2	6 5

Situation 1.3.3 (Bailey, 1992): A consumer research organization wishes to compare ten new brands of vacuum cleaners. The organization has bought one sample of each brand. Some housewives have agreed to compare the vacuum cleaners. Each housewife will use two vacuum cleaners in her home for a week and give one a score. Thus, at most five housewives can test cleaners in any one week. Moreover, to allow for housewife effects, it is best that each housewife tests every cleaner and therefore take part in the trial for five weeks. The following design is possible, with rows representing weeks, columns housewives and symbols as cleaners.

Rows (Weeks)	Columns (Housewives)				
	I	II	III	IV	V
I	1 10	7 5	9 3	6 4	8 2
II	2 7	1 9	5 6	8 3	10 4
III	3 6	2 4	1 8	10 5	9 7
IV	9 4	8 6	2 10	1 7	3 5
V	8 5	3 10	4 7	9 2	1 6

This situation arises frequently in consumer testing, when only one object of each of nk brands is available. Only one of each brand can therefore be used at any one time, and the trial is completed most quickly if every brand is used in every time-period. Consumers test k objects per time-period in their own homes: typically k = 2 or 3 or 4. To eliminate consumer effects, each of n consumers participates in the trial for n weeks, with time-period, consumers and brands forming the rows, columns and symbols of a row-column design.

Situation 1.3.4 (Bailey, 1992): An experiment was conducted where ten treatments are to be applied to sugar beet, which is grown in a 5 × 10 rectangular array of plots. Each plot is a single long North-South row of sugar beet, so the 10 plots in a single row of the rectangle are close to each other and these rows are regarded as a nuisance factor. The beet is sown from five seed-drills on an arm, which protrudes from the right of the tractor. The tractor drives Northwards up the left-hand side of the array, sowing seed in the first five columns, then turns round and drives Southwards down the right-hand side of the array, sowing seed in the last five columns. Thus, the first and last columns are sown by the same drill and drills form a second nuisance factor. The following row-column design for ten treatments in five rows, five columns and the intersection of each row and each column contains a cell of two units (k = 2) can be used:

Rows (Plots of Sugar Beet)	Columns (Drill)									
	I	II	III	IV	V	V	IV	III	II	I
I	1	5	3	4	2	8	6	9	7	10
II	2	1	5	3	4	10	8	6	9	7
III	6	4	8	10	9	7	5	1	2	3
IV	9	8	10	7	5	3	1	2	6	4
V	8	3	7	2	1	6	9	4	10	5

Situation 1.3.5 (Bailey, 1992): Sometimes the effects of some treatments may persist during the next experiment. This situation happens if the experimental units are trees, but it can also occur on arable crops if the treatments affect the soil directly, by inhibiting and

encouraging nematode growth. An experiment was conducted where suppose that in the first year, five varieties of potato were compared in five replicates. In the second year, a single standard variety is grown and ten chemicals are tested for their ability to control nematodes. First year's varieties will affect the number of nematodes in the soil, but it is assumed that there is no interaction between those varieties and second year's chemicals. Chemicals can be applied to smaller area of land than varieties, so each plot from first year is split into two for the chemicals. Use of a row-column design with rows representing old replicates, column as varieties and symbols as chemicals, ensures that each chemical occurs once in each old replicate and once on soil that had each variety in first year. The design given below could be used:

Rows (Replicates)	Columns (Varieties)				
	I	II	III	IV	V
I	1 10	7 5	9 3	6 4	8 2
II	2 7	1 9	5 6	8 3	10 4
III	3 6	2 4	1 8	10 5	9 7
IV	9 4	8 6	2 10	1 7	3 5
V	8 5	3 10	4 7	9 2	1 6

Situation 1.3.6 (Edmondson, 1998): An experiment was conducted to compare the colour intensities of apple sauce. The treatments consist of all combinations of 12 blends of apple sauce with 4 concentration of cinnamon. Treatments could be stored for 4 different lengths of time. A GRC design as shown below was used in which rows represented cinnamon concentrations, columns as storage times and symbols as blends.

Rows (Cinnamon Concentrations)	Columns (Storage Time)			
	I	II	III	IV
I	1 5 9	2 6 10	3 7 11	4 8 12
II	2 7 10	1 8 9	4 5 12	3 6 11

III	3 8 12	4 7 11	1 6 10	2 5 9
IV	4 6 11	3 5 12	2 8 9	1 7 10

This arrangement ensures that each of the 48 treatments occurred once and that both treatment factors were orthogonal to storage times. Part of the interaction between blends and concentrations was totally confounded with storage times.

Trojan squares are a special class of semi-Latin square designs based on sets of mutually orthogonal superimposed Latin squares and have been shown to be maximally efficient for pair-wise treatment comparisons in the plots-within-blocks stratum (Bailey, 1992). Trojan squares of size (n × n)/k are based on k sets of n × n orthogonal Latin squares and have a natural factorial treatment structure. Following is an example of a non-randomized Trojan square design of size (4 × 4)/2 constructed by superimposing two mutually orthogonal Latin square of size 4 × 4, one with 1, 2, 3, 4 and the other with 5, 6, 7, 8 symbols. This design could be extended to a design of size (4 × 4)/3 by superimposing an additional mutually orthogonal Latin square of size 4 × 4 but no further Trojan extension is possible, there being only three mutually orthogonal Latin squares of size 4 × 4.

Rows	Columns			
	I	II	III	IV
I	1 5	2 6	3 7	4 8
II	2 7	1 8	4 5	3 6
III	3 8	4 7	1 6	2 5
IV	4 6	3 5	2 8	1 7

The above design is complete row-wise and column-wise. However, cell-wise the design is a two associate Partially Balanced Incomplete Block (PBIB) design following a Group Divisible (GD) association scheme with the symbols from the same square as first associates and those from the other square(s) as second associates. A randomized layout of the design is:

Rows	Columns			
	I	II	III	IV
I	5 2	1 6	7 4	8 3
II	1 7	8 2	3 5	4 6
III	8 4	7 3	2 6	5 1
IV	3 6	4 5	1 8	2 7

Complete Trojan squares of size (n × n)/k have n^2 cells of size k and require n replicates of nk treatments. Sometimes, design or cost constraints make complete Trojan squares impossible and then incomplete. Trojan squares of size [(n-1) × n]/k or of size [n × (n-1)]/k can be useful. Such incomplete Trojan squares can be constructed by omitting any complete row or any complete column from any Trojan design of size (n × n)/k. For example, omitting the first row of the (4 × 4)/2 standard Trojan square gives the incomplete Trojan square of size (3 × 4)/2 as given below:

Rows	Columns			
	I	II	III	IV
I	2 7	1 8	4 5	3 5
II	3 8	4 7	4 6	2 5
III	4 6	3 5	2 8	1 7

The above design is complete in rows. If we consider the columns as blocks, then the given design is a PBIB design with GD association scheme. The association scheme is based on the deleted row (column). Again if each cell in the given design is considered as blocks, it leads to a PBIB design with three associate classes following rectangular association scheme.

For example, if an experiment is taken to compare eight nutrient regimes for glasshouse tomato. There were eight beds of plants and these were divided into four pairs of adjacent

beds to form the four columns of the design. Each bed was subdivided into three sets of eight adjacent plots running across the beds of the design used as the rows of the design. So, this situation gives rise to an incomplete Trojan square of size $(3 \times 4)/2$.

The block design from a semi-Latin square by ignoring the rows and columns is an incomplete block design. The block design of a Trojan square with k = n - 1 is necessarily a rectangular lattice design; therefore these are called as Latinized rectangular lattice. Generalizing this idea, Williams (1986) called semi-Latin squares as Latinized incomplete-block designs.

The closest relationship of semi-Latin (Trojan) squares is with partially balanced incomplete block designs with two associate classes. The relationship may be formalized as follows:

If there is k mutually orthogonal $n \times n$ Latin squares that can be used to produce a design with n replicates of nk treatments on n^2 blocks of k plots each, then this design will be semi-regular partially balanced design. The treatments from the same group never concur in the same block ($\lambda_1 = 0$) and that treatments from different groups concur in just one block ($\lambda_2 = 1$). Further, its blocks can be arranged in complete replicates in two mutually orthogonal ways. From this point of view the design may be regarded as a semi-Latin (Trojan) square with n rows and nk columns.

Suppose that R, C and S are treatment factors with n, n and nk levels respectively. If there are no interactions between the factors, then the three main effects may be orthogonally estimated from the fractional replicate consisting of the n^2k combinations in a semi-Latin square. Thus, a semi-Latin square is an asymmetrical orthogonal array with n^2k assemblies (or plots/ units), three constraints (factors R, C and S) and strength 2. This means that, for every pair of treatment factors, each pair of levels occurs equally often in the fraction.

1.4 Motivation of the Study

In general, the GRC designs developed in the literature are to study all possible pair-wise treatment comparisons. However, there may arise experimental situations where it is

desired to compare treatments belonging to two disjoint sets. The two sets are disjoint in the sense that there are no common treatments between the two. The interest here is to estimate the contrasts of the type ($\tau_s - \tau_{s'}$) with as high precision as possible, τ_s and $\tau_{s'}$ belongs to 1st and 2nd set of treatments respectively. For example, in agricultural experiments the aim is to test a set of new varieties of a crop with already existing varieties and to determine which of the varieties performs better in comparison to the existing varieties. The designs that are efficient for making all pair-wise comparisons may not be efficient for these subset of comparisons. GRC designs for comparing treatments belonging to two disjoint sets needed to be investigated.

The presence of missing observations, outliers in the data, etc. are some of the disturbances that may occur during experimentation. These disturbances may lead to less precise comparisons among treatments tried in the experiment. In order to overcome such situations, designs which are insensitive or robust against missing observations were required.

Further, the GRC designs developed in the literature are for single factor experiments Situations may arise wherein the experiment consist of more than one factor with each factor having more than one levels. A lot of work has been done for designs with factorial treatment structure under block and row-column setup. It was required to develop some methods of constructing GRC designs for multi-factor experiments.

A number of GRC designs are developed in the literature. For easy accessibility and quick reference of these designs by the experimenters, web solution for the generation of GRC designs was needed to be developed.

Considering the above, the present investigation was taken up with the following objectives:

1. To study generalized row-column designs for comparing treatments belonging to two disjoint sets.
2. To investigate the robustness of generalized row-column designs against missing observation(s).
3. To develop methods of constructing generalized row-column designs with factorial treatment structure.

4. To develop web solution for the generation of generalized row-column designs.

1.5 Scope of the Thesis

An introduction to GRC designs has been given in the present chapter and various experimental situations have also been described. Chapter II gives a detailed background of the topic by reviewing the work done on GRC designs. The general methodology consisting of the model, the information matrix pertaining to treatment effects, definitions and the procedure for working out the efficiency has been given in Chapter III.

Chapter IV contains the results and discussion related to the four objectives. Balanced Bipartite Generalized Row-Column (BBP-GRC) designs have been defined for the situation when the interest is to compare treatments belonging to two disjoint sets. Methods of constructing series of BBP-GRC have been described in which the contrasts of interest related to first set versus second set of treatments is estimated more precisely. Robustness of different classes of GRC designs against missing of one or more observations has been investigated and the efficiency of the residual designs have been reported and summarized in this chapter. Generalized confounded row-column (GCRC) designs, generalized partially confounded row-column (GPCRC) designs and fractional GCRC designs have been defined for the situations wherein the experiment consist of more than one factor with each factor having more than one levels. Methods of obtaining these designs have been given ensuring all lower order interactions including main effects to be estimable. In the end of Chapter IV, a description of the online software developed named WebGRC has been given that provides the online generation of randomized layout of GRC designs.

The results obtained have been summarized and concluded in Chapter V followed by an abstract and list of references. SAS codes have been developed for obtaining the information matrices and also the efficiencies of the GRC designs which are given in the Appendix at the end.

CHAPTER II

BACKGROUND

As described in Chapter I, GRC designs have been studied in the literature as semi-Latin square/ Trojan square designs. An (n × n)/k semi-Latin square is an arrangement of nk symbols (treatments) in an (n × n) square array such that each row-column intersection contains k symbols and each symbol occurs once in each row and each column. An array for an (n × n)/k semi-Latin square is as shown below.

Rows	Columns		
	1	. . .	N
1	1 . . . k	. . .	1 . . . k
. . .	. . .	. . .	. . .
n	1 . . . k	. . .	1 . . . k

The comparison of 12 treatments in an ordinary Latin square requires 144 plots, but semi-Latin square can be used with 72, 48, 36 or 24 plots.

Harshbarger and Davis (1952) first discussed Trojan squares but then it was named as Latinized Near Balanced Rectangular Lattices having k = n-1. These designs treat specifically the cases where the number of treatments is the product of two consecutive integers k (k-1), but they can be extended to other cases where the integer are not consecutive. They described the use of Latinized Near Balanced Rectangular Lattices in food industry. Rojas and White (1957) discussed some other uses of semi-Latin squares in the agronomic research.

Darby and Gilbert (1958) discussed the general case for k < n and introduced the name Trojan square designs where k > 2 and designs of the Latinized Rectangular Lattice type are described as Trojan squares for any 1 < k < n.

The varieties concurrence graph of an orthogonal semi-Latin square consists of n complete graphs on k vertices, with all edges of multiplicity n while that of Trojan square is a complete k-partite graph on k sets of n vertices. The analysis of this can be performed like a Latin square design with n rows, n columns and nk treatments.

Trojan squares of size (n × n)/ k are based on k sets of n × n orthogonal Latin squares and have factorial type treatment structure. For designs where Youden rectangles exist, an efficient (m × n)/k incomplete Trojan square can be obtained by superimposing k suitable (m × n) Youden rectangles and regarding each row-by-column intersection of the superimposed design as a block of size k. Williams (1986) generalized the notion and called semi-Latin squares as Latinized incomplete-block designs. Andersen and Hilton (1980) called semi-Latin squares as (1, 1, k) Latin rectangles.

The combinatorial properties of semi-Latin squares and related designs are discussed by Preece and Freeman (1983). They discussed the relationship between semi-Latin square, Trojan square and certain partially balanced incomplete block designs with two associate classes. They examined some relevant semi-regular designs of Clatworthy (1973) which can be obtained from semi-Latin squares.

Bailey (1988) discussed further construction for a range of semi-Latin and Trojan square designs through the procedure of inflation, superposition, product method, deletion augmentation and transversal augmentation. In inflation procedure, an n × n Latin square is taken and each letter is replaced by k new letters. This gives an (n × n)/ k semi-Latin square. Take an (n × n)/s semi-Latin square with s letters per cell. Replace each letter by r new letters. This gives an (n × n)/ (sr) semi-Latin square. It is the r-fold inflation of the original square. Inflated Latin square with n = 4 and k = 3 is as follows:

Rows	Columns			
	I	II	III	IV
I	A α a	B β b	C γ c	D δ d
II	D δ d	A α a	B β b	C γ c
III	C γ c	D δ d	A α a	B β b
IV	B β b	C γ c	D δ d	A α a

Bailey (1992) gave methods of constructing a range of semi-Latin and Trojan square designs. and She concluded that the Trojan squares are the optimal choice of semi-Latin squares for pair-wise comparisons of treatment means and also discussed their efficiencies. These are particularly suitable for crop research experiments either in field or in the glasshouse. Situation 1.3.2 mentioned in Chapter I is an example from real designed experiments.

Bailey (1992) recommended that the appropriate randomization of a semi-Latin square involve firstly, randomizing independently the rows and columns, and then randomizing independently the plots/ units within each cell. Based on the randomization procedure, the semi-Latin squares are recognized and statistically analyzed as a three-block-structured design, where the three types of blocks are the rows, columns and row-column intersections. Treatments are orthogonal to both rows and columns, which simply means that the treatments of a semi-Latin square are all accommodated in each row and each column.

Trojan square is generally the most appropriate choice of semi-Latin square for crop research and many applications of these designs are given by Edmondson (1998). Simple incomplete Trojan squares obtained by omitting a single row from a complete Trojan square have been discussed by Edmondson (1998) and are of considerable practical utility, but they are not sufficiently general for all purposes. However, unlike simple incomplete Trojan squares, not all generalised incomplete Trojan squares of given size are equally efficient and criteria are needed for discriminating between different designs.

Cheng and Bailey (1991) showed that every Trojan square is A-, D- and E-optimal among semi-Latin squares, indeed among all incomplete-block designs of that size. However, there are no Trojan squares when n = 6 and this size is needed for some applications. Bailey and Royle (1997) studied optimal semi-Latin squares with side six and block size two. The contrasts properties of the optimal semi-Latin squares with side six and block size two was investigated by Uto and Ekpenyong (2014) with a view to discriminating amongst them.

Some reference of semi-Latin squares and Trojan squares can be found in Dean *et al.* (2015). Several methods of construction of semi-Latin squares have given by Bedford and

Whitaker (2001). Bailey and Monod (2001) defined semi-Latin rectangle as a row-column design in which each row-column intersection has the same size, say k where k > 1, every treatment appears the same number of times in each row and every treatment appears the same number of times in each column. They gave an application of semi-Latin rectangle in plant disease experiments.

Dharmalingam (2002) gave an application of Trojan square designs and used it to obtain partial triallel crosses. SahaRay (2001) studied designs with unequal row and column sizes.

Edmondson (2002) constructed generalized incomplete Trojan square designs, denoted by (m × n)/k where m denotes the number of replicates of nk treatments, based on a set of k cyclic generators. The design is complete if rows are considered as blocks. If the columns are considered as blocks, then it gives a PBIB design with two associate classes. Again, if each cell in the given design is considered as blocks, it leads to a PBIB design with $\frac{v}{2}$ [or $\frac{v-1}{2}$] associate classes if v is even [or odd], where v = nk is the number of treatments.

A (4 × 4)/4 semi-Latin square is an arrangement of 16 treatments in an (4 × 4) array such that each row-column intersection contains 4 symbols and each symbol occurs once in each row and each column (Bailey and Chigbu, 1997). There existed three optimal (4 × 4)/4 semi-Latin squares for sixteen treatments in blocks of size four. Since these squares do not have the same concurrences, there was a need for distinguishing one square from the others and determining the most preferred square in a given context. Chigbu (2003) obtained the best of the three optimal (4 × 4)/4 semi-Latin squares by finding and comparing the variances of elementary contrasts of treatments for the squares.

Parsad (2006) discussed a method of constructing semi-Latin square with v = 2n treatments in n rows, n columns and cell size k = 2 by developing initial column.

Jaggi *et al.* (2010) defined generalized incomplete Trojan-Type designs to be a row-column design in which each cell, corresponding to the intersection of row and column, contains more than one treatment and the rows are incomplete. A method of constructing

generalized incomplete Trojan-Type design was developed and some properties of this class of designs are discussed. The generalized incomplete Trojan-Type designs were obtained for any number of treatments ≥ 6 (v = sm). Further, the cell size of these designs can be chosen depending on the experimental resources available. Considering rows as blocks and cells as blocks, the designs are found to be partially balanced for estimating the elementary treatment contrasts. Following is a generalized incomplete Trojan-Type design in eight rows, two columns with cells containing 4 units:

Rows	Columns							
	I				II			
I	1	2	3	4	5	6	7	8
II	3	4	5	6	7	8	9	10
III	5	6	7	8	9	10	11	12
IV	7	8	9	10	11	12	13	14
V	9	10	11	12	13	14	15	16
VI	11	12	13	14	15	16	1	2
VII	13	14	15	16	1	2	3	4
VIII	15	16	1	2	3	4	5	6

Varghese and Jaggi (2011) obtained generalized row-column designs with unequal cell sizes. Following is a generalized row-column design in eight rows, two columns with two types of cell sizes, column one with cell size 4 and column two with cell size 2:

Rows	Columns					
	I				II	
I	1	2	3	4	5	6
II	2	3	4	5	6	7
III	3	4	5	6	7	8
IV	4	5	6	7	8	1
V	5	6	7	8	1	2
VI	6	7	8	1	2	3
VII	7	8	1	2	3	4
VIII	8	1	2	3	4	5

It was showed that these designs can be advantageously used for obtaining mating plans, like partial diallel cross (PDC) or partial triallel cross (PTC) by considering treatments in the design as individual parental lines in the breeding programme and by making crosses between lines within each cell. Thus, the cell contents can be directly used to obtain the crosses and hence the method is quite simple and results in small number of crosses. The plans obtained from such designs are uniquely determined. Datta *et al*. (2014) obtained some methods of constructing row-column designs with multiple units per cell that are structurally incomplete. Datta *et al.* (2015) developed methods of constructing row-column designs with multiple units per cell with equal/ unequal cell sizes that are structurally complete, i.e. all the cells corresponding to the intersection of row and column receive at least two treatments.

In general, the GRC designs developed in the literature are to study all possible pair-wise treatment comparisons. However, there may arise experimental situations where it is desired to compare treatments belonging to two disjoint sets. The earliest work on comparing treatments from one set (test treatments) with one or more replications of treatment in second set (control) was carried out by Dunnett (1955, 1964). Dunnett (1955) also proposed (but did not solve) the problem of optimally allocating experimental units to control and test treatments so as to maximize the probability associated with the joint confidence statement concerning the many-to-one comparisons between the mean of the control treatment and the means of the test treatments. This optimal allocation problem was solved by Bechhofer and his coworkers (1969, 1970, 1981). A lot of work has been done in block design setting for comparing treatments from one set with a single treatment from other set.

Robson (1961) studied that a BIB design consisting of all the treatments including a control can be used for such situations. Cox (1958, page 238) noted that the BIB designs were not appropriate for the multiple comparisons with the control problem because of the special role played by the control treatment. Cox advocated augmenting a BIB design in test treatments with one or more replications of control in each block as a means of getting good designs. Cox did not give any analytical details of the proposed design.

Pearce (1960) proposed a class of designs for comparing test treatments with a control and gave their analysis for the one-way elimination of heterogeneity model. Pesek (1974)

has given logical details for a special case of Cox's designs in which control treatment is employed once in each block. He showed this design to be more efficient than a BIB design for comparison with a control, but is less efficient for pair-wise comparisons between the test treatments.

Das (1958) gave a class of block designs for treatment-control comparisons by augmenting one or more replication of a control in each block of the incomplete block designs and by adding some more blocks containing all the treatments. These designs were called as reinforced incomplete block designs.

Bechhofer and Tamhane (1981) gave a general class of block designs that is appropriate for making test treatment control comparisons. They referred such designs as Balanced Test Treatments Incomplete Block (BTIB) designs since these are balanced with respect to test treatments. Bechhofer and Tamhane (1981) also gave some methods of construction of these designs.

Constantine (1983) showed that a BIB design in test treatments augmented by a replication of the control in each block is A-optimal in the class of designs with exactly one replication of the control in each block. Jacroux (1984) showed that Constantine's conclusion remains valid even when the BIB designs are replaced by some group divisible designs.

An algorithm for finding A-optimal designs for the test treatments-control comparison was suggested by Hedayat and Majumdar (1984). Hedayat and Majumdar (1988) have given a detailed review on the optimality of designs for comparing test treatments with controls under 0-, 1- and 2-way elimination of heterogeneity models. A special class of General Efficiency Balance (GEB) designs, making test treatments-control comparisons was pointed out by Prasad (1989). Jacroux (1990) has studied in detail some optimal designs for comparing a set of test treatments with a set of controls under 0-way elimination of heterogeneity model. Parsad (1991) has investigated certain aspects of optimality of incomplete block designs for comparing a set of treatments (called test treatments) with one standard treatment for designs with unequal block sizes.

Jaggi (1992) studied optimality of one-way heterogeneity designs for comparing two disjoint sets of treatments. Parsad and Gupta (1994) studied optimality aspects of designs for making test treatments-control treatments comparisons and test vs test comparison under fixed and mixed effects models.

Majumdar and Tamhane (1996) considered the problem of the design and analysis of experiments for comparing several treatments with a control when heterogeneity is to be eliminated in two directions.

Jaggi *et al*. (1996) gave A-efficient block designs for comparing two disjoint sets of treatments. Parsad *et al*. (1996) obtained trace optimal designs with unequal block sizes for comparing two disjoint sets of treatments. Jaggi and Gupta (1997) obtained A-optimal block designs with unequal block sizes for comparing two disjoint sets of treatments. Gupta *et al*. (1998) studied weighted A-optimal row-column designs for making treatment-control and treatment-treatment comparisons. Ramana (1995) studied optimality aspects of designs for making test treatments-control treatments comparisons and test vs test comparison under fixed and mixed effects models.

Srivastava *et al.* (2000) studied optimality of block designs for making test treatments-control comparisons. Gupta and Parsad (2001) gave an overview of block designs for comparing test treatments with control treatments. Jacroux (2003) constructed A-optimal designs for comparing two sets of treatments. Hedayat and Yang (2005) obtained optimal and efficient cross over designs for comparing test treatments with a control treatment.

Abeynayake and Jaggi (2009) gave an overview of block designs for test treatments – control(s) comparisons and described some classes of row-column designs which are balanced for test treatments vs. control comparisons.

Some work has been done in row-column (RC) design for comparing treatments from one set with a single treatment from other set. Majumdar and Tamhane (1996) considered the problem of the design and analysis of experiments for comparing several treatments with a control when heterogeneity is to be eliminated in two directions. Parsad and Gupta (2001) have obtained some balanced bipartite row-column designs. Sarkar *et al*. (2013) obtained some general methods of constructing Balanced Test-control row-

column (BTRC) designs in complete/incomplete rows/columns and also obtained a class of BTRC designs with empty nodes.

In a well-planned experimental work, situation may arise where some observations are lost or destroyed or unavailable due to certain reasons that are beyond the control of the experimenter. Unavailability of the observations destroys the orthogonality and the balance of the design and also affects the inference. The commonly used statistical procedures are valid only when the underlying crucial assumptions like adequacy of underlying model, independence and normality of observations etc. are satisfied.

In designed experiments, one may have the data, which do not follow the crucial assumptions or ideal conditions on which the statistical procedures are developed. It may be due to the disturbances, which may occur during experimentation. The commonly known disturbances which may affect the inferences to any extent are presence of outliers in the data, loss of information in the form of missing observations, inadequacy of fitted model, presence of a systematic trend in the experimental units, exchange or interchange of treatments during the experiments, etc. These disturbances may render even the best statistical procedures poor. In order to circumvent such situations one has to think of designs, which are insensitive, or robust against such disturbances so that inferences drawn by the use of these designs are least affected due to such disturbances. The term robustness was introduced in statistical literature by Box and Hay (1953), but it was Fisher (1935) who produced the initial idea of robustness in his classical book "Design of experiments". A lot of work has been done to investigate robustness on block design set up.

The relative loss in efficiency due to missing data was first investigated by Jone (1976). He considered the problem of efficiency of the residual design after deleting one treatment from a BIB design. A lower bound to the efficiency E_{min} and the maximum efficiency E_0 (the maximum efficiency that would be obtained if the residual designs were balanced) were obtained. It was also proved that for a symmetrical BIB design, the efficiency of the residual design is E_{min} and is independent of the choice of the treatment to be omitted.

Ghosh (1978) introduced the robustness property of designs against non-availability of

data in the sense that, when t (a positive integer) observations are missing, all parameters are still estimable in the model assumed. Ghosh (1982) studied the robustness of block designs using connectedness property. Ghosh *et al*. (1983) investigated the robustness of PBIB designs based on association scheme with m classes against the unavailability of data.

Baksalary and Tabis (1987) derived three sufficient conditions for a block design to be maximally robust and applied these conditions to examine the robustness of certain variance-balanced designs.

Mukerjee and Kageyama (1990) evaluated the exact efficiency of the residual design in case of singular, semi regular and regular group-divisible (GD) designs with $\lambda_1 = 0$ and gave lower and upper bounds for the efficiency of regular GD designs with $\lambda_1 > 0$ for the loss of all observations of a complete block.

Dey *et al*. (1991) gave a review on robustness of designs. Gupta and Srivastava (1992) have investigated the robustness of (i) binary balanced block designs when all the observations in $m \geq 1$ disjoint blocks are lost, (ii) resolvable BIB designs against the loss of one complete replication (iii) Augmented BIB designs against the loss of all the observations in a block. Gupta and Srivastava (1992) have also shown that the variance balanced designs listed by Gupta and Jones (1983) are fairly robust against the loss of two disjoint blocks.

Ghosh *et al*. (1992) obtained the exact efficiencies when the design is BIB design or a singular semi-regular or regular GD design with $\lambda_1 = 0$ for the loss of a single observation, for regular GD designs with $\lambda_1 > 0$ and worked out the lower and upper bounds for the efficiency against the loss of a single observation.

Chakraborty (1996) studied robustness of block designs with nested rows and columns. Srivastava *et al.* (1996) investigated the robustness for the loss of a single observation pertaining to a test treatment in augmented BIB (ABIB) designs with a single control in each block. These designs are found to be fairly robust. Dey *et al.* (1997) further investigated the robustness when any $1 \leq t \leq k$ observations are lost in a block. They

derived the exact expression for efficiency of a BIB design and a lower bound to binary proper block design when $1 \leq t \leq k$ observations are lost, where k is the block size of a proper design.

Lal (1998) studied the robustness against loss of observations lying anywhere in the designs for one-way elimination of heterogeneity. The necessary and sufficient condition of robustness of general linear model has been simplified for general block design. Thus a simpler sufficient condition in terms of minimum eigenvalue of C-matrix of original design has been developed.

There is some work done related to study of robustness of RC designs. Low *et al*. (1999) showed that a cross over design based on a Williams Latin Square of order 4 can suffer substantial loss of efficiency if some observations in the final period are unavailable.

Varghese *et al.* (2002) showed that Williams square change-over designs are robust against missing of last α [$\leq$ v -1: v being the number of period in the design for v treatments] observations from an experimental unit.

Lal *et al.* (2003) investigated the robustness of Youden square and Latin square designs against the loss of any t ($\geq$ l) observations in a column/row and for the loss of any two observations in the design as per connectedness criterion. Bhar (2014) defined E-efficiency criterion and obtained lower bound of this criterion for the loss of any t observations in binary variance balanced block design.

The GRC designs developed in the literature are for single factor experiments. Situations may arise wherein the experiment consist of more than one factor with each factor having more than one levels. A lot of work has been done for designs with factorial treatment structure under block and row-column setup.

Yates (1937) defined row-column designs for factorial experiments known as Quasi-Latin or Lattice squares. Rao (1946) gave a method for constructing partially confounded square row-column designs. Cochran and Cox (1957) listed some row-column designs for factorial experiments. John and Lewis (1983) obtained factorial row-column designs

using generalized cyclic method of construction by augmenting row and column component designs.

Bailey and Patterson (1991) showed that two-replicate resolvable row-column designs are combinatorially equivalent to a single replicate row-column designed for two factors.

Wright *et al*. (2005) extended the result given by Bailey and Patterson (1991) for resolvable row-column designs with more than two replications. Choi and Gupta (2008) considered confounded row-column designs for symmetric factorial experiments and gave some methods to construct these designs by confounding appropriate interactions over rows and columns.

Row-column designs with the column component design resolvable in 2 replicates were constructed by Jarrett *et al*. (1997) and used it for factorial set up.

Bose and Dey (2009) have shown the correspondence of a crossover design balanced for first residuals to a v^2 factorial experiment arranged in a row-column design assuming that the first row has only main effects of one factor and there are no interactions in the first row. Here, the direct effects of treatments are considered as levels of first factor and the residual effects are considered as the levels of second factor.

Lahiri *et al*. (2010) suggested a design for soil test crop response experiment where there were 3 levels of organic manure and 3 strips of fertility gradients and corresponding to manure-fertility combination different factorial points from 3^3 factorial were applied according to the choice of the experimenters. Sarkar (2011) obtained some series of symmetric/ asymmetric factorial row-column design. Dash *et al*. (2013) constructed row-column designs for estimation of main effects and two factor interaction effects in 2^n factorial microarray experiments.

Online generation of experimental designs provides an easy accessibility to the users. In this direction a lot of work has been done at IASRI. Taksande *et al*. (2012) developed software solution for the generation of partial diallel crosses. Sharma *et al*. (2013) developed web solution for generating partially balanced incomplete block designs. Jaggi *et al*. (2015) developed web-enabled software for generation of experimental designs

balanced for indirect effects of treatments. Many other open sources and commercial packages are also available for generation of readymade layouts of designs based on different situations [for example AgroPlotter (2002), webPD (2015) etc.]

CHAPTER III

MATERIALS AND METHODS

Generalized Row-Column (GRC) designs are used when the number of treatments is substantially large with limited number of replicates controlling heterogeneity in two directions. Here, the general methodology has been described related to GRC designs under different objectives of the study.

3.1 Experimental Setup and Model

A GRC design is considered here with v treatments arranged in p rows, q columns and in each row-column intersection (i.e. cells) there are k units or plots resulting in total n= pqk experimental units or observations. The following three-way classified model with treatments, rows and columns is considered:

$$Y_{l(ij)} = \mu + \tau_{l(ij)} + \alpha_i + \beta_j + e_{l(ij)}; \quad \text{...(3.1.1a)}$$

$$i = 1,2,\ldots,p;\ j = 1,2,\ldots,q;\ l = 1,2,\ldots,k,$$

where $Y_{l(ij)}$ is the response from the l^{th} unit corresponding to the intersection of i^{th} row and j^{th} column. μ is the general mean, $\tau_{l(ij)}$ is the effect of the treatment appearing in the l^{th} unit corresponding to the intersection of i^{th} row and j^{th} column, α_i is the i^{th} row effect and β_j is the j^{th} column effect. $e_{l(ij)}$ is the error term identically and independently distributed and following normal distribution with mean zero and constant variance.

The above model can be written in matrix notation as follows:

$$\mathbf{Y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e}, \quad \text{...(3.1.1b)}$$

where $\mathbf{Y}$ is a n × 1 vector of observations, μ is the grand mean, $\mathbf{1}$ is the n × 1 vector of ones, $\boldsymbol{\Delta}'$ is n × v incidence matrix of observations versus treatments, $\boldsymbol{\tau}$ is a v × 1 vector of treatment effects, $\mathbf{D}_1'$ is n × p incidence matrix of observations versus rows, $\boldsymbol{\alpha}$ is p × 1 vector of row effects, $\mathbf{D}_2'$ is n × q incidence matrix of observations versus columns, $\boldsymbol{\beta}$ is q × 1 vector of column effects and $\mathbf{e}$ is n × 1 vector of random errors with $E(\mathbf{e}) = 0$ and $D(\mathbf{e}) = \sigma^2 \mathbf{I}_n$. Further, $\boldsymbol{\Delta}'\mathbf{1}_v = \mathbf{D}_1'\mathbf{1}_p = \mathbf{D}_2'\mathbf{1}_q = \mathbf{1}_n$.

This model can be written as

$$\mathbf{Y} = \mathbf{X}_1\theta_1 + \mathbf{X}_2\theta_2 + \mathbf{e},$$

where

$$\mathbf{X_1} = [\ \boldsymbol{\Delta}'\ ], \quad \mathbf{X_2} = [\mathbf{1} \quad \mathbf{D}_1' \quad \mathbf{D}_2']',$$

with $\boldsymbol{\theta}_1 = (\boldsymbol{\tau})$ as the vector of parameter of interest and $\theta_2 = (\mathbf{1} \quad \boldsymbol{\alpha} \quad \boldsymbol{\beta})'$ as the vector of nuisance parameters. The information matrix for treatment effects can be obtained as

$$\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1$$

Here,

$$\mathbf{X}_1'\mathbf{X}_1 = \boldsymbol{\Delta}\boldsymbol{\Delta}' = \mathbf{R}_\tau,$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{pmatrix}\boldsymbol{\Delta}\mathbf{1} & \boldsymbol{\Delta}\mathbf{D}_1' & \boldsymbol{\Delta}\mathbf{D}_2'\end{pmatrix} = \begin{pmatrix}\mathbf{r}_\tau & \mathbf{N_1} & \mathbf{N_2}\end{pmatrix}$$

and

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{pmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' \\ \mathbf{D}_1\mathbf{1} & \mathbf{D}_1\mathbf{D}_1' & \mathbf{D}_1\mathbf{D}_2' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2'\end{pmatrix} = \begin{pmatrix} n & \mathbf{k}_\alpha' & \mathbf{k}_\beta' \\ \mathbf{k}_\alpha & \mathbf{K}_\alpha & \mathbf{W} \\ \mathbf{k}_\beta & \mathbf{W}' & \mathbf{K}_\beta\end{pmatrix},$$

where, $\mathbf{r}_\tau = (r_{\tau 1}, r_{\tau 2}, \ldots, r_{\tau v})'$ is the v × 1 replication vector of treatments, $\mathbf{k}_\alpha = (k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})'$ is the p × 1 vector of row sizes and $\mathbf{k}_\beta = (k_{\beta 1}, k_{\beta 2}, \ldots, k_{\beta q})'$ is the q × 1 vector of column sizes. Further, $\mathbf{K}_\alpha$ = diag $(k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})$, the diagonal matrix of row-sizes, $\mathbf{K}_\beta$ = diag $(k_{\beta 1}, k_{\beta 2}, \ldots, k_{\beta q})$ as the diagonal matrix of column-sizes. $\mathbf{N}_1$ is the v× p incidence matrix of treatments versus rows, $\mathbf{N}_2$ is the v× q incidence matrix of treatments versus columns and **W** is the incidence matrix of rows versus columns.

The inverse of $\mathbf{X}_2'\mathbf{X}_2$ is obtained as follows:

$$(\mathbf{X}_2'\mathbf{X}_2)^{-} = = \begin{pmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{K}_{\boldsymbol{\alpha}}^{-} + \mathbf{K}_{\boldsymbol{\alpha}}^{-}\mathbf{F}\mathbf{Z}^{-}\mathbf{F} & -\mathbf{F}\mathbf{Z}^{-} \\ \mathbf{0} & -\mathbf{Z}^{-}\mathbf{F}' & \mathbf{Z}^{-}\end{pmatrix}.$$

Here, $\mathbf{F} = \mathbf{K}_{\boldsymbol{\alpha}}^{-}\mathbf{W}$ and $\mathbf{Z} = \mathbf{K}_\beta - \mathbf{W}'\mathbf{K}_\alpha^{-}\mathbf{W}$.

The information matrix for a GRC design is thus obtained as

$$\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1$$

$$= \mathbf{R}_\tau - \mathbf{N}_1\mathbf{K}_\alpha^{-}\mathbf{N}_1' + \mathbf{N}_1\mathbf{K}_\alpha^{-}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_1' - \mathbf{N}_2\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_1' - \mathbf{N}_1\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_2' + \mathbf{N}_2\mathbf{Z}^{-}\mathbf{N}_2' \quad ...(3.1.2)$$

The v × v matrix **C** is symmetric, non-negative definite with zero row and column sums.

Variance Balanced: A design is said to be variance balanced if it permits the estimation of all estimable normalized treatment contrasts with same variance. A connected design is variance balanced if and only if all the non-zero eigen values of the **C**-matrix are equal.

A GRC design is variance balanced if and only if its **C**-matrix expressed in Eq. 3.1.2 has all its diagonal elements equal and its off-diagonal elements equal i.e. **C** matrix is given by

$$\mathbf{C} = (a-b)\,\mathbf{I} + b\mathbf{J}\mathbf{J}'$$

where a and b are scalars.

Partially Variance Balanced: A GRC is said to be partially variance balanced if the variance of every estimable elementary contrasts among treatment effects are estimated with different variances. The variances depends upon the association among the treatments.

The methodology adopted under different objectives in terms of the definitions and construction of GRC designs is now described.

3.2 GRC Designs for Two Disjoint Sets

GRC designs for comparing treatments belonging to two disjoint sets are to be constructed such that the contrasts pertaining to treatments of first set vs. treatments of second set is estimated with as high precision as possible. The following block designs are used in the construction of GRC designs for comparing treatments belonging to two disjoint sets:

3.2.1 Balanced Incomplete Block (BIB) Design

Balanced Incomplete Block (BIB) design is the arrangement of v treatments in b blocks each of size k (<v) such that

i. Each treatment occurs at most once in a block
ii. Each treatment occurs in exactly r blocks
iii. Each pair of treatments occurs together in exactly λ blocks

The symbols v, b, r, k, λ are called the parameters of the design. These parameters satisfy the following relations

i. $vr = bk$
ii. $\lambda(v-1) = r(k-1)$
iii. $b \geq v$

Example 3.2.1.1: Following is a BIB design for v = b = 7, r = k = 3 and λ = 1:

1 2 4
2 3 5
3 4 6
4 5 7
5 6 1
6 7 2
7 1 3

The complement of a design is obtained by replacing treatments in a block by those which do not occur in the block. The parameters of complementary BIB design is given as v, b, (b-r), (v-k), (b-2r+λ). The following is the complement design with parameters 7, 7, 4, 4, 2 of the BIB design in Example 3.2.1.1:

3 5 6 7
1 4 6 7
1 2 5 7
1 2 3 6
2 3 4 7
1 3 4 5
2 4 5 6

3.2.2 Partially Balanced Incomplete Block (PBIB) design

Consider an association scheme with m classes ($m \geq 2$). A partially balanced incomplete block (PBIB) design, based on m class association scheme, is the arrangement of v treatments in b blocks such that

i. each block contains k (< v) distinct treatments
ii. each treatment occurs in r blocks
iii. if the treatments α and β are mutually i^{th} associates in the association scheme, then α and β occur together in λ_i blocks, where the integer λ_i does not depend on the pair (α, β) so long as they are mutually i^{th} associates, i = 1,2,...,m. Further not all λ_i's are equal.

The parameters of a PBIB design are v, b, r, k, λ_i. These parameters satisfy the following relations

i. vr = bk
ii. $\sum_{i=1}^{m} n_i \lambda_i = r(k-1)$

Example 3.2.2.1: The following is a PBIB design for v = 8, b = 8, r = 3, k = 3, $\lambda_1 = 0$ and $\lambda_2 = 1$:

1 2 3
4 5 6
1 4 7
2 5 8
1 6 8
3 5 7
2 6 7
3 4 8

The association scheme is given as below.

7 8
3 6
2 4
1 5

Treatments that appear in same row are first associates and others are second associates.

3.2.2.1 Two-Class Association Schemes

Association schemes of PBIB designs have also been used to construct GRC designs for two sets of treatments. These association schemes (Dey, 2010) are described below.

I. Group Divisible Association Scheme

Let there be v = pq treatments (p, q integers; p > 1, q > 1) arranged in a rectangular array with p rows and q columns. Two treatments are first associates if they belong to the same row of the array and are second associates otherwise. Here, $n_1 = q-1$ and $n_2 = (p-1)q$.

Example 3.2.2.1.1: Let p = 2 and q = 3 resulting in v = 6 treatments that are arranged in two rows and three columns as:

1 2 3
4 5 6

The two types of associates are obtained as follows:

Treatment	1st Associates	2nd Associates
1	2, 3	4, 5, 6
2	1, 3	4, 5, 6
3	1, 2	4, 5, 6
4	5, 6	1, 2, 3
5	4, 6	1, 2, 3
6	4, 5	1, 2, 3

II. Triangular Association Scheme

Let there are $v = \frac{n(n-1)}{2}$ treatments arranged in a square array of side n such that the positions on the principal diagonal of the array are left blank, the $\frac{n(n-1)}{2}$ positions above the principal diagonal are filled up by the v treatment symbols and the position below the principal diagonal are filled up by the v symbols in such a manner that the resultant arrangement is symmetrical about principal diagonal. The two-class triangular (T_2) association scheme then has the following association rule: two treatments are first associates if they belong to the same row or same column of the array and are second associates otherwise. Here, $n_1 = 2(n-2), n_2 = \frac{(n-2)(n-3)}{2}$.

Example 3.2.2.1.2: Let n = 5. The association scheme for v = 10 treatments can be given as below:

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	8	*	10
4	7	9	10	*

The 1st and 2nd associates of treatments 1, 2,..., 10 are as follows:

Treatment	**1st Associates**	**2nd Associates**
1	2, 3, 4, 5, 6, 7	8, 9, 10
2	1, 3, 4, 5, 8, 9	6, 7, 10
3	1, 2, 4, 6, 8, 10	5, 7, 9
4	1, 2, 3, 7, 9, 10	5, 6, 8
5	1, 6, 7, 2, 8, 9	3, 4, 10
6	1, 5, 7, 3, 8, 10	2, 4, 9
7	1, 5, 6, 4, 9, 10	2, 3, 8
8	2, 5, 9, 3, 6, 10	1, 4, 7
9	2, 5, 8, 4, 7, 10	3, 1, 6
10	3, 6, 8, 4, 7, 9	1 ,2, 5

III. Latin-Square (L_2) Association Scheme

Let $v = s^2$ symbols are arranged into an $s \times s$ square array. Two symbols are defined to be first associates if they occur in the same row or column of the array otherwise they are second associates. The parameters of the L_2 association scheme are given as $v= s^2$, $n_1= 2(s-1)$, $n_2= (s-1)^2$.

Example 3.2.2.1.3: Let s = 4. The association scheme for v = 16 treatments can be given as below:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

The two types of associates are obtained as follows:

Treatment	1st Associates	2nd Associates
1	2, 3, 4, 5, 9, 13	6, 7, 8, 10, 11, 12, 14, 15, 16
2	1, 3, 4, 6, 10, 14	5, 7, 8, 9, 11, 12, 13, 15, 16
3	1, 2, 4, 7, 11, 15	5, 6, 8, 9, 10, 12, 13, 14, 16
4	1, 2, 3, 8, 12, 16	5, 6, 7, 9, 10, 11, 13, 14, 15
5	1, 6, 7, 8, 9, 13	2, 3, 4, 10, 11, 12, 14, 15, 16
6	2, 5, 9, 7, 8, 13	1, 3, 4, 9, 11, 12, 13, 15, 16
7	5, 6, 8, 3, 11, 15	1, 2, 4, 9, 10, 12, 13, 14, 16
8	5, 6, 7, 4, 12, 15	1, 2, 3, 9, 10, 11, 13, 14, 15
9	1, 5, 13, 10, 11, 12	2, 3, 4, 6, 7, 8, 14, 15, 16
10	2, 6, 14, 9, 11, 12	1, 3, 4, 5, 7, 8, 13, 15, 16
11	3, 7, 15, 9, 10, 12	1, 2, 4, 5, 6, 8, 13, 14, 16
12	4, 8, 16, 9, 10, 11	1, 2, 3, 5, 6, 7, 13, 14, 15
13	1, 5, 9, 14, 15, 16	2, 3, 4, 6, 7, 8, 10, 11, 12
14	13, 15, 16, 2, 6, 10	1, 3, 4, 5, 7, 8, 9, 11, 12
15	3, 7, 11, 13, 14, 16	1, 2, 4, 5, 6, 8, 9, 10, 12
16	4, 8, 12, 13, 14, 15	1, 2, 3, 5, 6, 7, 9, 10, 11

The variances of the GRC designs for two sets of treatments in case of partially variance balanced design depends on the association between the treatments in the design. Each treatment has exactly n_i i^{th} associates, the value of i depends on the number of associates of treatments.

General form of the information matrix of treatment effects for GRC designs with two sets of treatments has been derived for some classes of designs in Chapter 4. SAS codes have been written in PROC IML to calculate the information matrix (**C**-matrix) of treatment effects for a GRC design with two sets of treatments under the three-way model, study the properties of the designs and calculate variance of estimate of elementary treatment contrast for comparing the treatments from first set with the treatments from same set, the treatments from first set with the treatments from second set and the average variance.

3.3 Robustness of GRC Designs

The presence of missing observations, outliers in the data, etc. are some of the disturbances that may occur during experimentation. These disturbances may lead to less precise comparisons among treatments tried in the experiment. A lot of work has been done on robustness of designs in block set up or row-column set up.

A GRC design is robust against loss of observations, if the loss of efficiency of the residual design as compared to the original design is small. If C_d is the information matrix for estimating the treatment effects of GRC design d and C_{d^*} is that of the residual design d* after the observations are lost, then the efficiency E of the residual design relative to the original design is given by

$$E = \frac{\text{Harmonic mean of non-zero eigen values of } C_{d^*}}{\text{Harmonic mean of non-zero eigen values of } C_d}$$

A GRC design is said to be robust if the efficiency of the resulting design after loss of information is more than 90%.

SAS code has been written in PROC IML to calculate the information matrix (**C**-matrix) of treatment effects, its eigen-values and the harmonic mean of non-zero eigen-values of **C**-matrix of original design and the residual design for GRC design.

3.4 Generalized Row-Column Designs with Factorial Treatment Structure

Factorial experiments are used when there are more than one factor each at more than one levels and the effects of varying the levels of the various factors affecting the process output are investigated. In case of symmetrical factorial, the number of levels of each of the factors in an experiment is the same.

In this study, GRC designs for factorial treatment structure considering the symmetrical factorial have been developed. The technique of confounding has been used for developing GRC designs which helps in reducing the row/ column size by taking one or more interaction contrasts identical with row/ column contrasts. Choi and Gupta (2008) developed a method for obtaining confounded row-column design making use of classical method of confounding.

Example 3.4.1: The following is a 2^4 confounded row-column design with p = q= 2^2 and k = 1.

Column Key Block	Row Key Block			
	0000	1100	0011	1111
0000	0000	1100	0011	1111
0110	0110	1010	0101	1001
1101	1101	0001	1110	0010
1011	1011	0111	1000	0100

The design is developed by obtaining two key blocks confounding different effects, one for row and other for column. The contents of the row-column design are developed by adding the corresponding combinations mod 2. Here, the effects confounded row-wise are F_1F_2, F_3F_4, $F_1F_2F_3F_4$ and column wise are $F_1F_2F_3$, $F_2F_3F_4$, F_1F_4.

The general rule of confounding is that for s^n treatment combinations in s^r plots per block, the total number of effects confounded is $\frac{s^{n-r}-1}{s-1}$ with (n-r) independent effects and $\frac{s^{n-r}-1}{s-1}$ - (n-r) as generalized effects. The contents are obtained by confounding higher order interactions. The confounded row-column designs have been developed in this study with more than one combination in each row-column intersection.

In factorial experiments, for large number of treatment combinations it is very difficult to the experimenter to conduct an experiment involving a single replication e.g. seven factors each at three levels a complete factorial experiment would mean testing 2187 treatment combinations in a single replication. The technique of recovering useful information by observing only a part of the complete factorial is known as fractional factorial, a concept introduced by D.J. Finney (1945). The interaction(s) which (are) is confounded for obtaining the fraction is (are) said to form the identity group of interaction(s) or defining contrasts(s).

Example 3.4.2: Let there are four factors F_1, F_2, F_3, F_4 each at two levels. An experimenter can only take 8 combinations out of these 16 combinations. Let $I \equiv F_1F_2F_3F_4$ to get two blocks of size 8. Thus, the key block obtained is as follows:

	F_1	F_2	$F_2F_3F_4$	$F_1F_2F_3F_4$
1001	+	-	+	-
0101	-	+	-	-
0011	-	-	-	-
1100	+	+	+	-
1010	+	-	+	-
0110	-	+	-	-
1111	+	+	+	-
0000	-	-	-	-

It can be seen that the main effect F_1 and interaction $F_2F_3F_4$ is estimated by same contrast in the $\frac{1}{2}\left(2^4\right)$ factorial. The various other aliases for $\frac{1}{2}\left(2^4\right)$ factorial are as follows:

$F_2 \equiv F_1F_3F_4$

$F_3 \equiv F_1F_2F_4$

$F_4 \equiv F_1F_2F_3$

$F_1F_2 \equiv F_3F_4$

$F_1F_3 \equiv F_2F_4$

$F_1F_4 \equiv F_2F_3$

$\mu \equiv F_1F_2F_3F_4$

Series of fractional confounded GRC designs have been developed so that useful information can be obtained by observing only a part of the complete factorial.

3.5 Web Solution for of Generalized Row-Column Designs

A large number of experimental designs under different situations have been developed in the literature. For ready referencing and potential use of these designs, online software for

generation of randomized layout of these designs is highly desirable. Online generation of designs are very much useful for the experimenters in providing a readymade solution. A large number of GRC designs are developed in the literature. In this study, a web solution for generation of GRC designs has been developed which will help the experimenters for an easy accessibility and quick reference of these designs. An online catalogue is also prepared for easy selection of the design.

Architecture of Web Solution for GRC Designs

The web solution for generation of GRC designs has been developed using client–server architecture along with an online catalogue of the designs within a permissible range. There are three main components i.e. user interface management, input data management and statistical engine for generation of GRC designs. At client side any communication to software from users is handled by user interface and input data handling is done by data management module. Statistical engine which hold the several procedures required for generation is implemented at server side. User interface has been separated from the statistical engine to free software developers from interface problem. Hyper Text Markup Language (HTML) and Cascading Style Sheets (CCS) have been used to develop the user interface management. ASP.NET has been used to develop input data management component. Web generation engine has been consructed using C# language. This engine contains the Dynamic Link Libraries (DLL) for generation and randomization of designs. Web generation of GRC Design has been developed for web platform and programming has been done with the ASP.NET and C# programming language. C# provides a complete set of tools for creation of rapid and powerful graphical user interface (GUI) based web applications. Microsoft Visual Studio 2010 integrated development environment has been used as a platform for development of the software. Fig. 3.1 shows the architecture of the software.

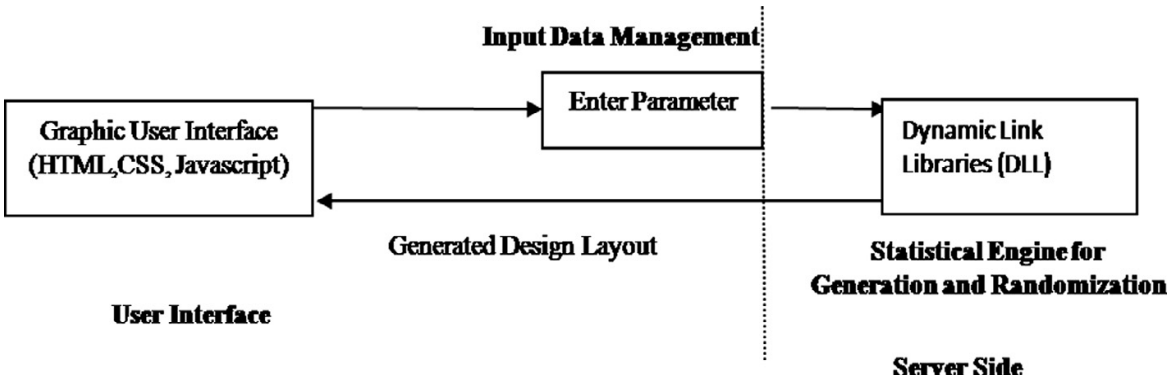

Fig. 3.1: Architecture for web generation of GRC design

CHAPTER IV

RESULTS AND DISCUSSIONS

The results of the present study on generalized row-column designs for single and multi-factor experiments are described in this chapter along with the discussion under each objective.

4.1 Generalized Row-Column Design for Treatments Belonging to Two Disjoint Sets

In this section, generalized row-column design setting when there are two sets of treatments that are disjoint, one set consisting of test treatments and the other of control treatments has been discussed. The two sets are disjoint in the sense that there are no common treatments between the two. The interest here is to estimate the contrasts of the type $(\tau_s - \tau_{s'})$ with as high precision as possible, τ_s and $\tau_{s'}$ belongs to 1^{st} and 2^{nd} set of treatments respectively. For example, in agricultural experiments the aim is to test a set of new varieties of a crop with a set of already existing varieties and to determine which of the varieties performs better in comparison to the existing variety. In the following section experimental setup of GRC design for treatments belonging to two disjoint sets has been discussed and series of GRC designs for comparing a set of test treatments to a set of control treatments have been developed.

4.1.1 Experimental Setup and Model for Two Disjoint Sets of Treatments

A GRC design is considered with $v = v_1 + v_2$ (v_1 treatments from first set, also called test treatments and v_2 treatments from second set, also called control treatments) treatments arranged in p rows, q columns and in each row-column intersection (i.e. cells) there are k units or plots resulting in total $n = pqk$ experimental units or observations. The following three-way classified model with treatments, rows and columns is considered:

$$Y_{l(ij)} = \mu + \tau_{l(ij)} + \alpha_i + \beta_j + e_{l(ij)}; \qquad \text{...(4.1.1a)}$$

$$i = 1,2,\ldots,p;\ j = 1,2,\ldots,q;\ l = 1,2,\ldots,k,$$

where $Y_{l(ij)}$ is the response from the l^{th} unit corresponding to the intersection of i^{th} row and j^{th} column. μ is the general mean, $\tau_{l(ij)}$ is the effect of the treatment appearing in the l^{th} unit

corresponding to the intersection of i^{th} row and j^{th} column, α_i is the i^{th} row effect and β_j is the j^{th} column effect. $e_{l(ij)}$ is the error term that is identically and independently distributed and following normal distribution with mean zero and constant variance σ^2.

The above model can be written in matrix notation as follows:

$$\mathbf{Y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e}, \qquad \text{...(4.1.1b)}$$

where $\mathbf{Y}$ is a n × 1 vector of observations, μ is the grand mean, $\mathbf{1}$ is the n × 1 vector of ones, $\boldsymbol{\Delta}'$ is n × v incidence matrix of observations versus treatments, $\boldsymbol{\tau}$ is a v × 1 vector of treatment effects, $\mathbf{D}_1'$ is n × p incidence matrix of observations versus rows, $\boldsymbol{\alpha}$ is p × 1 vector of row effects, $\mathbf{D}_2'$ is n × q incidence matrix of observations versus columns, $\boldsymbol{\beta}$ is q × 1 vector of column effects and $\mathbf{e}$ is n × 1 vector of random errors with E($\mathbf{e}$) = 0 and D($\mathbf{e}$) = $\sigma^2\mathbf{I}_n$. Further, $\boldsymbol{\Delta}'\mathbf{1}_v = \mathbf{D}_1'\mathbf{1}_p = \mathbf{D}_2'\mathbf{1}_q = \mathbf{1}_n$,

$\boldsymbol{\Delta}\mathbf{D}_1' = \mathbf{N}_1 = \begin{bmatrix} \mathbf{N}_{11} \\ \mathbf{N}_{12} \end{bmatrix}$, $(v_1 + v_2) \times p$ matrix with $\mathbf{N}_{11}$ as the incidence of treatments of first set versus row and $\mathbf{N}_{12}$ as the incidence of treatments of second set versus row,

$\boldsymbol{\Delta}\mathbf{D}_2' = \mathbf{N}_2 = \begin{bmatrix} \mathbf{N}_{21} \\ \mathbf{N}_{22} \end{bmatrix}$, $(v_1 + v_2) \times q$ matrix with $\mathbf{N}_{21}$ as the incidence of first set of treatments versus column and $\mathbf{N}_{22}$ as the incidence of second set of treatments versus column and

$\mathbf{W}$ is the incidence matrix of rows versus columns.

$\mathbf{r} = [\mathbf{r}_{\tau 1}' \ \ \mathbf{r}_{\tau 2}']'$ is the $(v_1 + v_2) \times 1$ replication vector of treatments with $\mathbf{r}_{\tau 1}$ as the replication vector of first set treatments and $\mathbf{r}_{\tau 2}$ as the replication vector of second set treatments and

$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix}$ with $\mathbf{R}_1$ ($\mathbf{R}_2$) as the diagonal matrix of replication of first (second) set of treatments.

$\mathbf{k}_\alpha = (k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})'$ is the p × 1 vector of row sizes with $\mathbf{K}_\alpha$ = diag $(k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})$, the diagonal matrix of row-sizes.

$\mathbf{k}_\beta = (k_{\beta 1}, k_{\beta 2}, \ldots, k_{\beta q})'$ is the q × 1 vector of column sizes with $\mathbf{K}_\beta$ = diag $(k_{\beta 1}, k_{\beta 2}, \ldots, k_{\beta q})$ as the diagonal matrix of column-sizes.

Model (4.1.1b) can be re-written as

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \mathbf{e},$$

where

$$\mathbf{X_1} = [\boldsymbol{\Delta}'], \quad \mathbf{X_2} = [\mathbf{1} \quad \mathbf{D}_1' \quad \mathbf{D}_2']',$$

with $\boldsymbol{\theta}_1 = (\boldsymbol{\tau})$ as the vector of parameters of interest and $\boldsymbol{\theta}_2 = (\mathbf{1} \quad \boldsymbol{\alpha} \quad \boldsymbol{\beta})'$ as the vector of nuisance parameters. The information matrix for treatment effects can be obtained as

$$\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1$$

Here,

$$\mathbf{X}_1'\mathbf{X}_1 = \boldsymbol{\Delta}\boldsymbol{\Delta}' = \mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & 0 \\ 0 & \mathbf{R}_2 \end{pmatrix},$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{pmatrix} \boldsymbol{\Delta}\mathbf{1} & \boldsymbol{\Delta}\mathbf{D}_1' & \boldsymbol{\Delta}\mathbf{D}_2' \end{pmatrix} = \begin{pmatrix} \mathbf{r} & \mathbf{N_1} & \mathbf{N_2} \end{pmatrix}$$

and

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{pmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' \\ \mathbf{D_1}\mathbf{1} & \mathbf{D_1}\mathbf{D}_1' & \mathbf{D_1}\mathbf{D}_2' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2' \end{pmatrix} = \begin{pmatrix} \mathrm{n} & \mathbf{k}_\alpha' & \mathbf{k}_\beta' \\ \mathbf{k}_\alpha & \mathbf{K}_\alpha & \mathbf{W} \\ \mathbf{k}_\beta & \mathbf{W}' & \mathbf{K}_\beta \end{pmatrix}.$$

The inverse of $\mathbf{X}_2'\mathbf{X}_2$ is obtained using the following result:

$$\text{If } \mathbf{X} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{pmatrix} \text{ then } \mathbf{X}^{-} = \begin{pmatrix} \mathbf{A}^{-} + \mathbf{F}\mathbf{E}^{-}\mathbf{F}' & -\mathbf{F}\mathbf{E}^{-} \\ -\mathbf{E}^{-}\mathbf{F}' & \mathbf{E}^{-} \end{pmatrix}$$

$$\text{where } \mathbf{F} = \mathbf{A}^{-}\mathbf{B} \text{ and } \mathbf{E} = \mathbf{D} - \mathbf{B}'\mathbf{A}^{-}\mathbf{B}$$

Here, $\mathbf{F} = \mathbf{K}_\alpha^{-}\mathbf{W}$ and $\mathbf{E} = \mathbf{K}_\beta - \mathbf{W}'\mathbf{K}_\alpha^{-}\mathbf{W} = \mathbf{Z}$ (say).

$$\text{Now, } (\mathbf{X}_2'\mathbf{X}_2)^{-} = = \begin{pmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{K}_\alpha^{-} + \mathbf{K}_\alpha^{-}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}' & -\mathbf{F}\mathbf{Z}^{-} \\ \mathbf{0} & -\mathbf{Z}^{-}\mathbf{F}' & \mathbf{Z}^{-} \end{pmatrix}.$$

The information matrix of a GRC design for two sets of treatments is thus obtained as

$$\mathbf{C} = \begin{pmatrix} \mathbf{R}_t - \mathbf{K_{11}} & -\mathbf{K_{12}} \\ -\mathbf{K_{21}} & \mathbf{R}_c - \mathbf{K_{22}} \end{pmatrix} \qquad \ldots(4.1.2)$$

where,

$$\mathbf{K}_{11} = \mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' - \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}' + \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}'$$

$$\mathbf{K}_{12} = \mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}' - \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}' - \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}' + \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{22}'$$

$$\mathbf{K}_{21} = \mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{11}' + \mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' - \mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' - \mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{21}' + \mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{N}_{21}'$$

$$\mathbf{K}_{22} = \mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}' + \mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}' - \mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}' - \mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}' + \mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{N}_{22}' \quad \text{...(4.1.3)}$$

The $(v_1 + v_2) \times (v_1 + v_2)$ matrix **C** is symmetric, non-negative definite with zero row and column sums. Considering this information matrix, the GRC design for two disjoint sets of treatments is now defined.

Definition 4.1.1: A GRC design with p rows, q columns and intersection of each row-column having k units in a cell is said to be a Balanced Bipartite Generalized Row-Column (BBP-GRC) design for comparing a set of v_1 treatments to a set of v_2 treatments if and only if its **C** matrix is of the form

$$\mathbf{C} = \begin{bmatrix} (f_1 - f_2)\mathbf{I}_{v_1} + f_2\mathbf{1}_{v_1}\mathbf{1}_{v_1}' & f_3\mathbf{1}_{v_1}\mathbf{1}_{v_2}' \\ f_3\mathbf{1}_{v_2}\mathbf{1}_{v_1}' & (f_4 - f_5)\mathbf{I}_{v_2} + f_5\mathbf{1}_{v_2}\mathbf{1}_{v_2}' \end{bmatrix}$$

such that $f_1 + (v_1-1)f_2 + f_3v_2 = 0$ and $f_4 + (v_2-1)f_5 + f_3v_1 = 0$ where f_1, f_2, f_3, f_4 and f_5 are scalars. The parameters of a BBP-GRC design can be represented as v_1, v_2, p, q, r_1 (replication of treatments of first set), r_2 (replication of treatments of second set) and k.

Some methods of constructing BBP-GRC designs are now described along with the examples.

4.1.2 Methods of Constructing BBP-GRC Designs

Method 4.1.2.1: Consider any GRC design with parameter v*, p*, q*, r* and k*. Out of v* treatments, cu treatments ($c > 1$, $u > 1$) are taken such that $cu \leq (v^*-2)$ and these cu treatments are divided into c sets of size u each. Replace all the treatments of 1st set of size u with 1st control treatment, 2nd set with 2nd control treatment and so on cth set with cth control treatment. The resulting design is a BBP-GRC design for comparing $v_1 = (v^* - cu)$ treatments of first set, $v_2 = c$ treatments of second set in $p = p^*$ rows, $q = q^*$ columns, $r_1 = r^*$, $r_2 = ur^*$ and $k = k^*$.

Example 4.1.2.1.1: Consider the following GRC design (Datta, 2012) with parameters v*= 7, p* = 3, q* = 7, r* = 6 and k* = 2.

Rows	Columns						
	I	II	III	IV	V	VI	VII
I	1 7	2 1	3 2	4 3	5 4	6 4	7 6
II	2 6	3 7	4 1	5 2	6 3	7 4	1 5
III	3 5	4 6	5 7	6 1	7 2	1 3	2 4

Let u = 2 and c = 2, replace the last set of 2 treatments (6, 7) with one control (5) and second last set of 2 treatments (4, 5) with another control (4). The design so obtained is a BBP-GRC design for comparing a set of v_1 = 3 (1, 2, 3) treatments of first set replicated r_1= 6 times with v_2 = 2 (4, 5) treatments of second set replicated r_2 = 12 times in p = p* = 3 rows, q = q* = 7 columns and cell size k = 2. The design is as shown below.

Rows	Columns						
	I	II	III	IV	V	VI	VII
I	1 5	2 1	3 2	4 3	4 4	5 4	5 5
II	2 5	3 5	4 1	4 2	5 3	5 4	1 4
III	3 4	4 5	4 5	5 1	5 2	1 3	2 4

The information matrix for estimating treatment effects with respect to both the sets of treatments is obtained as follows:

$$\mathbf{C} = \begin{pmatrix} 5.833\mathbf{I}_3 - 0.833\mathbf{J}_{3\times 3} & \text{-}1.667\mathbf{J}_{3\times 2} \\ \text{-}\mathbf{1}.667\mathbf{J}_{2\times 3} & 11.666\mathbf{I}_2 - 3.333\mathbf{J}_{2\times 2} \end{pmatrix}$$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.343\sigma^2$, $s \neq s' = 1, 2, ..., v_1$,

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.257\sigma^2$, $s \neq s'$, $s = 1, 2, ..., v_1, s' = v_1+1, ..., v_2$

and average variance is $0.274\,\sigma^2$.

It can be seen that the contrasts pertaining to first set of treatments with second set of treatments is estimated with less variance.

Method 4.1.2.2: Consider a Latin square of order v and another orthogonal Latin square of the same order. Renumber the v treatments of the second Latin square by v+1, v+2,..., 2v. Superimpose the second Latin square on the first leaving the first row of both the squares. Each cell of the first row of both the Latin squares is augmented by a single control treatment numbered as (2v+1). Place these 2 rows along with the superimposed arrangement. This will result in a BBP-GRC design with parameters $v_1 = 2v$, $v_2 = 1$, $p = v+1$, $q = v$, $r_1 = v$, $r_2 = 2v$ and $k = 2$. The design so obtained is partially balanced with respect to the first set of 2v treatments following a group divisible association scheme. The 2v treatments are arranged in two rows of size v each as shown below.

1	2	...	v
v+1	v+2	...	2v

The treatments in the same row are first associates to each other and the treatments in the other row are second associates.

Example 4.1.2.2.1: Consider a Latin square of order 5 with treatments numbered as 1, 2, 3, 4, 5 along with another orthogonal Latin square of same order with treatments numbered as 6, 7, 8, 9, 10. The first row of both the Latin squares is taken as row number I and II of BBP-GRC design and augment treatment 11 in each cell. Row III to VI are obtained by superimposing two Latin squares and retaining rows other than first. The following arrangement of BBP-GRC design is obtained with parameters $v_1 = 10$, $v_2 = 1$, $p = 6$, $q = 5$, $r_1 = 5$, $r_2 = 10$ and $k = 2$:

Rows	Columns				
	I	II	III	IV	V
I	6 11	7 11	8 11	9 11	10 11
II	1 11	2 11	3 11	4 11	5 11
III	2 8	3 9	4 10	5 6	1 7
IV	3 10	4 6	5 7	1 8	2 9
V	4 7	5 8	1 9	2 10	3 6
VI	5 9	1 10	2 6	3 7	4 8

The information matrix for estimating the two sets of treatment effects is obtained as follows:

$$\mathbf{C}=\begin{pmatrix} 4.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 \\ -0.5 & 4.5 & -0.5 & -0.5 & -0.5 & -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 \\ -0.5 & -0.5 & 4.5 & -0.5 & -0.5 & -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 \\ -0.5 & -0.5 & -0.5 & 4.5 & -0.5 & -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 & 4.5 & -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 \\ -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & 4.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\ -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 & 4.5 & -0.5 & -0.5 & -0.5 & -0.5 \\ -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 & -0.5 & 4.5 & -0.5 & -0.5 & -0.5 \\ -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 & -0.5 & -0.5 & 4.5 & -0.5 & -0.5 \\ -0.4 & -0.4 & -0.4 & -0.4 & -0.4 & -0.5 & -0.5 & -0.5 & -0.5 & 4.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 5 \end{pmatrix}$$

The treatments are arranged as given below:

1	2	3	4	5
6	7	8	9	10

The various associates of the treatments are as follows:

Treatments	1st Associate	2nd Associate
1	2 3 4 5	6 7 8 9 10
2	1 3 4 5	6 7 8 9 10
3	1 2 4 5	6 7 8 9 10
4	1 2 3 5	6 7 8 9 10
5	1 2 3 4	6 7 8 9 10
6	7 8 9 10	1 2 3 4 5
7	6 8 9 10	1 2 3 4 5
8	6 7 9 10	1 2 3 4 5
9	6 7 8 10	1 2 3 4 5
10	6 7 8 9	1 2 3 4 5

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.405\sigma^2$, $s \neq s' = 1, 2, ..., v_1$, s and s' are 1st associate from first set,

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.400\sigma^2$, $s \neq s' = 1,2, ..., v_1$, s and s' are 2nd associate from first set,

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.382\sigma^2$, $s \neq s'$, $s= 1,2, ..., v_1$, $s'= v_1+1, ..., v_2$, 1st associate of first set with second set,

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.4000\sigma^2$, $s \neq s'$, $s= 1,2, ..., v_1$, $s'= v_1+1, ..., v_2$, 2nd associate of first set with second set,

and average variance is $0.401\sigma^2$.

Method 4.1.2.3: Consider a Balanced Incomplete Block (BIB) design with parameters v*, b*, r*, k*, λ* and it's complementary with parameter v*, b*, b* - r*, v* - k*, v* - 2r*+λ*. Arrange the blocks of the BIB design in the first row giving rise to q = b* columns. The blocks obtained from the complement are arranged in the second row.

Case I: If v* > 2k* then augment $v_2 = v^* - 2k^*$ treatments of second set to all the cells of the first row and replace these treatments by last v* - 2k* treatments of first set. The resulting design will be a BBP-GRC design with parameters $v_1 = 2k^*$, $v_2 = v^* - 2k^*$, p = 2, q = b*, $r_1 = b^*$, $r_2 = 2b^*$ and $k = v^* - k^*$.

Case II: If v* < 2k* then augment $v_2 = 2k^* - v^*$ control treatments to all the cells of the second row and replace these treatments by last 2k* - v* treatments of first set. The resulting design will be a BBP-GRC design with parameters $v_1 = 2k^*$, $v_2 = 2k^* - v^*$, p = 2, q = b*, $r_1 = b^*$, $r_2 = 2b^*$ and $k = v^* - k^*$.

Special Case I: Consider a BIB design of the form $v^* = s^2$, b* = s(s+1), r* = s+1, k* = s, λ* = 1 which can be obtained by using mutually orthogonal Latin squares. A BBP-GRC design is obtained using the above method with $v = v_1 + v_2$, where $v_1 = 2s$ and $v_2 = s(s-2)$ treatments arranged in p = 2 rows, q = s(s+1) columns and in each row-column intersection (i.e. cells) there are k = s(s-1) units or plots resulting in total $n = 2s^2(s^2-1)$ experimental units or observations.

The structure of the various incidence matrices as per model (4.1.1b) of the design obtained is as follows:

$$\mathbf{N}_1 = \begin{pmatrix} \mathbf{N}_{11} \\ \mathbf{N}_{12} \end{pmatrix} = \begin{pmatrix} (s+1)\mathbf{1}_{v_1} & (s^2-1)\mathbf{1}_{v_1} \\ (s+1)^2\mathbf{1}_{v_2} & (s^2-1)\mathbf{1}_{v_2} \end{pmatrix},$$

$$\mathbf{N}_2 = \begin{pmatrix} \mathbf{N}_{21} \\ \mathbf{N}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{J}_{v_1 \times q} \\ 2\mathbf{J}_{v_2 \times q} \end{pmatrix}$$

and $\mathbf{W} = s\mathbf{J}_{p \times q}$

So, $$\mathbf{N}_1\mathbf{N}_1' = \begin{pmatrix} \mathbf{N}_{11}\mathbf{N}_{11}' & \mathbf{N}_{11}\mathbf{N}_{12}' \\ \mathbf{N}_{12}\mathbf{N}_{11}' & \mathbf{N}_{12}\mathbf{N}_{12}' \end{pmatrix} = \begin{pmatrix} (s+1)^2\left[1+(s-1)^2\right]\mathbf{J}_{v_1 \times v_1} & \left[(s+1)^3+(s^2-1)^2\right]\mathbf{J}_{v_1 \times v_2} \\ \left[(s+1)^3+(s^2-1)^2\right]\mathbf{J}_{v_2 \times v_1} & \left[2(s+1)^2(s^2+1)\right]\mathbf{J}_{v_2 \times v_2} \end{pmatrix}$$

and $\mathbf{N}_2\mathbf{N}_2' = \begin{pmatrix} \mathbf{N}_{21}\mathbf{N}_{21}' & \mathbf{N}_{21}\mathbf{N}_{22}' \\ \mathbf{N}_{22}\mathbf{N}_{21}' & \mathbf{N}_{22}\mathbf{N}_{22}' \end{pmatrix} = \begin{pmatrix} s(s+1)\mathbf{J}_{v_1\times v_1} & 2s(s+1)\mathbf{J}_{v_1\times v_2} \\ 2s(s+1)\mathbf{J}_{v_2\times v_1} & 4s(s+1)\mathbf{J}_{v_2\times v_2} \end{pmatrix}.$

Also, $\mathbf{R} = \begin{pmatrix} s(s+1)\mathbf{I}_{v_1\times v_1} & 0 \\ 0 & 2s(s+1)\mathbf{I}_{v_2\times v_2} \end{pmatrix},$

$\mathbf{K}_\alpha = kq\mathbf{I}_p = s^2(s+1)\mathbf{I}_p$ and $\mathbf{K}_\beta = kp\mathbf{I}_p = 2s\mathbf{I}_q$.

Further,

$$\mathbf{Z} = \mathbf{K}_\beta - \mathbf{W}'\mathbf{K}_\alpha^{-}\mathbf{W} = 2s(s-1)\left[\mathbf{I}_q - \frac{1}{q}\mathbf{J}_q\right]$$

$$\Rightarrow \mathbf{Z}^{-} = \frac{1}{2s(s-1)}\mathbf{I}_q$$

$$\mathbf{F} = \mathbf{K}_\alpha^{-}\mathbf{W} = \frac{1}{s(s+1)}\mathbf{J}_{p\times q},\quad \mathbf{F}\mathbf{Z}^{-} = \frac{1}{2s^2(s^2-1)}\mathbf{J}_{p\times q},\quad \mathbf{F}\mathbf{Z}^{-}\mathbf{F}' = \frac{1}{2s^2(s^2-1)}\mathbf{J}_{p\times p}.$$

The information matrix for BBP-GRC design is obtained as follows:

$$\mathbf{C} = \begin{pmatrix} \mathbf{R}_1 - \mathbf{K}_{11} & -\mathbf{K}_{12} \\ -\mathbf{K}_{21} & \mathbf{R}_2 - \mathbf{K}_{22} \end{pmatrix}$$

$$\mathbf{K}_{11} = \mathbf{N}_{11}\mathbf{K}_\alpha^{-}\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' - \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}' + \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}'$$

$$\mathbf{N}_{11}\mathbf{K}_\alpha^{-}\mathbf{N}_{11}' = \frac{(s+1)\left[1+(s-1)^2\right]}{s^2(s^2-1)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' = \left((s+1)\mathbf{1}_{v_1} \quad (s^2-1)\mathbf{1}_{v_1}\right)\times\frac{1}{2s^2(s^2-1)}\mathbf{J}_{p\times p}\times\begin{pmatrix}(s+1)\mathbf{1}_{v_1}' \\ (s^2-1)\mathbf{1}_{v_1}'\end{pmatrix}$$

$$= \frac{s^2(s+1)^2}{2s^2(s^2-1)}\mathbf{J}_{v_1\times v_1} = \frac{(s+1)}{2(s-1)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' = \mathbf{J}_{v_1\times q}\times\frac{1}{2s^2(s^2-1)}\mathbf{J}_{q\times p}\times\begin{pmatrix}(s+1)\mathbf{1}_{v_1}' \\ (s^2-1)\mathbf{1}_{v_1}'\end{pmatrix}$$

$$= \frac{(s+1)}{2(s-1)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}^{'}=\left((s+1)\mathbf{1}_{v_1}\quad (s^2\text{-}1)\mathbf{1}_{v_1}\right)\times\frac{1}{2s^2(s^2\text{-}1)}\mathbf{J}_{p\times q}\times\begin{pmatrix}(s+1)\mathbf{1}_{v_1}^{'}\\(s^2\text{-}1)\mathbf{1}_{v_1}^{'}\end{pmatrix}$$

$$=\frac{(s+1)\left[1+(s-1)^2\right]}{s^2(s^2\text{-}1)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}^{'}=\frac{(s+1)}{2(s\text{-}1)}\mathbf{J}_{v_1\times v_1}$$

$$\Rightarrow\mathbf{K}_{11}=\frac{(s+1)\left[1+(s-1)^2\right]}{s^2(s\text{-}1)}\mathbf{J}_{v_1\times v_1}$$

Similarly,

$$\mathbf{K}_{12}=\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}+\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}+\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}$$

$$\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}=\frac{\left[(s+1)^3+(s^2\text{-}1)^2\right]}{s^2(s^2\text{-}1)}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\frac{(s+1)}{s\text{-}1}\mathbf{J}_{v_1\times v_2}$$

$$\Rightarrow\mathbf{K}_{12}=\frac{\left[(s+1)^3+(s^2\text{-}1)^2\right]}{s^2(s^2\text{-}1)}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{K}_{21}=\frac{\left[(s+1)^3+(s^2\text{-}1)^2\right]}{s^2(s^2\text{-}1)}\mathbf{J}_{v_2\times v_1}$$

$$\mathbf{K}_{22}=\mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}+\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}+\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}+\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}$$

$$\mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}=\frac{\left[2(s+1)(s^2+1)\right]}{s^2(s\text{-}1)}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\frac{2(s+1)}{s\text{-}1}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\frac{2(s+1)}{s\text{-}1}\mathbf{J}_{v_2\times v_2}$$

$$\Rightarrow\mathbf{K}_{22}=\frac{\left[2(s+1)(s^2+1)\right]}{s^2(s\text{-}1)}\mathbf{J}_{v_2\times v_2}$$

The information matrix for BBP-GRC design is thus obtained as

$$\mathbf{C}=\begin{pmatrix} s(s+1)\mathbf{I}_{v_1\times v_1}-\dfrac{(s+1)\left[1+(s-1)^2\right]}{s^2(s-1)}\mathbf{J}_{v_1\times v_1} & -\dfrac{\left[(s+1)^3+(s^2-1)^2\right]}{s^2(s^2-1)}\mathbf{J}_{v_1\times v_2} \\ -\dfrac{\left[(s+1)^3+(s^2-1)^2\right]}{s^2(s^2-1)}\mathbf{J}_{v_2\times v_1} & 2s(s+1)\mathbf{I}_{v_2\times v_2}-\dfrac{\left[2(s+1)(s^2+1)\right]}{s^2(s-1)}\mathbf{J}_{v_2\times v_2} \end{pmatrix} \quad \ldots(4.1.4)$$

Example 4.1.2.3.1: Consider a BIB design with parameters as v* = 9, b* = 12, r* = 4, k* = 3, λ* = 1. Arrange the blocks of this BIB design in the first row and its complementary in the second row. Since here v* > 2k*, augment v_2 = v* - 2k* = 3 treatments of second set to all the cells of the first row and replace these treatments by last 3 treatments of first set. The resulting design is a BBP-GRC design with parameter v_1 = 6 (numbered as 1, 2, 3, 4, 5, 6), v_2 = 3 (numbered as 7, 8, 9), p = 2, q = 12, r_1 = 12, r_2 = 24 and k = 6.

Rows	Columns											
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
I	1 2 3 7 8 9	4 5 6 7 8 9	7 8 9 7 8 9	1 4 7 7 8 9	2 5 8 7 8 9	3 6 9 7 8 9	1 6 8 7 8 9	2 4 9 7 8 9	3 5 7 7 8 9	1 5 9 7 8 9	2 6 7 7 8 9	3 4 8 7 8 9
II	4 5 6 7 8 9	1 2 3 7 8 9	1 2 3 4 5 6	2 3 5 6 8 9	1 3 4 6 7 9	1 2 4 5 7 8	2 3 4 5 7 9	1 3 5 6 7 8	1 2 4 6 8 9	2 3 4 6 7 8	1 3 4 5 8 9	1 2 5 6 7 9

The information matrix for estimating treatment effects of first and second set is obtained from (4.1.4) is as follows:

$$\mathbf{C}=\begin{pmatrix} 12\mathbf{I}_{6\times 6}-1.111\mathbf{J}_{6\times 6} & -1.778\mathbf{J}_{6\times 3} \\ -1.778\mathbf{J}_{3\times 6} & 24\mathbf{I}_{3\times 3}-4.444\mathbf{J}_{3\times 3} \end{pmatrix}$$

$$V(\hat{\tau}_s-\hat{\tau}_{s'})=0.1667\sigma^2,\ s\neq s'=1,2,\ldots,v_1$$

$$V(\hat{\tau}_s-\hat{\tau}_{s'})=0.1285\sigma^2,\ s\neq s',\ s=1,2,\ldots,v_1, s'=v_1+1,\ldots,v_2$$

Average variance is $0.1406\,\sigma^2$.

Special Case II: Consider a BIB design of the form v*, $b^*={}^{v^*}C_2=\dfrac{v^*(v^*-1)}{2}$, r* = v*-1, k* = 2, λ* = 1. A BBP-GRC design with v_1 = 4, v_2 = v* - 4 treatments arranged in p = 2

rows, q= $\frac{v^*(v^*-1)}{2}$ columns and in each row-column intersection there are k = v*-2 units resulting in total n= v*(v*-1)(v*-2) experimental units or observations.

The structure of the incidence matrices as per model (4.1.1b) of the design obtained is as follows:

$$\mathbf{N}_1 = \begin{pmatrix} \mathbf{N}_{11} \\ \mathbf{N}_{12} \end{pmatrix} = \begin{pmatrix} (v^*-1)\mathbf{1}_{v_1} & \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_1} \\ \frac{(v^*-1)(v^*+2)}{2}\mathbf{1}_{v_2} & \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_2} \end{pmatrix}$$

$$\mathbf{N}_2 = \begin{pmatrix} \mathbf{N}_{21} \\ \mathbf{N}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{J}_{v_1 \times q} \\ 2\mathbf{J}_{v_2 \times q} \end{pmatrix}$$

$$\mathbf{W} = (v^*-2)\mathbf{J}_{p\times q}$$

Thus,

$$\mathbf{N}_1\mathbf{N}_1' = \begin{pmatrix} \mathbf{N}_{11}\mathbf{N}_{11}' & \mathbf{N}_{11}\mathbf{N}_{12}' \\ \mathbf{N}_{12}\mathbf{N}_{11}' & \mathbf{N}_{12}\mathbf{N}_{12}' \end{pmatrix} = \begin{pmatrix} \frac{(v^*-1)^2\left[4+(v^*-2)^2\right]}{4}\mathbf{J}_{v_1\times v_1} & \frac{(v^*-1)^2(v^{*2}-2v^*+8)}{4}\mathbf{J}_{v_1\times v_2} \\ \frac{(v^*-1)^2(v^{*2}-2v^*+8)}{4}\mathbf{J}_{v_2\times v_1} & \frac{(v^*-1)^2(v^{*2}+4)}{2}\mathbf{J}_{v_2\times v_2} \end{pmatrix}$$

and $$\mathbf{N}_2\mathbf{N}_2' = \begin{pmatrix} \mathbf{N}_{21}\mathbf{N}_{21}' & \mathbf{N}_{21}\mathbf{N}_{22}' \\ \mathbf{N}_{22}\mathbf{N}_{21}' & \mathbf{N}_{22}\mathbf{N}_{22}' \end{pmatrix} = \begin{pmatrix} \frac{v^*(v^*-1)}{2}\mathbf{J}_{v_1\times v_1} & v^*(v^*-1)\mathbf{J}_{v_1\times v_2} \\ v^*(v^*-1)\mathbf{J}_{v_1\times v_2} & 2v^*(v^*-1)\mathbf{J}_{v_2\times v_2} \end{pmatrix}.$$

Here, $\mathbf{R} = \begin{pmatrix} \frac{v^*(v^*-1)}{2}\mathbf{I}_{v_1\times v_1} & 0 \\ 0 & v^*(v^*-1)\mathbf{I}_{v_2\times v_2} \end{pmatrix}$,

$$\mathbf{K}_\alpha = kq\mathbf{I}_p = \frac{v^*(v^*-1)(v^*-2)}{2}\mathbf{I}_p \text{ and } \mathbf{K}_\beta = kp\mathbf{I}_p = 2(v^*-2)\mathbf{I}_q.$$

Further,

$$\mathbf{Z} = \mathbf{K}_{\beta} - \mathbf{W}'\mathbf{K}_{\alpha}^{-}\mathbf{W} = 2(v^*-2)\left[\mathbf{I}_{\mathbf{q}} - \frac{2}{v^*(v^*-1)}\mathbf{J}_{\mathbf{q}}\right]$$

$$\Rightarrow \mathbf{Z}^{-} = \frac{1}{2(v^*-2)}\mathbf{I}_{\mathbf{q}}$$

$$\mathbf{F} = \mathbf{K}_{\alpha}^{-}\mathbf{W} = \frac{2}{v^*(v^*-1)}\mathbf{J}_{p\times q},\ \mathbf{F}\mathbf{Z}^{-}\mathbf{F}' = \frac{1}{v^*(v^*-1)(v^*-2)}\mathbf{J}_{p\times p},\ \mathbf{F}\mathbf{Z}^{-} = \frac{1}{v^*(v^*-1)(v^*-2)}\mathbf{J}_{p\times q}$$

Thus, the information matrix for BBP-GRC design obtained is

$$\mathbf{C} = \begin{pmatrix} \mathbf{R}_1 - \mathbf{K}_{11} & -\mathbf{K}_{12} \\ -\mathbf{K}_{21} & \mathbf{R}_2 - \mathbf{K}_{22} \end{pmatrix}$$

$$\mathbf{K}_{11} = \mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' - \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}' + \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}'$$

$$\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{11}' = \frac{(v^*-1)\left[4+(v^*-2)^2\right]}{2v^*(v^*-2)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' = \left((v^*-1)\mathbf{1}_{v_1} \quad \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_1}\right) \times \frac{1}{v^*(v^*-1)(v^*-2)}\mathbf{J}_{p\times p} \times \begin{pmatrix} (v^*-1)\mathbf{1}_{v_1}' \\ \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_1}' \end{pmatrix}$$

$$= \frac{(v^*-1)[4+(v^*-2)^2]}{4v^*(v^*-2)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' = \mathbf{J}_{v_1\times q} \times \frac{1}{v^*(v^*-1)(v^*-2)}\mathbf{J}_{q\times p} \times \begin{pmatrix} (v^*-1)\mathbf{1}_{v_1}' \\ \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_1}' \end{pmatrix} = \frac{v^*(v^*-1)}{4(v^*-2)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}' = \left((v^*-1)\mathbf{1}_{v_1} \quad \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_1}\right) \times \frac{1}{v^*(v^*-1)(v^*-2)}\mathbf{J}_{\mathbf{p}\times\mathbf{q}} \times \begin{pmatrix} (v^*-1)\mathbf{1}_{v_1}' \\ \frac{(v^*-1)(v^*-2)}{2}\mathbf{1}_{v_1}' \end{pmatrix}$$

$$= \frac{(v^*-1)[4+(v^*-2)^2]}{4v^*(v^*-2)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}' = \frac{v^*(v^*-1)}{4(v^*-2)}\mathbf{J}_{v_1\times v_1}$$

$$\Rightarrow \mathbf{K}_{11} = \frac{(v^*-1)\left[4+(v^*-2)^2\right]}{2v^*(v^*-2)}\mathbf{J}_{v_1\times v_1}$$

Similarly,

$$\mathbf{K}_{12}=\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}+\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}-\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}+\mathbf{N}_{21}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}$$

$$\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}=\frac{(v^{*}\text{-}1)(v^{*}\text{-}2v^{*}\text{+}8)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\frac{(v^{*}\text{-}1)(v^{*2}\text{-}2v^{*}\text{+}8)}{4v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}=\frac{v^{*}(v^{*}\text{-}1)}{2(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2},\ \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\frac{(v^{*}\text{-}1)(v^{*2}\text{-}2v^{*}\text{+}8)}{4v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}=\frac{v^{*}(v^{*}\text{-}1)}{2(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2}$$

$$\Rightarrow\mathbf{K}_{12}=\frac{(v^{*}\text{-}1)(v^{*}\text{-}2v^{*}\text{+}8)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{K}_{21}=\frac{(v^{*}\text{-}1)(v^{*}\text{-}2v^{*}\text{+}8)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_1}$$

$$\mathbf{K}_{22}=\mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}+\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}+\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}+\mathbf{N}_{22}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}$$

$$\mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}=\frac{(v^{*}\text{-}1)(v^{*2}\text{+}4)}{v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2},\ \mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\frac{(v^{*}\text{-}1)(v^{*2}\text{+}4)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}=\frac{v^{*}(v^{*}\text{-}1)}{(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2},\ \mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\frac{(v^{*}\text{-}1)(v^{*2}\text{+}4)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{22}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}=\frac{v^{*}(v^{*}\text{-}1)}{(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2}$$

$$\Rightarrow\mathbf{K}_{22}=\frac{(v^{*}\text{-}1)(v^{*2}\text{+}4)}{v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2}$$

So the information matrix for this series of BBP-GRC design is obtained as

$$\mathbf{C}=\begin{pmatrix}\frac{v^{*}(v^{*}\text{-}1)}{2}\mathbf{I}_{v_1\times v_1}-\frac{(v^{*}\text{-}1)\left[4+(v^{*}-2)^{2}\right]}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_1} & -\frac{(v^{*}\text{-}1)(v^{*}\text{-}2v^{*}\text{+}8)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_1\times v_2}\\ -\frac{(v^{*}\text{-}1)(v^{*}\text{-}2v^{*}\text{+}8)}{2v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_1} & v^{*}(v^{*}\text{-}1)\mathbf{I}_{v_2\times v_2}-\frac{(v^{*}\text{-}1)(v^{*2}\text{+}4)}{v^{*}(v^{*}\text{-}2)}\mathbf{J}_{v_2\times v_2}\end{pmatrix}$$

…(4.1.5)

Example 4.1.2.3.2: Consider a BIB design with parameters as v* = 6, b* = 15, r* = 5, k* = 2, λ* = 1. Arrange the blocks of the BIB design as per the above mentioned method. The resulting design is a BBP-GRC design with parameter v_1 = 4 (1, 2, 3, 4), v_2 = 2 (5, 6), p = 2, q = 15, r_1 = 15, r_2 = 30 and k = 4.

Rows	Columns														
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV
I	1 2 5 6	1 3 5 6	1 4 5 6	1 5 5 6	1 6 5 6	2 3 5 6	2 4 5 6	2 5 5 6	2 6 5 6	3 4 5 6	3 5 5 6	3 6 5 6	4 5 5 6	4 6 5 6	5 6 5 6
II	3 4 5 6	2 4 5 6	2 3 5 6	2 3 4 6	2 3 4 5	1 4 5 6	1 3 5 6	1 3 4 6	1 3 4 5	1 2 5 6	1 2 4 6	1 2 4 5	1 2 3 6	1 2 3 5	1 2 3 4

The information matrix as obtained in (4.1.5) is

$$\mathbf{C}=\begin{pmatrix} 15\mathbf{I}_{4\times4}-2.083\mathbf{J}_{4\times4} & -3.333\mathbf{J}_{4\times2} \\ -3.333\mathbf{J}_{2\times4} & 30\mathbf{I}_{2\times2}-8.333\mathbf{J}_{2\times2} \end{pmatrix}$$

$V(\hat{\tau}_s-\hat{\tau}_{s'})=0.1333\sigma^2$, $s \neq s' = 1,2,..., v_1$

$V(\hat{\tau}_s-\hat{\tau}_{s'})=0.1042\sigma^2$, $s \neq s'$, $s= 1,2,..., v_1, s'= v_1+1,..., v_2$

Average variance is $0.1133\,\sigma^2$.

Remark: If we consider a Partially Balanced Incomplete Block (PBIB) design with parameter v*, b*, r*, k*, λ_i (i = 1, 2,..., m) and its complement and use the same method as given above, the resulting design will be a BBP-GRC.

Example 4.1.2.3.3: Consider a group divisible (GD) design with parameter v* = 12, b* = 9, r* = 3, k* = 4, λ_1 = 0, λ_2 = 1. Arrange the blocks of the GD design and its complement design as described in the above method. The resulting design is a BBP-GRC design with parameter v_1 = 8 (1, 2, 3, 4, 5, 6, 7, 8), v_2 = 4 (9, 10, 11, 12), p = 2, q = 9, r_1 = 9, r_2 = 18 and k = 8.

Rows	Columns								
	I	II	III	IV	V	VI	VII	VIII	IX
I	1 4 7 10 9 10 11 12	1 5 8 11 9 10 11 12	1 6 9 12 9 10 11 12	2 4 8 12 9 10 11 12	2 5 9 10 9 10 11 12	2 6 7 11 9 10 11 12	3 4 9 11 9 10 11 12	3 5 7 12 9 10 11 12	3 6 8 10 9 10 11 12
II	2 3 5 6 8 9 11 12	6 2 12 7 3 4 9 10	2 3 7 8 4 5 10 11	1 3 7 9 5 6 10 11	1 3 7 8 4 6 11 12	1 3 8 9 4 5 10 12	1 2 7 8 5 6 10 12	1 2 8 9 4 6 10 11	1 2 4 5 7 9 11 12

The information matrix for estimating treatment effects of first and second set is obtained as follows:

$$\mathbf{C}=\begin{pmatrix} 9\mathbf{I}_{4\times4} - 0.625\mathbf{J}_{4\times4} & -\mathbf{J}_{4\times4} \\ -\mathbf{J}_{4\times4} & 18\mathbf{I}_{4\times4} - 2.5\mathbf{J}_{4\times4} \end{pmatrix}$$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.222\sigma^2$, $s \neq s' = 1,2,..., v_1$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.017\sigma^2$, $s \neq s'$, $s = 1,2,..., v_1, s' = v_1+1,..., v_2$

Average variance is $0.187\,\sigma^2$.

Method 4.1.2.4: **Case I:** Consider a two-class association scheme for v* treatments with number of first associates as n_1 and number of second associates as n_2. Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column.

i) If $|n_1+1-n_2|$ is even, then augment $v_2 = (n_1+1-n_2)/2$ new treatments two times in each cell of column which has lesser cell size. The resulting design will be a BBP-GRC design with parameter $v_1 = v^*$, $v_2 = (n_1+1-n_2)/2$, $p = v^*$, $q = 2$, $r_1 = v^*$, $r_2 = v^*v_2$ and $k = n_1+1$.

ii) If $|n_1+1-n_2|$ is odd then augment one new treatment (n_1+1-n_2) number of times in each cell of column which has lesser cell size. The resulting design will be a BBP-GRC design with parameter $v_1 = v^*$, $v_2=1$, $p = v^*$, $q = 2$, $r_1 = v^*$, $r_2 = v^*(n_1+1-n_2)$ and $k = n_1+1$.

The design obtained is variance balanced with respect to the first set and second set of treatments.

Special Case: Consider a triangular association scheme with $v^*=\frac{n(n-1)}{2}$, $n_1=2(n-2)$ and $n_2=\frac{(n-2)(n-3)}{2}$. Following above procedure, the resulting BBP-GRC design will have $v_1=v^*=\frac{n(n-1)}{2}$, $v_2=\frac{n_1+n_2-1}{2}=\frac{9n-n^2-12}{4}$, $p=v^*=\frac{n(n-1)}{2}$, $q = 2$, $r_1=v^*=\frac{n(n-1)}{2}$, $r_2=pv_2=\frac{n(n-1)(9n-n^2+12)}{8}$ and $k=n_1+1=(2n-3)$.

Here, $$\mathbf{R}=\begin{pmatrix} \frac{n(n-1)}{2}\mathbf{I}_{v_1\times v_1} & 0 \\ 0 & \frac{n(n-1)(9n-n^2-12)}{8}\mathbf{I}_{v_2\times v_2} \end{pmatrix}$$

$$\mathbf{K}_\alpha = kq\mathbf{I}_p = 2(2n-3)\mathbf{I}_p \text{ and } \mathbf{K}_\beta = kp\mathbf{I}_p = \frac{n(n-1)(2n-3)}{2}\mathbf{I}_q$$

$$\mathbf{N_1}=\begin{pmatrix}\mathbf{N}_{11}\\ \mathbf{N}_{12}\end{pmatrix}=\begin{pmatrix}\mathbf{J}_{v_1\times p}\\ \frac{9n-n^2-12}{4}\mathbf{J}_{v_2\times p}\end{pmatrix}$$

$$\mathbf{N_2}=\begin{pmatrix}\mathbf{N}_{21}\\ \mathbf{N}_{22}\end{pmatrix}=\begin{pmatrix}(2n-3)\mathbf{1}_{v_1} & \frac{(n-2)(n-3)}{2}\mathbf{1}_{v_1}\\ 0 & \frac{n(n-1)(9n-n^2-12)}{8}\mathbf{1}_{v_2}\end{pmatrix}$$

$$\mathbf{W} = (2n-3)\,\mathbf{J}_{p\times q}$$

So, $$\mathbf{N}_1\mathbf{N}_1' = \begin{pmatrix}\mathbf{N}_{11}\mathbf{N}_{11}' & \mathbf{N}_{11}\mathbf{N}_{12}'\\ \mathbf{N}_{12}\mathbf{N}_{11}' & \mathbf{N}_{12}\mathbf{N}_{12}'\end{pmatrix}=\begin{pmatrix}\frac{n(n-1)}{2}\mathbf{J}_{v_1\times v_1} & \frac{n(n-1)(9n-n^2-12)}{8}\mathbf{J}_{v_1\times v_2}\\ \frac{n(n-1)(9n-n^2-12)}{8}\mathbf{J}_{v_2\times v_1} & \frac{n(n-1)(9n-n^2-12)^2}{32}\mathbf{J}_{v_2\times v_2}\end{pmatrix}$$

and

$$\mathbf{N}_2\mathbf{N}_2' = \begin{pmatrix}\mathbf{N}_{21}\mathbf{N}_{21}' & \mathbf{N}_{21}\mathbf{N}_{22}'\\ \mathbf{N}_{22}\mathbf{N}_{21}' & \mathbf{N}_{22}\mathbf{N}_{22}'\end{pmatrix}=\begin{pmatrix}\left[(2n-3)^2+\frac{(n-2)^2(n-3)^2}{4}\right]\mathbf{J}_{v_1\times v_1} & \frac{n^2(n-1)(n-2)(n-3)}{4}\mathbf{J}_{v_1\times v_2}\\ \frac{n^2(n-1)(n-2)(n-3)}{4}\mathbf{J}_{v_1\times v_2} & \frac{n^2(n-1)^2(9n-n^2-12)}{64}\mathbf{J}_{v_2\times v_2}\end{pmatrix}$$

Further,

$$\mathbf{Z} = \mathbf{K}_{\beta} - \mathbf{W}'\mathbf{K}_{\alpha}^{-}\mathbf{W} = \frac{n(n-1)(2n-3)}{2}\left[\mathbf{I}_q - \frac{2}{v^*(v^*-1)}\mathbf{J}_q\right] \Rightarrow \mathbf{Z}^{-} = \frac{2}{n(n-1)(2n-3)}\mathbf{I}_q$$

$$\mathbf{F} = \mathbf{K}_{\alpha}^{-}\mathbf{W} = \frac{1}{2}\mathbf{J}_{p\times q}$$

$$\mathbf{F}\mathbf{Z}^{-}\mathbf{F}' = \frac{1}{n(n-1)(2n-3)}\mathbf{J}_{p\times p}$$

$$\mathbf{F}\mathbf{Z}^{-} = \frac{1}{n(n-1)(2n-3)}\mathbf{J}_{p\times q}$$

Also

$$\mathbf{K}_{11} = \mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' - \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' + \mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}' + \mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}'$$

$$\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{11}' = \frac{n(n-1)}{4(2n-3)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' = \frac{1}{2(2n-3)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{11}' = \left((2n-3)\mathbf{1}_{v_1} \quad \frac{(n-2)(n-3)}{2}\mathbf{1}_{v_1}\right) \times \frac{1}{n(n-1)(2n-3)}\mathbf{J}_{q\times p} \times \mathbf{J}_{p\times v_1}$$

$$= \frac{n(n-1)}{4(2n-3)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{11}' = \mathbf{J}_{v_1\times p} \times \frac{1}{n(n-1)(2n-3)}\mathbf{J}_{\mathbf{p}\times\mathbf{q}} \times \mathbf{J}_{p\times v_1}$$

$$= \frac{1}{2(2n-3)}\mathbf{J}_{v_1\times v_1}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{N}_{21}' = \frac{2\left[(2n-3)^2 + \frac{(n-2)^2(n-3)^2}{4}\right]}{n(n-1)(2n-3)}\mathbf{J}_{v_1\times v_1}$$

$$\Rightarrow \mathbf{K}_{11} = \frac{2\left[(2n-3)^2 + \frac{(n-2)^2(n-3)^2}{4}\right]}{n(n-1)(2n-3)}\mathbf{J}_{v_1\times v_1}$$

Similarly,

$$\mathbf{K}_{12}=\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}+\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}-\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}+\mathbf{N}_{21}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}$$

$$\mathbf{N}_{11}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}=\frac{\text{n(n-1)(9n-n}^{2}\text{-12)}}{\text{16(2n-3)}}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\frac{\text{(9n-n}^{2}\text{-12)}}{\text{4(2n-3)}^{2}}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}=\frac{\text{n(n-1)(9n-n}^{2}\text{-12)}}{\text{16(2n-3)}}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{11}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\frac{\text{(9n-n}^{2}\text{-12)}}{\text{4(2n-3)}^{2}}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{N}_{21}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}=\frac{\text{(n-2)(n-3)(9n-n}^{2}\text{-12)}}{\text{8(2n-3)}}\mathbf{J}_{v_1\times v_2}$$

$$\Rightarrow\mathbf{K}_{12}=\frac{\text{(n-2)(n-3)(9n-n}^{2}\text{-12)}}{\text{8(2n-3)}}\mathbf{J}_{v_1\times v_2}$$

$$\mathbf{K}_{21}=\frac{\text{(n-2)(n-3)(9n-n}^{2}\text{-12)}}{\text{8(2n-3)}}\mathbf{J}_{v_2\times v_1}$$

$$\mathbf{K}_{22}=\mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}+\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}-\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}+\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}+\mathbf{N}_{22}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}$$

$$\mathbf{N}_{12}\mathbf{K}_{\alpha}^{-}\mathbf{N}_{12}^{'}=\frac{\text{n(n-1)(9n-n}^{2}\text{-12)}}{\text{64(2n-3)}}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{F}'\mathbf{N}_{12}^{'}=\frac{\text{n}^{2}\text{(n}-1)^{2}}{\text{8(2n-3)}}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{22}\mathbf{Z}^{-}\mathbf{F}'\,\mathbf{N}_{12}^{'}=\frac{\text{n(n-1)(9n-n}^{2}\text{-12)}}{\text{64(2n-3)}}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{12}\mathbf{F}\mathbf{Z}^{-}\mathbf{N}_{22}^{'}=\frac{\text{n}^{2}\text{(n}-1)^{2}}{\text{8(2n-3)}}\mathbf{J}_{v_2\times v_2}$$

$$\mathbf{N}_{22}\mathbf{Z}^{-}\,\mathbf{N}_{22}^{'}=\frac{\text{n(n-1)(9n-n}^{2}\text{-12)}}{\text{32(2n-3)}}\mathbf{J}_{v_2\times v_2}$$

$$\Rightarrow\mathbf{K}_{22}=\frac{\text{n(n-1)(9n-n}^{2}\text{-12)}}{\text{32(2n-3)}}\mathbf{J}_{v_2\times v_2}$$

The information matrix for BBP-GRC design thus obtained is

$$\mathbf{C}=\begin{pmatrix} \frac{n(n-1)}{2}\mathbf{I}_{v_1\times v_1}-\frac{2\left[(2n-3)^2+\frac{(n-2)^2(n-3)^2}{4}\right]}{n(n-1)(2n-3)}\mathbf{J}_{v_1\times v_1} & -\frac{(n-2)(n-3)(9n-n^2-12)}{8(2n-3)}\mathbf{J}_{v_1\times v_2} \\ -\frac{(n-2)(n-3)(9n-n^2-12)}{8(2n-3)}\mathbf{J}_{v_2\times v_1} & \frac{n(n-1)(9n-n^2-12)}{8}\mathbf{I}_{v_2\times v_2}-\frac{n(n-1)(9n-n^2-12)}{32(2n-3)}\mathbf{J}_{v_2\times v_2} \end{pmatrix}$$

…(4.1.6)

Example 4.1.2.4.1: Consider a triangular association scheme with $v^*=10,\ n_1=6, n_2=3.$ Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. Here, $|n_1+1-n_2|= 4$, so augment $v_2 = 2$ new treatments two times in each cell of the second column. The resulting design is a BBP-GRC design with $v_1=v^*=10,\ v_2=2,\ p=10,\ q=2, r_1=10,\ r_2=20$ and $k=7.$

Rows	Columns													
	I							II						
I	1	2	3	4	5	6	7	8	9	10	11	11	12	12
II	2	1	3	4	5	8	9	6	7	10	11	11	12	12
III	3	1	2	4	6	8	10	5	7	9	11	11	12	12
IV	4	1	2	3	7	9	10	5	6	8	11	11	12	12
V	5	1	6	7	2	8	9	3	4	10	11	11	12	12
VI	6	1	5	7	3	8	10	2	4	9	11	11	12	12
VII	7	1	5	6	4	9	10	2	3	8	11	11	12	12
VIII	8	2	5	9	3	6	10	1	4	7	11	11	12	12
IX	9	2	5	8	4	7	10	3	1	6	11	11	12	12
X	10	3	6	8	4	7	9	1	2	5	11	11	12	12

The information matrix for estimating treatment effects of first and second set is obtained as follows:

$$\mathbf{C}=\begin{pmatrix} 10\mathbf{I}_{10\times 10}-0.8286\mathbf{J}_{10\times 10} & -0.8571\mathbf{J}_{10\times 2} \\ -0.8571\mathbf{J}_{2\times 10} & 20\mathbf{I}_{2\times 2}-5.714286\mathbf{J}_{2\times 2} \end{pmatrix}$$

$V(\hat{\tau}_s-\hat{\tau}_{s'})=0.200\sigma^2,\ s\neq s'=1,2,...,\ v_1$

$V(\hat{\tau}_s-\hat{\tau}_{s'})=0.173\sigma^2,\ s\neq s',\ s=1,2,...,\ v_1, s'=v_1+1,...,v_2$

Average variance is $0.190\,\sigma^2$.

Example 4.1.2.4.2: Consider a triangular association scheme with $v^*=15, n_1=8, n_2=6$. Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. Here, $|n_1+1-n_2|= 3$, so augment $v_2 = 1$ new treatment in each cell of the second column thrice as it has lesser cell sizes. The resulting design will be a BBP-GRC design with parameter $v_1 = v^*=15$, $v_2 =1$, $p =15$, $q=2$, $r_1 =15$, $r_2 =45$ and $k=9$.

Rows	Columns																	
	I									II								
I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	16	16
II	2	1	3	4	5	6	10	11	12	7	8	9	13	14	15	16	16	16
III	3	1	2	4	5	7	10	13	14	6	8	9	11	12	15	16	16	16
IV	4	1	2	3	5	8	11	13	15	6	7	9	10	12	14	16	16	16
V	5	1	2	3	4	9	12	14	15	6	7	8	10	11	13	16	16	16
VI	6	1	7	8	9	2	10	11	12	3	4	5	13	14	15	16	16	16
VII	7	1	6	8	9	3	10	13	14	2	4	5	11	12	15	16	16	16
VIII	8	1	6	7	9	4	11	13	15	2	3	5	10	12	14	16	16	16
IX	9	1	6	7	8	5	12	14	15	2	3	4	10	11	13	16	16	16
X	10	2	6	11	12	3	7	13	14	1	4	5	8	9	15	16	16	16
XI	11	2	6	10	12	4	8	13	15	1	3	5	7	9	14	16	16	16
XII	12	2	6	10	11	5	9	14	15	1	3	4	7	8	13	16	16	16
XIII	13	3	7	10	14	4	8	11	15	1	2	5	6	9	12	16	16	16
XIV	14	3	7	10	13	5	9	12	15	1	2	4	6	8	11	16	16	16
XV	15	4	8	11	13	5	9	12	14	1	2	3	6	7	10	16	16	16

The information matrix for estimating treatment effects of first and second set is obtained as follows:

$$\mathbf{C}=\begin{pmatrix} 15\mathbf{I}_{15\times 15} - 0.867\mathbf{J}_{15\times 15} & -2\mathbf{1}_{15} \\ -2\mathbf{1}'_{15} & 30 \end{pmatrix}$$

$$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.133\sigma^2,\ s \neq s' = 1,2,..., v_1$$

$$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.096\sigma^2,\ s \neq s',\ s= 1,2,..., v_1, s'= v_1+1,..., v_2$$

Average variance is $0.129\,\sigma^2$.

Case II: Consider a two-class association scheme (v^*, n_1, n_2). Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. Then augment v_2 new treatments in each cell of both the columns. The resulting design will be a BBP-GRC design with parameter $v_1 = v^*$, v_2, $p = v^*$, $q = 2$, $r_1 = v^*$, $r_2 = 2v^*$, $k_1 = n_1+v_2+1$ and $k_2 = n_2+v_2$, where k_1 and k_2 are two different cell sizes. The design so obtained will have unequal cell sizes and is variance balanced with respect to the first set and second set of treatments.

Example 4.1.2.4.3: Consider a group divisible association scheme with $v^*=12, n_1=3, n_2=8$. Arranging the first associates along with the corresponding treatment in the first column, the second associates in the second column and augmenting 2 new treatments in each cell of both the columns, the following BBP-GRC design is obtained with parameters $v_1 = 12$, $v_2 = 2$, $p = 12$, $q = 2$, $r_1 = 12$, $r_2 = 24$, $k_1 = 6$ and $k_2 = 10$.

Rows	Columns															
	I						II									
I	1	2	3	4	13	14	5	6	7	8	9	10	11	12	13	14
II	2	1	3	4	13	14	5	6	7	8	9	10	11	12	13	14
III	3	1	2	4	13	14	5	6	7	8	9	10	11	12	13	14
IV	4	1	2	3	13	14	5	6	7	8	9	10	11	12	13	14
V	5	6	7	8	13	14	1	2	3	4	9	10	11	12	13	14
VI	6	5	7	8	13	14	1	2	3	4	9	10	11	12	13	14
VII	7	5	6	8	13	14	1	2	3	4	9	10	11	12	13	14
VIII	8	5	6	7	13	14	1	2	3	4	9	10	11	12	13	14
IX	9	10	11	12	13	14	1	2	3	4	5	6	7	8	13	14
X	10	9	11	12	13	14	1	2	3	4	5	6	7	8	13	14
XI	11	9	10	12	13	14	1	2	3	4	5	6	7	8	13	14
XII	12	9	10	11	13	14	1	2	3	4	5	6	7	8	13	14

The information matrix for estimating treatment effects of first and second set is obtained as follows:

$$\mathbf{C}=\begin{pmatrix} 12\mathbf{I}_{12\times12} - 0.756\mathbf{J}_{12\times12} & -1.467\,\mathbf{J}_{12\times2} \\ -1.467\,\mathbf{J}_{2\times12} & 24\mathbf{I}_{2\times2} - 3.2\mathbf{J}_{2\times2} \end{pmatrix}$$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.183\sigma^2$, $s \neq s'$, $s, s' = 1,2,..., v_1$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.134\sigma^2$, $s \neq s'$, $s = 1,2,..., v_1, s' = v_1+1,..., v_2$

Average variance is $0.155\,\sigma^2$.

The series can also be obtained by arranging the first associates in the first column and the second associates in the second column and augmenting v_2 new treatments in each cell of both the columns resulting in BBP-GRC design with incomplete rows.

Example 4.1.2.4.4: The following is a BBP-GRC design with parameter $v_1 = 12$, $v_2 = 2$, $p = 12$, $q = 2$, $r_1 = 11$, $r_2 = 24$, $k_1 = 5$ and $k_2 = 10$:

Rows	Columns													
	I					II								
I	2	3	4	13	14	5	6	7	8	9	10	11	12	13	14
II	1	3	4	13	14	5	6	7	8	9	10	11	12	13	14
III	1	2	4	13	14	5	6	7	8	9	10	11	12	13	14
IV	1	2	3	13	14	5	6	7	8	9	10	11	12	13	14
V	6	7	8	13	14	1	2	3	4	9	10	11	12	13	14
VI	5	7	8	13	14	1	2	3	4	9	10	11	12	13	14
VII	5	6	8	13	14	1	2	3	4	9	10	11	12	13	14
VIII	5	6	7	13	14	1	2	3	4	9	10	11	12	13	14
IX	10	11	12	13	14	1	2	3	4	5	6	7	8	13	14
X	9	11	12	13	14	1	2	3	4	5	6	7	8	13	14
XI	9	10	12	13	14	1	2	3	4	5	6	7	8	13	14
XII	9	10	11	13	14	1	2	3	4	5	6	7	8	13	14

The information matrix for estimating treatment effects of first and second set is obtained as follows:

$$\mathbf{C} = \begin{pmatrix} 10.93\mathbf{I}_{12\times12} - 0.678\mathbf{J}_{12\times12} & -1.41\mathbf{J}_{12\times2} \\ -1.41\mathbf{J}_{2\times10} & 24\mathbf{I}_{2\times2} - 3.6\mathbf{J}_{2\times2} \end{pmatrix}$$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.183\sigma^2$, $s \neq s' = 1,2,..., v_1$

$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.134\sigma^2$, $s \neq s'$, $s = 1,2,..., v_1, s' = v_1+1,..., v_2$

Average Variance $0.169\,\sigma^2$

It is seen that in all the methods obtained above for constructing BBP-GRC designs, the contrast for first set versus second set of treatments is estimated more precisely.

4.2 Robustness of GRC Designs Against Missing Observation(s)

Here in this section, the robustness of different classes of GRC designs (Bailey, 1992; Jaggi *et al.*, 2010; Datta, 2012; Datta *et al.*, 2015) against missing of one or more observations within a cell as per the efficiency criteria, as defined in Chapter 3, has been investigated. We consider a design be highly robust against missing observation(s) if the loss in efficiency of the residual design is not more than 5% and robust if the loss in efficiency of the residual design is between 5% to 10%.

A list of robust GRC design has prepared giving the parameters and the efficiency of the designs. A SAS code (given in the Appendix) has been written in PROC IML to calculate the harmonic mean of non-zero eigen-values of information matrix of original design and the residual design under the three-way model for GRC design described in Chapter 3.

Series I: Bailey (1992) defined semi-Latin square (n × n / k) as an arrangement of v = nk treatments in n rows and n columns and intersection of each row and column containing k units each. These semi-Latin squares are constructed by superimposing k number of Latin squares of order n and symbols of each Latin square are represented by different symbols.

Example I.1: Following is a semi-Latin square for v = 10 treatments arranged in 5 rows, 5 columns and intersection of each row-column having 2 units:

Rows	Columns									
	I		II		III		IV		V	
I	1	6	2	7	3	8	4	9	5	10
II	2	8	3	9	4	10	5	6	1	7
III	3	10	4	6	5	7	1	8	2	9
IV	4	7	5	8	1	9	2	10	3	6
V	5	9	1	10	2	6	3	7	4	8

Example I.2: Following is a semi-Latin square for v = 12 treatments arranged in 4 rows, 4 columns and intersection of each row-column having 3 units:

Rows	Columns			
	I	II	III	IV
I	1 5 9	2 8 11	3 6 12	4 7 10
II	2 6 10	1 7 12	4 5 11	3 8 9
III	3 7 11	4 6 9	1 8 10	2 5 12
IV	4 8 12	3 5 10	2 7 9	1 6 11

The robustness of this class of designs has been investigated against missing of some/ all observations of last column. Without loss of generality, the observations from units of last column are assumed to be missing as the columns can always be interchanged. Table 4.2.1 gives the parameters of the designs considered i.e., number of treatments ($v \leq 25$), number of rows (p), number of columns (q), replication (r), cell size (k) and the number of observation(s) missing with the unit/ cell number of the last column from which the observation(s) are missing along with the efficiency (E) of the residual design relative to the original design. The efficiency has been obtained by taking the ratio of harmonic means (HM) of information matrix $\mathbf{C}_d$ for treatment effects of original design with all observations to that of residual design $\mathbf{C}_{d^*}$ with missing observations.

Table 4.2.1: Parameters and efficiency of the residual design for Series I

S. No	**v**	**p**	**q**	**r**	**k**	**No. of observations missing**	**Unit/ Cell No.**	**HM ($C_{d)}$**	**HM (C_{d^*})**	**E**
1	6	3	3	3	2	1	last unit in last cell	3.00	2.67	0.89
2	6	3	3	3	2	2	both units in last cell	3.00	2.31	0.77
3	6	3	3	3	2	2	any two units from different cells	3.00	2.33	0.78
4	6	3	3	3	2	3	any three units from different cells	3.00	2.07	0.69
5	6	3	3	3	2	4	last two units from different	3.00	2.07	0.69

							cells and last cell total			
6	8	4	4	4	2	1	last unit in last cell	3.87	3.49	0.90
7	8	4	4	4	2	2	last cell total	3.87	3.31	0.86
8	8	4	4	4	2	2	any two observations from last units of last column	3.87	3.29	0.85
9	8	4	4	4	2	3	any three observations from last unit of last column	3.87	3.00	0.78
10	8	4	4	4	2	4	last two units from different cells and last cell total	3.87	2.80	0.72
11	8	4	4	4	2	5	last three units from different cells and last cell total	3.87	2.68	0.69
12	12	4	4	4	3	1	last unit in last cell	4.00	3.87	0.97
13	12	4	4	4	3	2	last any two units from last cell	4.00	3.74	0.94
14	12	4	4	4	3	3	last cell total	4.00	3.62	0.90
15	12	4	4	4	3	2	any two observations from last unit of last column	4.00	3.73	0.93
16	12	4	4	4	3	3	any three observations from last unit of last column	4.00	3.59	0.90
17	12	4	4	4	3	4	last unit from other cell and last cell total	4.00	3.46	0.86
18	12	4	4	4	3	6	last three units from different cells and last cell total	4.00	3.25	0.81

19	10	5	5	5	2	1	last unit in last cell	5.00	4.85	0.97
20	10	5	5	5	2	2	last cell total	5.00	4.70	0.94
21	10	5	5	5	2	2	any two observations from last unit of last column	5.00	4.68	0.94
22	10	5	5	5	2	3	any three observations from last unit of last column	5.00	4.49	0.90
23	10	5	5	5	2	4	any four observations from last unit of last column	5.00	4.28	0.86
24	10	5	5	5	2	5	last unit of each cell last column	5.00	4.06	0.81
25	10	5	5	5	2	6	last unit of last cell last column last cell total	5.00	3.94	0.79
26	15	5	5	5	3	1	last unit in last cell	5.00	4.91	0.98
27	15	5	5	5	3	2	any two observations from last cell	5.00	4.81	0.96
28	15	5	5	5	3	3	last cell total	5.00	4.72	0.94
29	15	5	5	5	3	2	any two observations from last unit of last column	5.00	4.81	0.96
30	15	5	5	5	3	3	any three observations from last unit of last column	5.00	4.70	0.94
31	15	5	5	5	3	4	any four observations from last unit of last column	5.00	4.58	0.92
32	15	5	5	5	3	5	last unit of last cell last column	5.00	4.46	0.89
33	15	5	5	5	3	7	last unit of last	5.00	4.46	0.89

							cell last column last cell total			
34	14	7	7	7	2	1	last unit in last cell	7.00	6.90	0.99
35	14	7	7	7	2	2	last cell total	7.00	6.81	0.97
36	14	7	7	7	2	2	any two observations from last unit of last column	7.00	6.80	0.97
37	14	7	7	7	2	3	any three observations from last unit of last column	7.00	6.80	0.97
38	14	7	7	7	2	4	any four observations from last unit of last column	7.00	6.56	0.94
39	14	7	7	7	2	5	any five observations from last unit of last column	7.00	6.44	0.92
40	14	7	7	7	2	6	any six observations from last unit of last column	7.00	6.30	0.90
41	14	7	7	7	2	7	last unit of last cell last column	7.00	6.17	0.88
42	14	7	7	7	2	8	last unit of last cell last column last cell total	7.00	6.17	0.88
43	21	7	7	7	3	1	last unit in last cell	7.00	6.94	0.99
44	21	7	7	7	3	2	any two observations from last cell	7.00	6.94	0.99
45	21	7	7	7	3	3	last cell total	7.00	6.82	0.97
46	21	7	7	7	3	2	any two observations from last unit of last column	7.00	6.87	0.98
47	21	7	7	7	3	3	any three	7.00	6.81	0.97

							observations from last unit of last column			
48	21	7	7	7	3	4	any four observations from last unit of last column	7.00	6.73	0.96
49	21	7	7	7	3	5	any five observations from last unit of last column	7.00	6.65	0.95
50	21	7	7	7	3	6	any six observations from last unit of last column	7.00	6.58	0.94
51	21	7	7	7	3	7	last unit of last cell last column	7.00	6.50	0.93
52	21	7	7	7	3	9	last unit of last cell last column last cell total	7.00	6.39	0.91

The efficiency of the designs obtained above in Table 4.2.1 has been summarized in Table 4.2.2. It is seen that the efficiency of the resultant design is quite high for most of the designs.

Table 4.2.2: Summary of efficiency

S. No.	Efficiency	No. of Designs
1	< 0.70	3
2	0.70 - 0.80	5
3	0.80 - 0.85	2
4	0.85 - 0.90	9
5	0.90 - 0.95	17
6	≥ 0.95	16

Out of 52 designs investigated, 16 designs have efficiency more than and equal to 95% and are highly robust where as there are 17 designs that have efficiency between 0.90 - 0.95 and are thus robust. There is a decreasing trend in efficiency with increase in number of missing observations. In fact, the intensity or the consequences depends upon the size

of the design. It is seen that smaller designs are more affected by the missing observations.

Series II: Jaggi *et al.* (2010) developed a series of generalized incomplete Trojan-type design for v = sm (s≥ 2, m distinct group), cells of size k with p =m rows and q columns.

Example II.1: Following is a generalized incomplete Trojan-type design for v = 16 treatments arranged in 8 rows, 2 columns and intersection of each row-column having k= 4 units:

Rows	Columns							
	I				II			
I	1	2	3	4	5	6	7	8
II	3	4	5	6	7	8	9	10
III	5	6	7	8	9	10	11	12
IV	7	8	9	10	11	12	13	14
V	9	10	11	12	13	14	15	16
VI	11	12	13	14	15	16	1	2
VII	13	14	15	16	1	2	3	4
VIII	15	16	1	2	3	4	5	6

The robustness of this class of designs has been investigated against missing of some/ all of observations pertaining to last column. Table 4.2.3 gives the parameters and efficiency of the residual design for this series of GRC designs.

Table 4.2.3: Parameters and efficiency of the residual design for Series II

S. No.	**v**	**p**	**q**	**r**	**k**	**No of Observations Missing**	**Cell/ Unit No**	**HM ($C_{d)}$**	**HM (C_{d*})**	**E**
1	16	8	2	4	4	1	last unit in last cell	3.60	3.51	0.97
2	16	8	2	4	4	2	any two unit from last cell	3.60	3.40	0.94
3	16	8	2	4	4	3	any three	3.60	3.32	0.92

							unit from last cell			
4	16	8	2	4	4	4	total last cell	3.60	3.22	0.89
5	16	8	2	4	4	2	any two observation from last unit of last column	3.60	3.22	0.89
6	16	8	2	4	4	3	any three observation from last unit of last column	3.60	3.20	0.89
7	16	8	2	4	4	4	any four observation from last unit of last column	3.60	2.99	0.83
8	16	8	2	4	4	5	any five observation from last unit of last column	3.60	2.99	0.83
9	16	8	2	4	4	6	any six observation from last unit of last column	3.60	2.60	0.72
10	16	8	2	4	4	7	any seven observation from last unit of last column	3.60	2.52	0.70
11	16	8	2	4	4	8	last unit of last cell last column	3.60	2.49	0.69
12	16	8	2	4	4	11	last unit of last cell last column last cell total	3.60	2.26	0.63
13	16	8	3	6	4	1	last	5.86	5.78	0.99
14	16	8	3	6	4	2	any two observation	5.86	5.70	0.97

							from last cell			
15	16	8	3	6	4	3	any three observation from last cell	5.86	5.62	0.96
16	16	8	3	6	4	4	total last cell	5.86	5.54	0.94
17	16	8	3	6	4	2	any two observation from last unit of last column	5.86	5.54	0.94
18	16	8	3	6	4	3	any three observation from last unit of last column	5.86	5.58	0.95
19	16	8	3	6	4	4	any four observation from last unit of last column	5.86	5.46	0.93
20	16	8	3	6	4	5	any five observation from last unit of last column	5.86	5.34	0.91
21	16	8	3	6	4	6	any six observation from last unit of last column	5.86	5.34	0.91
22	16	8	3	6	4	7	any seven observation from last unit of last column	5.86	5.08	0.87
23	16	8	3	6	4	8	last unit of last cell last column	5.86	4.95	0.84
24	16	8	3	6	4	11	last unit of last cell last column last cell total	5.86	4.73	0.81
25	6	6	2	4	2	1	last	3.57	3.28	0.92

26	6	6	2	4	2	2	total last cell	3.57	3.01	0.84
27	6	6	2	4	2	2	any two observation from last unit of last column	3.57	2.83	0.79
28	6	6	2	4	2	3	any three observation from last unit of last column	3.57	2.13	0.60
29	6	6	2	4	2	4	any four observation from last unit of last column	3.57	2.19	0.61
30	6	6	2	4	2	5	any five observation from last unit of last column	3.57	2.00	0.56
31	6	6	2	4	2	6	last unit of last cell last column	3.57	1.58	0.44
32	6	6	2	4	2	7	last unit of last cell last column last cell total	3.57	1.41	0.40
33	6	7	2	6	3	1	last unit in last cell	5.83	5.63	0.97
34	6	7	2	6	3	2	any three units in last cell	5.83	5.63	0.97
35	6	7	2	6	3	3	all the units in last cell	5.83	5.25	0.90
36	6	7	2	6	3	2	any two observation from last unit of last column	5.83	5.38	0.92
37	6	7	2	6	3	3	any three observation from last unit	5.83	5.38	0.92

							of last column			
38	6	7	2	6	3	4	any four observation from last unit of last column	5.83	4.70	0.81
39	6	7	2	6	3	5	any five observation from last unit of last column	5.83	4.40	0.75
40	6	7	2	6	3	6	any six observation from last unit of last column	5.83	4.40	0.75
41	6	7	2	6	3	7	last unit of last cell last column	5.83	4.28	0.73
42	6	7	2	6	3	9	last unit of last cell last column last cell total	5.83	3.83	0.66

It is seen from Table 4.2.3 that out of 42 design, 7 designs have efficiency more than and equal to 0.95 and are highly robust where as there are 11 designs that have efficiency between 90% to 95% and are thus robust. Here also there is a decreasing trend in efficiency with increase in number of missing observations. Smaller designs, in terms of the total number of units, are more affected by the missing observations.

Series III: Datta (2012) developed a series of GRC designs for $v = 2t + 1$ ($t > 1$) and cells of size two with $p = t$ rows of size $2(2t+1)$, $q = (2t+1)$ columns of size $2t$, $r = 2t$ and $k = 2$ by developing the following initial columns mod $(2t + 1)$:

1	2t + 1
2	2t
3	2t – 1
.	.

.	.
.	.
t	2t - (t - 2)

Example III.1: For t = 3, v = 7 and the contents of the initial column are as follows:

1 7

2 6

3 5

Developing these columns mod 7 results in the following GRC design in three rows of size 14, 7 columns of size 6 with 2 units per cell and replication of each treatment being 6:

Rows	Columns													
	I		II		III		IV		V		VI		VII	
I	1	7	2	1	3	2	4	3	5	4	6	5	7	6
II	2	6	3	7	4	1	5	2	6	3	7	4	1	5
III	3	5	4	6	5	7	6	1	7	2	1	3	2	4

The robustness of this class of designs has been investigated against missing of some/ all of observations pertaining to last column. Table 4.2.4 gives the parameters and efficiency of the residual design for this series of GRC designs.

Table 4.2.4: Parameters and efficiency of the residual design for Series III

S. No.	**v**	**p**	**q**	**r**	**k**	**No. of observations missing**	**Unit/ Cell No.**	**HM ($C_{d)}$**	**HM (C_{d*})**	**E**
1	5	2	4	4	2	1	last unit in last cell	3.75	3.41	0.91
2	5	2	4	4	2	2	both units in last cell	3.75	3.08	0.82
3	5	2	4	4	2	2	any two units from different cells	3.75	3.07	0.82
4	5	2	4	4	2	3	any three units from	3.75	2.79	0.74

							different cells			
5	7	3	7	6	2	1	last unit in last cell	5.83	5.63	0.96
6	7	3	7	6	2	2	both units in last cell	5.83	5.42	0.93
7	7	3	7	6	2	3	any three units from different cells	5.83	5.21	0.89
8	9	4	9	8	2	1	last unit in last cell	7.88	7.73	0.98
9	9	4	9	8	2	2	both units in last cell	7.88	7.58	0.96
10	9	4	9	8	2	4	any four observations from last unit of last column	7.88	7.23	0.92
11	9	4	9	8	2	5	any three observations from last unit and last cell total	7.88	7.09	0.90
12	11	5	11	10	2	1	last unit in last cell	9.90	9.78	0.99
13	11	5	11	10	2	2	both units in last cell	9.90	9.67	0.98
14	11	5	11	10	2	5	last unit of the cells	9.90	9.29	0.94
15	11	5	11	10	2	6	any four observations from last unit and last cell total	9.90	9.18	0.93
16	13	6	13	12	2	1	last unit in last cell	11.92	11.82	0.99
17	13	6	13	12	2	2	both units in last cell	11.92	11.73	0.98
18	13	6	13	12	2	6	last units of the cells	11.92	11.31	0.95
19	13	6	13	12	2	7	any five observations	11.92	11.22	0.94

							from last unit and last cell total			
20	15	7	15	14	2	1	last unit in last cell	13.93	13.85	0.99
21	15	7	15	14	2	2	both units in last cell	13.93	13.77	0.99
22	15	7	15	14	2	7	last units of the cells	13.93	13.34	0.96
23	15	7	15	14	2	8	any six observations from last unit and last cell total	13.93	13.27	0.95

It is seen from Table 4.2.4 that the efficiency of the resultant design is quite high for most of the designs. Out of 23 design, 11 designs have efficiency more than and equal to 0.95 and are highly robust where as there are 7 designs that have efficiency between 90% to 95% and are thus robust. Here also there is a decreasing trend in efficiency with increase in number of missing observations. Smaller designs, in terms of the total number of units, are more affected by the missing observations.

Series IV: Datta (2012) developed GRC designs with parameters v (even), p = (v-1) rows of size v, q = $\frac{v}{2}$ columns of size 2(v-1), r = (v-1) and k = 2 by developing following initial columns mod v:

1	v
v	2
2	v-1
v-1	3
.	.
.	.
.	.
$v - (\frac{v}{2}-2)$	$v - \frac{v}{2}$
$\frac{v}{2}$	$\frac{v}{2} + 1$

Example IV.1: For v = 8, following is a GRC design with cells containing 2 units in 7 rows of size 8 each and 4 columns of size 14 each:

Rows	Columns							
	I		II		III		IV	
I	1	8	2	1	3	2	4	3
II	8	2	1	3	2	4	3	5
III	2	7	3	8	4	1	5	2
IV	7	3	8	4	1	5	2	6
V	3	6	4	7	5	8	6	1
VI	6	4	7	5	8	6	1	7
VII	4	5	5	6	6	7	7	8

The efficiency of this class of design has been worked out against missing of some/ all of observations of last cell/ column. Table 4.2.5 contains the parameters ($v \leq 12$) and efficiency of the residual design for this series.

Table 4.2.5 Parameters and efficiency of the residual design for Series IV

S. No.	**v**	**p**	**q**	**r**	**k**	**No. of observations missing**	**Unit/ Cell No.**	**HM ($C_{d)}$**	**HM (C_{d*})**	**E**
1	6	5	3	5	2	1	last unit in last cell	4.41	4.06	0.92
2	6	5	3	5	2	2	both units in last cell	4.41	3.92	0.89
3	6	5	3	5	2	2	any two units from different cells	4.41	3.75	0.85
4	6	5	3	5	2	3	any three units from different cells	4.41	3.29	0.75
5	6	5	3	5	2	4	any four observations from last unit	4.41	2.53	0.57
6	6	5	3	5	2	5	last unit in each cell of last column	4.41	3.03	0.69

7	6	5	3	5	2	6	last unit in each cell of last column and total last cell	4.41	3.03	0.69
8	8	7	4	7	2	1	last unit in last cell	6.37	6.15	0.97
9	8	7	4	7	2	2	both units in last cell	6.37	6.04	0.95
10	8	7	4	7	2	2	any two units from different cells	6.37	5.98	0.94
11	8	7	4	7	2	3	any three units from different cells	6.37	5.73	0.90
12	8	7	4	7	2	4	any four observations from last unit of last column	6.37	5.28	0.83
13	8	7	4	7	2	5	any five observations from last unit of last column	6.37	5.25	0.83
14	8	7	4	7	2	6	any six observations from last unit of last column	6.37	4.83	0.76
15	8	7	4	7	2	7	last unit in each cell of last column	6.37	5.11	0.80
16	8	7	4	7	2	8	last unit in each cell of last column and total last cell	6.37	5.11	0.80
17	10	9	5	9	2	1	last unit in last cell	8.34	8.19	0.98
18	10	9	5	9	2	2	both units in last cell	8.34	8.09	0.97
19	10	9	5	9	2	2	any two units from different	8.34	8.06	0.97

							cells			
20	10	9	5	9	2	3	any three units from different cells	8.34	7.89	0.95
21	10	9	5	9	2	4	any four observations from last unit of last column	8.34	7.67	0.92
22	10	9	5	9	2	5	any five observations from last unit of last column	8.34	7.50	0.90
23	10	9	5	9	2	6	any six observations from last unit of last column	8.34	7.24	0.87
24	10	9	5	9	2	7	any seven observations from last unit of last column	8.34	7.21	0.86
25	10	9	5	9	2	8	any eight observations from last unit of last column	8.34	6.96	0.83
26	10	9	5	9	2	9	last unit in each cell of last column	8.34	7.11	0.85
27	10	9	5	9	2	10	last unit in each cell of last column and total last cell	8.34	7.11	0.85
28	12	11	6	11	2	1	last unit in last cell	10.32	10.21	0.99
29	12	11	6	11	2	2	total last cell	10.32	10.12	0.98
30	12	11	6	11	2	2	any two observations from last unit of last column	10.32	10.09	0.98
31	12	11	6	11	2	3	any three observations	10.3	9.99	0.97

							from last unit of last column	2		
32	12	11	6	11	2	4	any four observations from last unit of last column	10.32	9.99	0.97
33	12	11	6	11	2	5	any five observations from last unit of last column	10.32	9.73	0.94
34	12	11	6	11	2	6	any six observations from last unit of last column	10.32	9.54	0.92
35	12	11	6	11	2	7	any seven observations from last unit of last column	10.32	9.49	0.92
36	12	11	6	11	2	8	any eight observations from last unit of last column	10.32	9.28	0.90
37	12	11	6	11	2	9	any nine observations from last unit of last column	10.32	9.27	0.90
38	12	11	6	11	2	10	any ten observations from last unit of last column	10.32	9.10	0.88
39	12	11	6	11	2	11	last unit in each cell of last column	10.32	9.10	0.88
40	12	11	6	11	2	12	last unit in each cell of last column and total last cell	10.32	9.10	0.88
41	14	13	7	13	2	1	last unit in last cell	12.31	12.31	1.00
42	14	13	7	13	2	2	both units in last cell	12.31	12.14	0.99

43	14	13	7	13	2	2	any two units from different cells	12.31	12.12	0.98
44	14	13	7	13	2	3	any three units from different cells	12.31	12.04	0.98
45	14	13	7	13	2	4	any four observations from last unit of last column	12.31	12.04	0.98
46	14	13	7	13	2	5	any five observations from last unit of last column	12.31	11.82	0.96
47	14	13	7	13	2	6	any six observations from last unit of last column	12.31	11.70	0.95
48	14	13	7	13	2	7	any seven observations from last unit of last column	12.31	11.60	0.94
49	14	13	7	13	2	8	any eight observations from last unit of last column	12.31	11.45	0.93
50	14	13	7	13	2	9	any nine observations from last unit of last column	12.31	11.41	0.93
51	14	13	7	13	2	10	any ten observations from last unit of last column	12.31	11.27	0.92
52	14	13	7	13	2	11	any eleven observations from last unit of last column	12.31	11.26	0.91
53	14	13	7	13	2	12	any twelve observations from last unit of last column	12.31	11.13	0.90

54	14	13	7	13	2	13	last unit in each cell of last column	12.31	11.12	0.90
55	14	13	7	13	2	14	last unit in each cell of last column and total last cell	12.31	11.12	0.90

It is seen from the Table 4.2.5 that the efficiency of the resultant design is quite high for most of the designs. Out of 55 designs, 36 designs have efficiency more than 90% and are thus robust.

Series V: Datta *et al.* (2015) developed a method of constructing GRC designs with v (prime) treatments in p = 2 rows of size $\frac{kv(v-1)}{2}$, q = $\frac{v(v-1)}{2}$ columns of size 2k and each cell of size k.

Example V.1: Following is a GRC design with v = 5 treatments in 2 rows of size 20 each and 10 columns of size 4 each and cells containing 2 units:

Rows	Columns									
	I	II	III	IV	V	VI	VII	VIII	IX	X
I	1 2	2 3	3 4	4 5	5 1	1 3	2 4	3 5	4 1	5 2
II	2 3	3 4	4 5	5 1	1 2	3 5	4 1	5 2	1 3	2 4

Example V.2: For v = 5, a GRC design with cell size 3 is obtained in 2 rows of size 30 each and 10 columns of size 6 each as follows:

Rows	Columns									
	I	II	III	IV	V	VI	VII	VIII	IX	X
I	1 2 3	2 3 4	3 4 5	4 5 1	5 1 2	1 3 5	2 4 1	3 5 2	4 1 3	5 2 4
II	2 3 4	3 4 5	4 5 1	5 1 2	1 2 3	3 5 2	4 1 3	5 2 4	1 3 5	2 4 1

The robustness of this class of designs is investigated against missing of observations of last cell/ column. Table 4.2.6 lists the parameters and efficiency of the residual design for this series.

Table 4.2.6: Parameters and efficiency of the residual design for Series V

S. No.	v	p	q	r	k	No. of observations missing	Unit/ Cell No.	HM (C_d)	HM (C_{d^*})	E
1	5	2	10	8	2	1	last unit in last cell	6.25	5.90	0.94
2	5	2	10	8	2	2	both units in last cell	6.25	5.82	0.93
3	5	2	10	8	2	2	any two units from different cells	6.25	5.82	0.93
4	5	2	10	8	2	3	any three units from different cells	6.25	5.52	0.88
5	5	2	10	12	3	1	last unit in last cell	10.83	10.52	0.97
6	5	2	10	12	3	2	any two observations from last cell	10.83	10.21	0.94
7	5	2	10	12	3	3	total last cell	10.83	10.22	0.94
8	5	2	10	12	3	2	last column each cell last unit	10.83	10.33	0.95
9	5	2	10	12	3	4	last column each cell last unit and last cell total	10.83	9.93	0.92
10	5	2	10	16	4	1	last unit in last cell	15.63	15.33	0.98
11	5	2	10	16	4	2	any two observations from last cell	15.63	15.12	0.97
12	5	2	10	16	4	3	any three observations	15.63	14.94	0.96

							from last cell			
13	5	2	10	16	4	4	total last cell	15.63	14.80	0.95
14	5	2	10	16	4	2	last unit in each cell of last column	15.63	15.12	0.97
15	5	2	10	16	4	5	last unit in each cell of last column and last cell total	15.63	14.52	0.93
16	7	2	21	12	2	1	last unit in last cell	8.75	8.53	0.97
17	7	2	21	12	2	2	two observations from last cell	8.75	8.47	0.97
18	7	2	21	12	2	2	last column each cell last unit	8.75	8.47	0.97
19	7	2	21	12	2	3	last column each cell last unit and last cell total	8.75	8.28	0.95
20	7	2	21	18	3	1	last unit in last cell	15.17	14.96	0.99
21	7	2	21	18	3	2	any two observations from last cell	15.17	14.84	0.98
22	7	2	21	18	3	3	total last cell	15.17	14.76	0.97
23	7	2	21	18	3	2	last column each cell last unit	15.17	14.84	0.98
24	7	2	21	18	3	4	last column each cell last unit and last cell total	15.17	14.56	0.96
25	7	2	21	24	4	1	last unit in last cell	21.88	21.68	0.99
26	7	2	21	24	4	2	any two observations	21.88	21.54	0.98

							from last cell			
27	7	2	21	24	4	3	any three observations from last cell	21.88	21.42	0.98
28	7	2	21	24	4	4	total last cell	21.88	21.32	0.97
29	7	2	21	24	4	2	last column each cell last unit	21.88	21.54	0.98
30	7	2	21	24	4	5	last column each cell last unit and last cell total	21.88	21.14	0.97
31	7	2	21	30	5	1	last unit in last cell	28.70	28.51	0.99
32	7	2	21	30	5	2	any two observations from last cell	28.70	28.37	0.99
33	7	2	21	30	5	3	any three observations from last cell	28.70	28.23	0.98
34	7	2	21	30	5	4	any four observations from last cell	28.70	28.10	0.98
35	7	2	21	30	5	5	total last cell	28.70	27.99	0.98
36	7	2	21	30	5	2	each cell in last unit of last column	28.70	28.36	0.99
37	7	2	21	30	5	6	last unit in each cell of last column and total last cell	28.70	27.81	0.97
38	11	2	55	20	2	1	last unit in last cell	13.75	13.62	0.99
39	11	2	55	20	2	2	total last cell	13.75	13.59	0.99
40	11	2	55	20	2	2	each cell in last unit of last column	13.75	13.59	0.99

41	11	2	55	20	2	3	each cell in last unit of last column and total last cell	13.75	13.48	0.98
42	11	2	55	30	3	1	last unit in last cell	23.83	23.71	1.00
43	11	2	55	30	3	2	any two observations from last cell	23.83	23.64	0.99
44	11	2	55	30	3	3	total last cell	23.83	23.59	0.99
45	11	2	55	30	3	2	last unit in each cell of last column	23.83	23.64	0.99
46	11	2	55	30	3	4	last unit in each cell of last column and last cell total	23.83	23.48	0.99
47	11	2	55	40	4	1	last unit in last cell	34.35	34.24	1.00
48	11	2	55	40	4	2	any two observations from last cell	34.35	34.16	0.99
49	11	2	55	40	4	3	any three observations from last cell	34.35	34.08	0.99
50	11	2	55	40	4	4	total last cell	34.35	34.02	0.99
51	11	2	55	40	4	2	last unit in each cell of last column	34.35	34.16	0.99
52	11	2	55	40	4	5	last unit in each cell of last column and last cell total	34.35	33.92	0.99
53	11	2	55	50	5	1	last unit in last cell	45.04	44.93	1.00
54	11	2	55	50	5	2	any two	45.04	44.85	1.00

							observations from last cell			
55	11	2	55	50	5	3	any three observations from last cell	45.04	44.77	0.99
56	11	2	55	50	5	4	any four observations from last cell	45.04	44.69	0.99
57	11	2	55	50	5	5	total last cell	45.04	44.63	0.99
58	11	2	55	50	5	2	last unit in each cell of last column	45.04	44.85	1.00
59	11	2	55	50	5	6	last unit in each cell of last column and last cell total	45.04	44.53	0.99

It is seen from the Table 4.2.6 that the efficiency of the resultant design is quite high for most of the designs. Out of 59 designs, 51 designs have efficiency more than and equal to 0.95 and are highly robust where as there are 7 designs that have efficiency 0.90-0.95 and are thus robust. There are few designs with no loss of efficiency.

Series VI: Datta *et al.* (2015) developed this series of GRC design for unequal cell sizes. This design is developed by using a BIB design with parameters v*, b* (even), r*, k*, λ*. The resulting design have parameters v = v*, p = 2 rows of size $\frac{v^*b^*}{2}$, q = b* columns of size v*, r = b*, $k_1 = k^*$, and $k_2 = v^* - k^*$.

Example VI.1: Consider a BIB design with parameters v* = 5, b* = 10, r* = 4, k* = 2, λ* = 1. The following is a GRC design with parameters v = 5, p = 2 of size 25 each and q = 10 columns of size 5, r = 10, $k_1 = 2$ and $k_2 = 3$.

Rows	Columns									
	I	II	III	IV	V	VI	VII	VIII	IX	X
I	1 2	1 3	1 4	1 5	2 3	3 4 5	2 4 5	2 3 5	2 3 4	1 4 5
II	1 3 5	1 3 4	1 2 5	1 2 4	1 2 3	2 4	2 5	3 4	3 5	4 5

The following Table 4.2.7 the parameter of the GRC designs developed by Series V along with number of observation missing and the cell number from which the observations are missing, harmonic mean of non-zero eigen values of information matrix of original design and the residual design under the three-way model and The efficiency (E) of the residual design relative to the original design.

Table 4.2.7: Parameters and efficiency of the residual design for Series VI

S. No.	**v**	**p**	**q**	**r**	**k**	**No. of observation missing**	**Unit/ Cell No.**	**HM (C_d)**	**HM (C_{d^*})**	**E**
1	5	2	10	10	2 3	1	last unit in last cell	8.50	8.29	0.98
2	5	2	10	10	3 3	2	last any two units from last cell	8.50	8.17	0.96
3	5	2	10	10	4 3	3	last cell total	8.50	7.93	0.93
4	5	2	10	10	5 3	2	last unit of each cell of last column	8.50	8.01	0.94
5	5	2	10	10	6 3	5	last unit of each cell of last column and last cell total	8.50	7.69	0.91
6	9	2	12	12	3 6	1	last unit	12.00	11.86	0.99
7	9	2	12	12	4 6	2	last any two units from last cell	12.00	11.72	0.98
8	9	2	12	12	5 6	3	last any three units from last	12.00	11.59	0.97

							cell			
9	9	2	12	12	6 6	4	last any four units from last cell	12.00	11.47	0.96
10	9	2	12	12	7 6	5	last any five units from last cell	12.00	11.34	0.94
11	9	2	12	12	8 6	6	total last cell	12.00	11.21	0.93
12	9	2	12	12	9 6	2	last unit of each cell of last column	12.00	11.73	0.98
13	9	2	12	12	10 6	9	last unit of each cell of last column and last cell total	12.00	11.09	0.92
14	9	2	18	8	4 5	1	last unit	18.00	17.86	0.99
15	9	2	18	8	5 5	2	last any two units from last cell	18.00	17.73	0.99
16	9	2	18	8	6 5	3	any three units from last cell	18.00	17.60	0.98
17	9	2	18	8	7 5	4	last four units from last cell	18.00	17.47	0.97
18	9	2	18	8	8 5	5	total last cell	18.00	17.34	0.96
19	9	2	18	8	9 5	2	last unit of each cell of last column	18.00	17.73	0.99
20	9	2	18	8	10 5	9	last unit of each cell of last column and last cell total	18.00	17.35	0.96
21	10	2	30	30	3 7	1	last unit in last cell	29.76	29.64	1.00
22	10	2	30	30	4 7	2	last two units from	29.76	29.52	0.99

							last cell			
23	10	2	30	30	5 7	3	last any three units from last cell	29.76	29.40	0.99
24	10	2	30	30	6 7	4	last any four units from last cell	29.76	29.29	0.98
25	10	2	30	30	7 7	5	last any five units from last cell	29.76	29.19	0.98
26	10	2	30	30	8 7	6	last any six units from last cell	29.76	29.09	0.98
27	10	2	30	30	9 7	7	last cell total	29.76	28.95	0.97
28	10	2	30	30	10 7	2	last unit of each cell of last column	29.76	29.53	0.99
29	10	2	30	30	11 7	8	last unit of each cell of last column and last cell total	29.76	28.84	0.97

It is seen from the Table 4.2.7 that the efficiency of the resultant design is quite high for most of the designs. Out of 28 designs, 23 design have efficiency more than and equal to 95% and are highly robust and 5 designs are robust.

Thus all the series of GRC designs investigated are found to be robust against loss of observations.

4.3 Generalized Row-Column Designs with Factorial Treatment Structure

GRC designs developed in the literature are for single factor. There may be situations when more than one factor has to be taken in a single experiment, like if an experimenter is interested in studying the effect of two or more fertilizers on a crop or effect of various feeds and environments on milk yield of cows, statistically more efficient and informative method is to use factorial row-column designs, than to run separate experiments for each

factor (Cheng, 2014). Factorial RC designs need fewer experimental units and permit estimation of interactions between the factors. For example, in a veterinary trial involving 4 breeds and 4 age groups of calves to study the effects of 4 drugs at 2 levels each, the following factorial row-column design in 4 rows and 4 columns with one treatment combination per cell can be used:

Rows (Age)	Columns (Breed)			
	I	II	III	IV
I	1111	0100	0010	1001
II	0001	1010	1100	0111
III	1000	0011	0101	1110
IV	0110	1101	1011	0000

If the four drugs (factors) are represented as A, B, C and D, then in this design, effects ABC, BCD and AD are confounded row-wise and ABD, ACD and BC are confounded column-wise.

In the following section, GRC designs with factorial treatment structure are considered and confounded GRC designs for symmetrical factorial have been obtained.

4.3.1 Generalized Confounded Row-Column Design

Generalized Confounded Row-Column (GCRC) design is an arrangement of s^n factorial combinations in p (>1) rows and q (>1) columns such that the intersection of each row and column receives s^k ($k < n$) treatment combinations and is represented here as GCRC ($s^n//p,q//s^k$) design.

Consider a s^n factorial involving n factors F_1, F_2,..., F_n, having s levels each. The simplest way of obtaining a GCRC design is by arranging a s^n factorial in a Latin square of order s^{n-k} with s^k units in a row-column intersection in the following way:

- Obtain (s^n, s^k) factorial resulting in s^{n-k} blocks.
- Consider these blocks as the cells of the first column of a Latin square of order s^{n-k}.

- The remaining columns are obtained by arranging the cells in a the form of Latin square.
- This will result in a GCRC ($s^n//s^{n-k}, s^{n-k}//s^k$) design, i.e. s^n treatment combinations arranged in s^{n-k} rows and columns and intersection of each row and column containing s^k treatment combinations.

Example 4.3.1.1: The following is a GCRC design for 2^4 treatment combinations arranged in 2^2 number of rows and columns and intersection of each row and column consist of 2^2 treatment combinations represented as GCRC ($2^4//2^2, 2^2//2^2$) design:

Rows	Columns			
	I	II	III	IV
I	0000 1011 0111 1100	1000 0100 0011 1111	0001 1010 0110 1101	0010 1001 0101 1110
II	1000 0100 0011 1111	0001 1010 0110 1101	0010 1001 0101 1110	0000 1011 0111 1100
III	0001 1010 0110 1101	0010 1001 0101 1110	0000 1011 0111 1100	1000 0100 0011 1111
IV	0010 1001 0101 1110	0000 1011 0111 1100	1000 0100 0011 1111	0001 1010 0110 1101

Here, it is seen that row-wise and column-wise the design is complete i.e. all the 16 combinations are appearing in all the 4 rows and 4 columns exactly once. Cell-wise, the factorial effects $F_1F_2F_3$, $F_1F_2F_4$ and F_3F_4 are completely confounded.

Another arrangement of a GCRC ($2^4//2^2, 2^2//2^2$) design can be obtained by confounding the factorial effects $F_1F_3F_4$, $F_2F_3F_4$ and F_1F_2. These two arrangements together will ensure the estimation of all the factorial effects.

A method of constructing GCRC design with rows incomplete is now given below.

4.3.2 Method of Construction

Consider a s^n factorial involving n factors F_1, F_2, ..., F_n, having s levels each. The treatment combinations may be denoted by n- tuples $a_1, a_2, ..., a_n$, where the s levels of a factor are coded as 0,1,..., s-1 with $a_i \varepsilon \{0,1,..., s-1\}$, i= 1,2,...,n. The s^n combinations are

arranged in cells of size s^k (k<n) in $p = s^{n-k}$ rows and $q = s^{n-k-1}$ columns following the steps described below.

- Obtain (s^n, s^k) factorial resulting in s^{n-k} blocks saving main effects and lower order interactions.
- Consider these blocks as the cells of the first column of the s^{n-k-1} columns.
- Select any one of the higher order effects from first step.
- Develop the cells of the remaining columns by confounding the effect selected above row-wise.

The resultant arrangement is a GCRC ($s^n//s^{n-k}$,$s^{n-k-1}//s^k$) design. These designs will have incomplete rows of size s^{n-1} each and complete columns.

Remark: If a design is required in row size less than s^{n-1}, then more effects are to be confounded in rows.

Example 4.3.2.1: Consider a 2^4 factorial.

- A (2^4, 2^2) factorial resulting in 2^2 blocks is obtained by confounding F_1F_2, F_3F_4 and $F_1F_2F_3F_4$.
- These blocks constitute the cells of the first column.
- The second column is obtained by confounding $F_1F_2F_3F_4$ row-wise.

The resultant arrangement is a GCRC ($2^4//2^2$,$2//2^2$) design as given below.

Rows	Columns							
	I				II			
I	0000	1100	0011	1111	1010	0110	1001	0101
II	1000	0100	1011	0111	0010	1110	0001	1101
III	0010	1110	0001	1101	1000	0100	1011	0111
IV	1010	0110	1001	0101	0000	1100	0011	1111

It may be noted that row I and II constitutes a complete replicate and so is row III and IV. Thus, the design is resolvable row-wise.

Example 4.3.2.2: The following is a GCRC $(2^5//2^3,2^2//2^2)$ design is for 2^5 treatment combinations arranged in 2^3 rows, 2^2 columns and intersection of each row and column containing 2^2 combinations:

Rows	Columns			
	I	II	III	IV
I	00000 10101 01110 11011	00100 10001 01100 11111	11000 01101 10110 00011	10010 00111 11100 01001
II	10000 00101 11110 01010	01000 11101 00110 10011	00010 10111 01100 11001	00001 10100 01111 11010
III	01000 11101 00110 10011	00010 10111 01100 11001	00001 10100 01111 11010	10000 00101 11110 01010
IV	00100 10001 01100 11111	11000 01101 10110 00011	10010 00111 11100 01001	00000 10101 01110 11011
V	00010 10111 01100 11001	00001 10100 01111 11010	10000 00101 11110 01010	01000 11101 00110 10011
VI	00001 10100 01111 11010	10000 00101 11110 01010	01000 11101 00110 10011	00010 10111 01100 11001
VII	11000 01101 10110 00011	10010 00111 11100 01001	00000 10101 01110 11011	00100 10001 01100 11111
VIII	10010 00111 11100 01001	00000 10101 01110 11011	00100 10001 01100 11111	11000 01101 10110 00011

In the above design, the columns are complete, effect $F_1F_2F_4F_5$ is completely confounded row-wise and the effects confounded cell-wise are $F_1F_2F_4F_5$, $F_1F_2F_3$, $F_3F_4F_5$, $F_2F_3F_5$, $F_1F_3F_4$, F_1F_5, F_2F_4. In this design also, the rows form a resolvable structure with four groups I and II, III and IV, V and VI, VII and VIII having two rows each.

Example 4.3.2.3: The following is a GCRC $(3^4//3^2,3//3^2)$ design for 3^4 factorial arranged in 3^2 rows, 3 columns and intersection of each row and column consist of 3^2 combinations:

Rows	Columns		
	I	II	III
I	0000 1012 0122 1101 2021 0211 1220 2110 2202	0010 1022 0102 1111 2001 0221 1200 2120 2212	0020 1002 0112 1121 2011 0201 1210 2100 2222

II	1000 2012 1122 2101 0021 1211 2220 0110 0202	0100 1112 0222 1201 2121 0011 1020 2210 2002	0001 1010 0120 1102 2022 0212 1221 2111 2200
III	0100 1112 0222 1201 2121 0011 1020 2210 2002	0001 1010 0120 1102 2022 0212 1221 2111 2200	1000 2012 1122 2101 0021 1211 2220 0110 0202
IV	0010 1022 0102 1111 2001 0221 1200 2120 2212	0020 1002 0112 1121 2011 0201 1210 2100 2222	0000 1012 0122 1101 2021 0211 1220 2110 2202
V	0001 1010 0120 1102 2022 0212 1221 2111 2200	1000 2012 1122 2101 0021 1211 2220 0110 0202	0100 1112 0222 1201 2121 0011 1020 2210 2002
VI	1100 2112 1222 2201 0121 1011 2020 0210 0002	2000 0012 2122 0101 1021 2211 0220 1110 1202	0200 1212 0022 1001 2221 0111 1120 2010 2102
VII	2000 0012 2122 0101 1021 2211 0220 1110 1202	0200 1212 0022 1001 2221 0111 1120 2010 2102	1100 2112 1222 2201 0121 1011 2020 0210 0002
VIII	0200 1212 0022 1001 2221 0111 1120 2010 2102	1100 2112 1222 2201 0121 1011 2020 0210 0002	2000 0012 2122 0101 1021 2211 0220 1110 1202
IX	0020 1002 0112 1121 2011 0201 1210 2100 2222	0000 1012 0122 1101 2021 0211 1220 2110 2202	0010 1022 0102 1111 2001 0221 1200 2120 2212

It can be seen that cell-wise the effects confounded are $F_1F_2^2F_3^2, F_1F_3F_4^2, F_2F_3^2F_4^2, F_1^2F_2^2F_4^2$ and row-wise the highest order interaction among the cell-wise confounded effects i.e. $F_1^2F_2^2F_4^2$ is confounded.

4.3.3 Partial Confounding

In a single replicate design, the effects confounded in rows and cells cannot be estimated as they are completely confounded (Example 4.3.2.1 – 4.3.2.3). Also, the error variance σ^2 cannot be estimated from a single replicate design unless additional interactions may be assumed to be zero. A partially confounded design in two or more replications is therefore required to estimate σ^2 and some or all of the confounded interactions.

Generalized partially confounded row-column (GPCRC) design can be used in such situations.

Example 4.3.3.1: The design given below is a GCRC ($2^5//2^3,2^2//2^2$) design for 2^5 treatment combinations arranged in 2^3 rows, 2^2 columns and intersection of each row and column containing 2^2 combinations in 2 replication. In the first replication $F_2F_3F_4F_5$ is confounded row-wise and $F_1F_2F_3$, $F_1F_4F_5$, $F_1F_3F_5$, $F_1F_2F_4$, F_2F_5, F_3F_4, $F_2F_3F_4F_5$ are confounded cell-wise.

Rows	Columns			
	I	II	III	IV
I	00000 11001 10110 01111	10000 01001 00110 11111	01100 00101 11010 00011	01010 10011 11100 00101
II	10000 01001 00110 11111	01100 00101 11010 00011	01010 10011 11100 00101	00000 11001 10110 01111
III	01000 10001 11110 00111	00100 11101 10010 01011	00010 11011 10100 01101	00001 11000 10111 01110
IV	00100 11101 10010 01011	00010 11011 10100 01101	00001 11000 10111 01110	01000 10001 11110 00111
V	00010 11011 10100 01101	00001 11000 10111 01110	01000 10001 11110 00111	00100 11101 10010 01011
VI	00001 11000 10111 01110	01000 10001 11110 00111	00100 11101 10010 01011	00010 11011 10100 01101
VII	01100 00101 11010 00011	01010 10011 11100 00101	00000 11001 10110 01111	10000 01001 00110 11111
VIII	01010 10011 11100 00101	00000 11001 10110 01111	10000 01001 00110 11111	01100 00101 11010 00011

In the following second replication, $F_1F_3F_4F_5$ is confounded row-wise and $F_1F_2F_5$, $F_2F_3F_4$, $F_2F_3F_5$, $F_1F_2F_4$, F_1F_3, F_4F_5, $F_1F_3F_4F_5$ are confounded cell-wise:

Rows	Columns			
	I	II	III	IV
I	00000 11100 01011 10111	01000 10100 00011 11111	10010 01110 11001 00101	10001 01101 11010 00110
II	10000 01100 11011 00111	00100 11000 01111 10011	00010 11110 01001 10101	00001 11101 01010 10110
III	01000 10100 00011 11111	10010 01110 11001 00101	10001 01101 11010 00110	00000 11100 01011 10111
IV	00100 11000 01111 10011	00010 11110 01001 10101	00001 11101 01010 10110	10000 01100 11011 00111
V	00010 11110 01001 10101	00001 11101 01010 10110	10000 01100 11011 00111	00100 11000 01111 10011
VI	00001 11101 01010 10110	10000 01100 11011 00111	00100 11000 01111 10011	00010 11110 01001 10101
VII	10010 01110 11001 00101	10001 01101 11010 00110	00000 11100 01011 10111	01000 10100 00011 11111
VIII	10001 01101 11010 00110	00000 11100 01011 10111	01000 10100 00011 11111	10010 01110 11001 00101

In the above two replications two four factor interactions, eight three factor interactions and four two factor interactions are partially confounded. The effects which are confounded in replication 1 are estimable from replication 2 and vice versa.

4.3.4 Fractional GCRC Design

It may not be always feasible to take all the combinations in a single experiment. Hence, a series of fractional GCRC design is obtained here. First construct the half replicate of a 2^n factorial experiment by confounding highest order interaction. Then confound any disjoint set of interactions such that the generalized interaction is the highest order interaction provided none of the main effect is confounded to form the first column. Now, confound any another interaction row-wise (order should be the same with the lowest order interaction confounded in the columns) besides the interaction confounded in the columns such that its generalized interaction is highest order interaction and none of the main effect is confounded. This will result in GCRC design in half replicate arranging 2^n treatment combinations in 2 rows, 2 columns and intersection of each row and column

having 2^{n-3} units with some effects confounded row-wise, some in column-wise and some are in cells i.e. a GCRC ($2^n//2,2//2^{n-3}$) design ensuring the estimation of all main effects.

Example 4.3.4.1: The half replicate of a 2^4 factorial experiment confounding $I \equiv F_1F_2F_3F_4$ is as follows:

0000 1100 1010 1001 0110 0101 0011 1111

This half replicate of 2^4 factorial is arranged in 2 rows, 2 columns and intersection of each row and column consisting of 2 combinations i.e. GCRC ($2^4//2,2//2$) design is given as below.

Rows	Columns	
	I	II
I	0000 1111	1001 0110
II	1100 0011	1010 0101

It can be seen that the effects confounded column-wise are $I \equiv F_1F_2 \equiv F_3F_4 \equiv F_1F_2F_3F_4$, cell-wise $I \equiv F_1F_2 \equiv F_3F_4 \equiv F_1F_4 \equiv F_2F_4 \equiv F_1F_3 \equiv F_2F_3 \equiv F_1F_2F_3F_4$ and row-wise $I \equiv F_1F_4 \equiv F_2F_3 \equiv F_1F_2F_3F_4$.

Example 4.3.4.2: The following is the arrangement of half replicate of 2^5 factorial in 2 rows, 2 columns and intersection of each row and column containing 2^2 combinations, i.e. GCRC ($2^5//2,2//2^2$) design by confounding the effects $I \equiv F_1F_2F_3 \equiv F_4F_5 \equiv F_1F_2F_3F_4F_5$, $I \equiv F_1F_2F_3 \equiv F_4F_5 \equiv F_1F_4 \equiv F_2F_3F_4 \equiv F_1F_5 \equiv F_2F_3F_5 \equiv F_1F_2F_3F_4F_5$ and $I \equiv F_1F_4 \equiv F_2F_3F_5 \equiv F_1F_2F_3F_4F_5$ column-wise, cell-wise and row-wise respectively:

Rows	Columns							
	I				II			
I	00000	01100	11011	10111	10010	01001	00101	11110
II	11000	10100	00011	01111	10001	01010	00110	11101

Example 4.3.4.3: GCRC ($2^6//2,2//2^3$) design is obtained by arranging half replicate of 2^6 treatment combinations in 2 rows, 2 columns and intersection of each row and column containing 2^3 combinations:

Rows	Columns	
	I	II
I	000000 110000 001100 000011 111100 110011 001111 111111	100010 100001 010010 010001 101110 101101 011110 011101
II	101000 100100 011000 010100 101011 100111 011011 010111	001010 001001 000110 000101 111010 111001 110110 110101

It can be seen that the effects confounded column-wise are $I \equiv F_1F_2F_3F_4 \equiv F_5F_6 \equiv F_1F_2F_3F_4F_5F_6$, cell-wise $I \equiv F_1F_2F_3F_4 \equiv F_5F_6 \equiv F_3F_4 \equiv F_3F_4F_5F_6 \equiv F_1F_2 \equiv F_1F_2F_5F_6 \equiv F_1F_2F_3F_4F_5F_6$ and row-wise $I \equiv F_1F_2F_5F_6 \equiv F_3F_4 \equiv F_1F_2F_3F_4F_5F_6$.

The method discussed above for fractional GCRC design can be extended to the case of 3^n series and quarter replicate.

All the GCRC designs obtained here ensure that all lower order interactions including main effects are estimable.

Illustration

Consider the GCRC design with four factors given in Example 4.3.2.1. The layout along with the hypothetical data, within parenthesis, is as given below.

Rows	Columns							
	I				II			
I	0000 (36.58)	1100 (37.33)	0011 (37.13)	1111 (39.13)	1010 (37.31)	0110 (37.32)	1001 (37.43)	0101 (37.44)
II	1000 (37.03)	0100 (37.04)	1011 (38.11)	0111 (38.07)	0010 (37.16)	1110 (38.44)	0001 (37.29)	1101 (38.55)
III	0010 (37.25)	1110 (38.45)	0001 (37.23)	1101 (38.45)	1000 (37.58)	0100 (37.54)	1011 (38.57)	0111 (38.59)
IV	1010 (37.85)	0110 (37.81)	1001 (37.93)	0101 (37.91)	0000 (37.51)	1100 (38.32)	0011 (38.11)	1111 (38.13)

The data is analyzed using SAS 9.3. The Analysis of variance is as given below.

ANOVA

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Row	3	2.42	0.81	636.5	<.0001
Column	1	0.50	0.50	392.79	<.0001
F_1	1	3.53	3.53	2787.93	<.0001
F_2	1	3.41	3.41	2694.31	<.0001
$F_1 \times F_2$	1	0.39	0.39	307.44	<.0001
F_3	1	2.14	2.14	1687.43	<.0001
$F_1 \times F_3$	1	0.19	0.19	150.53	<.0001
$F_2 \times F_3$	1	0.19	0.19	148.1	<.0001
$F_1 \times F_2 \times F_3$	1	0.02	0.02	16.19	0.0014
F_4	1	2.85	2.85	2250.2	<.0001
$F_1 \times F_4$	1	0.18	0.18	145.69	<.0001
$F_2 \times F_4$	1	0.19	0.19	152.97	<.0001
$F_1 \times F_2 \times F_4$	1	0.01	0.01	4.15	0.0626
$F_3 \times F_4$	1	0.27	0.27	214.71	<.0001
$F_1 \times F_3 \times F_4$	1	0.01	0.01	9.79	0.008
$F_2 \times F_3 \times F_4$	1	0.01	0.01	9.18	0.0097
Error	13	0.02	0.0015		
Total	31	16.3			

Since the highest order interaction i.e. $F_1F_2F_3F_4$ is totally confounded in the rows so it is non estimable. The effects of row, column is significant. Among the treatment effects all main effects, two factor interactions and three factor interactions (except the effects $F_1F_2F_4$) are statistically significant.

In case of fractional factorial experiment as given in Example 4.3.4.2, a data set was analyzed and it was found that all the main effects and few lower order interactions were estimable.

4.4 Web Solution for Generation of Generalized Row-Column Designs

A number of GRC designs are available in the literature. For easy accessibility and quick reference of these designs by the experimenters, an online software ***Web Generation of***

Generalized Row-Column Designs (WebGRC) has been developed. Fig. 4.1 shows the hierarchical structure chart for the design of the software.

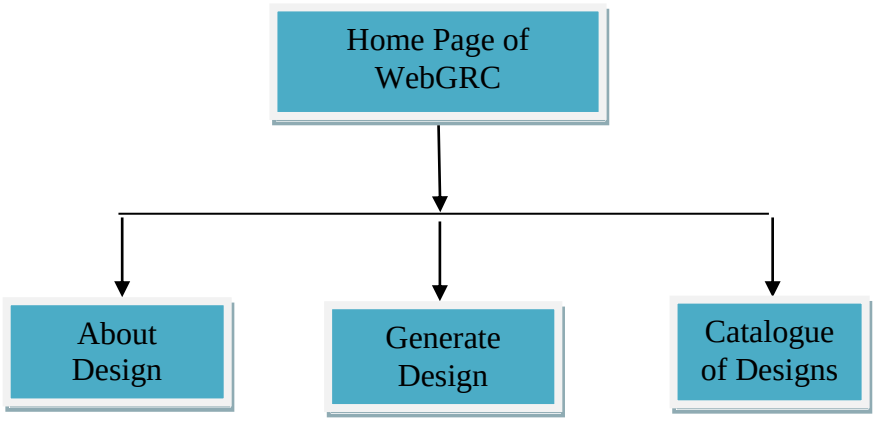

Fig. 4.1: Hierarchical structure of WebGRC

WebGRC generates design and randomized layout for various classes of GRC designs. It generates GRC Design for odd number of treatments (Datta, 2012), GRC designs for even number of treatments (Datta, 2012; Parsad, 2006). The webpage displays the layout plans along with the randomized layout for given number of treatments. The home page of the software is shown in Fig. 4.2.

In order to generate the design, user has to follow the following steps:

- Click on ‘Generate Design’ as shown in Fig. 4.2.
- Select ‘Even number of treatments: series 1’ under ‘Generate Design’.
- Enter the number of treatments (v) = 8 (say) as shown in Fig. 4.3.
- Click on ‘Generate Design’ and the generated design along with parameters v = 8, p = 7, q = 4, r = 7, k =2) will be displayed as shown in Fig. 4.4.
- Click on ‘Generate Randomized Layout’ to get a randomized layout of the design as shown in Fig. 4.5.
- Output can be exported to MS-Excel spread sheet for further use as shown in Fig. 4.6.

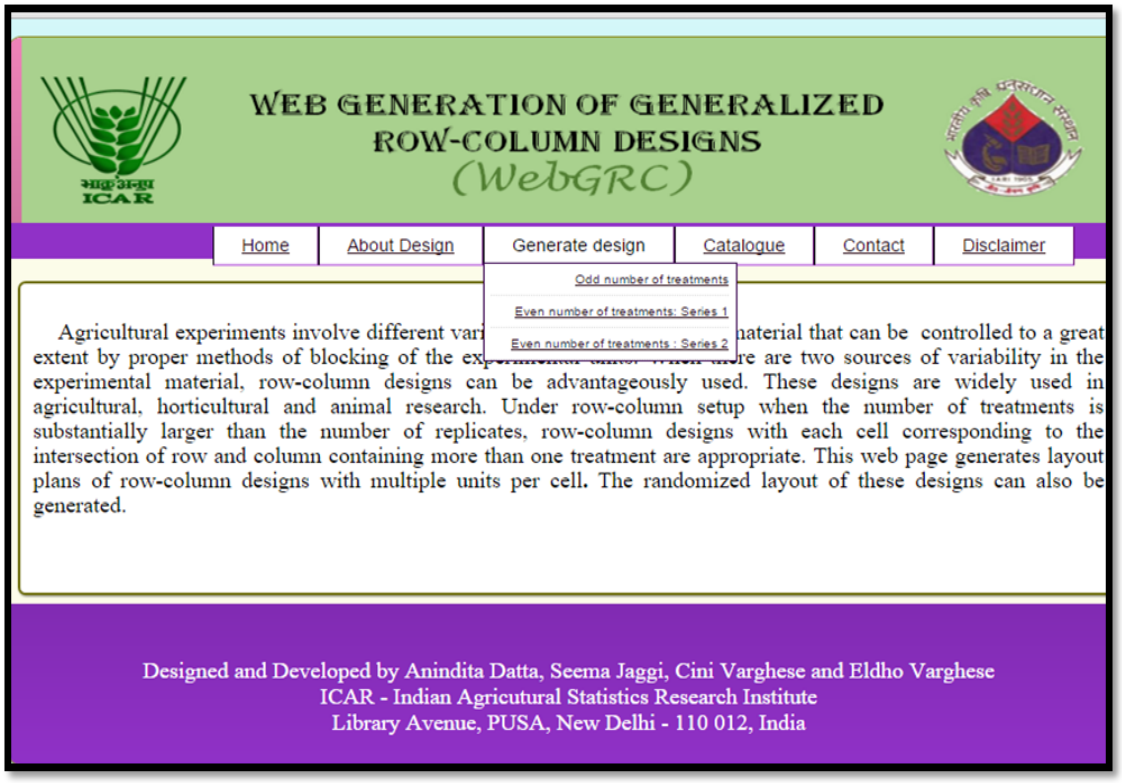

Fig. 4.2: GRC designs

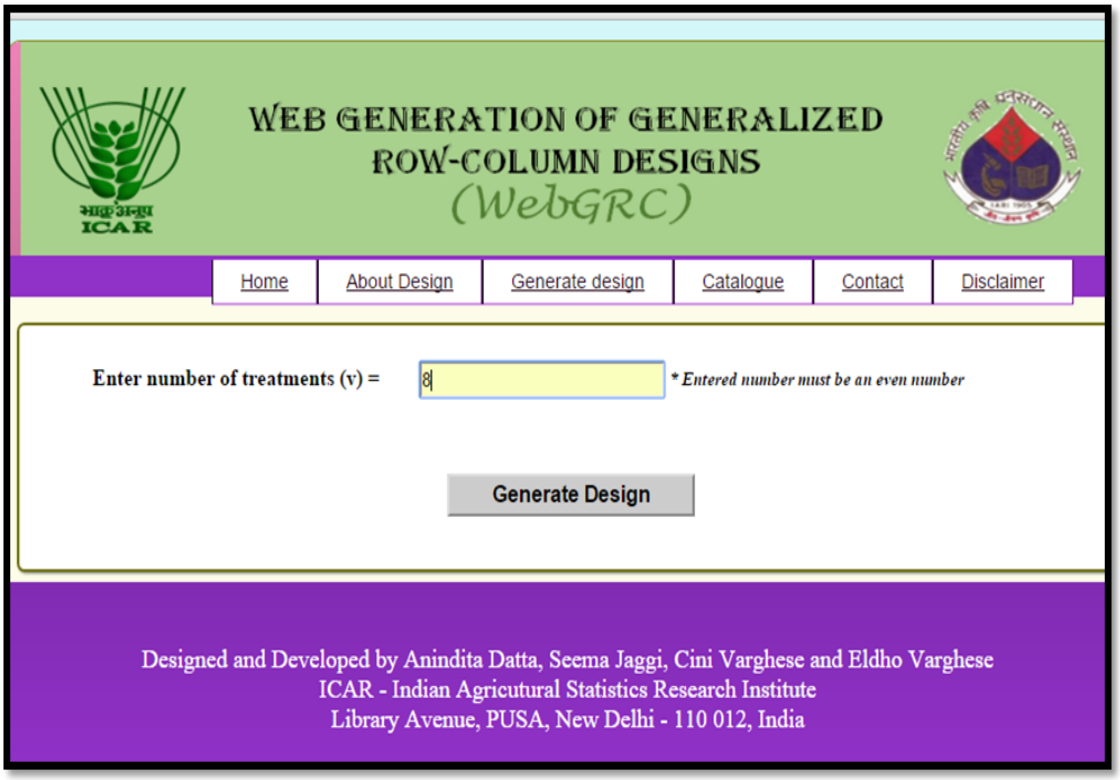

Fig. 4.3: Generation of GRC design

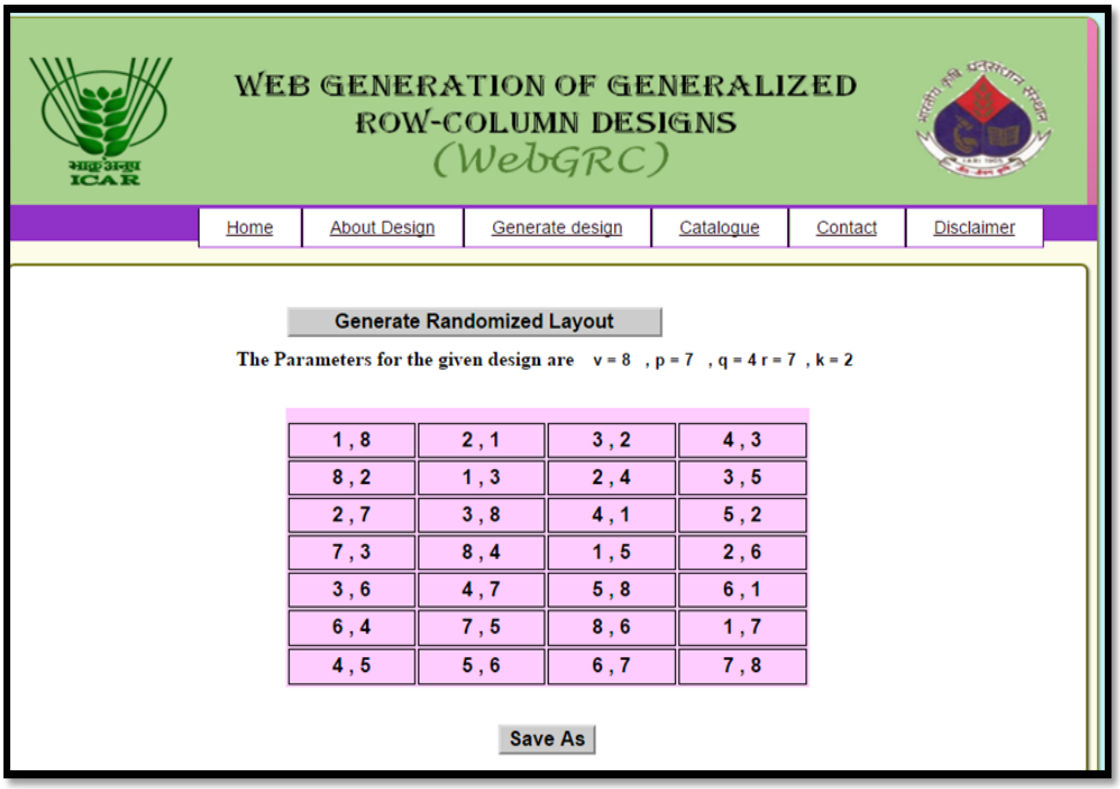

Fig. 4.4: GRC design for v = 8

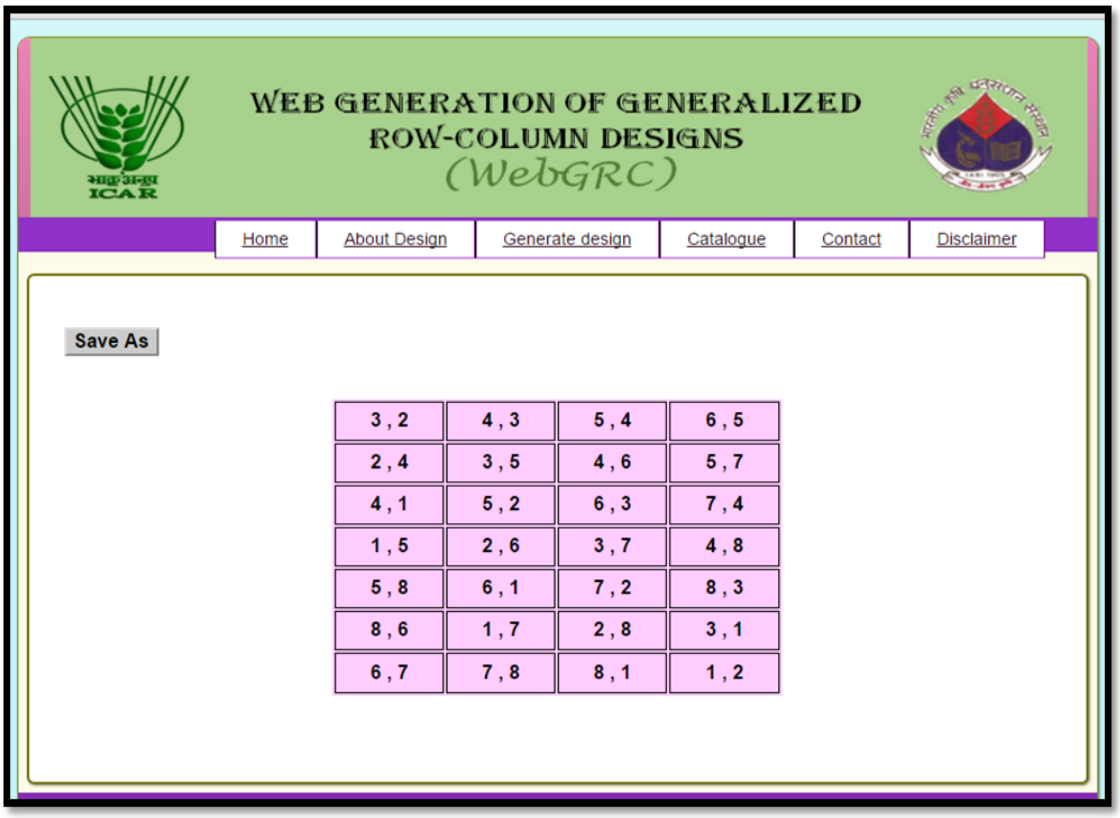

Fig. 4.5: Randomized layout of design for v = 8

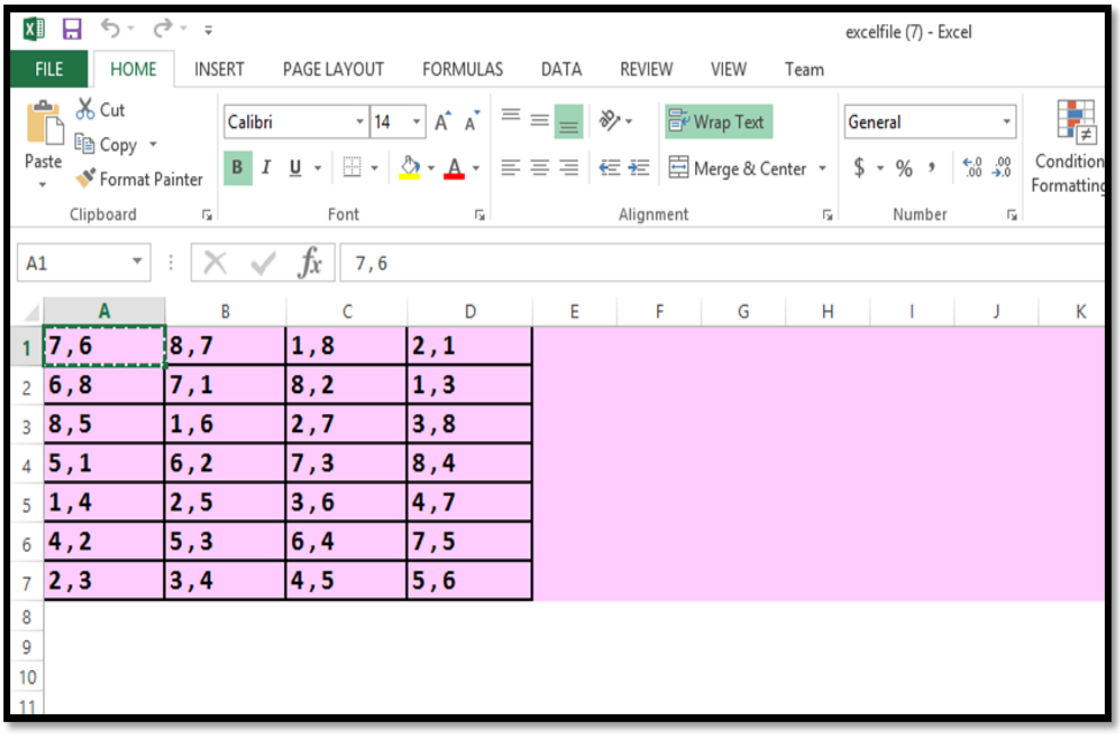

Fig. 4.6: Saving in excel

Similarly the design for v = 7 along with its randomized layout are shown in Fig. 4.7 and Fig. 4.8 respectively. Fig. 4.9 gives the layout of GRC design for v = 6.

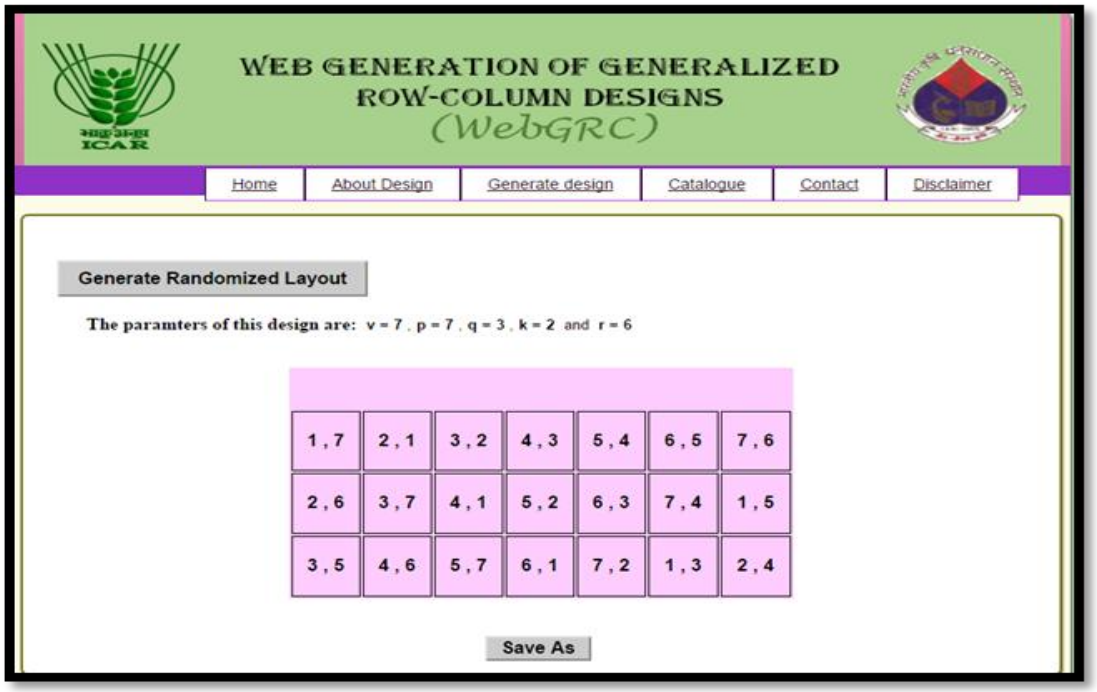

Fig. 4.7: GRC Design for v = 7

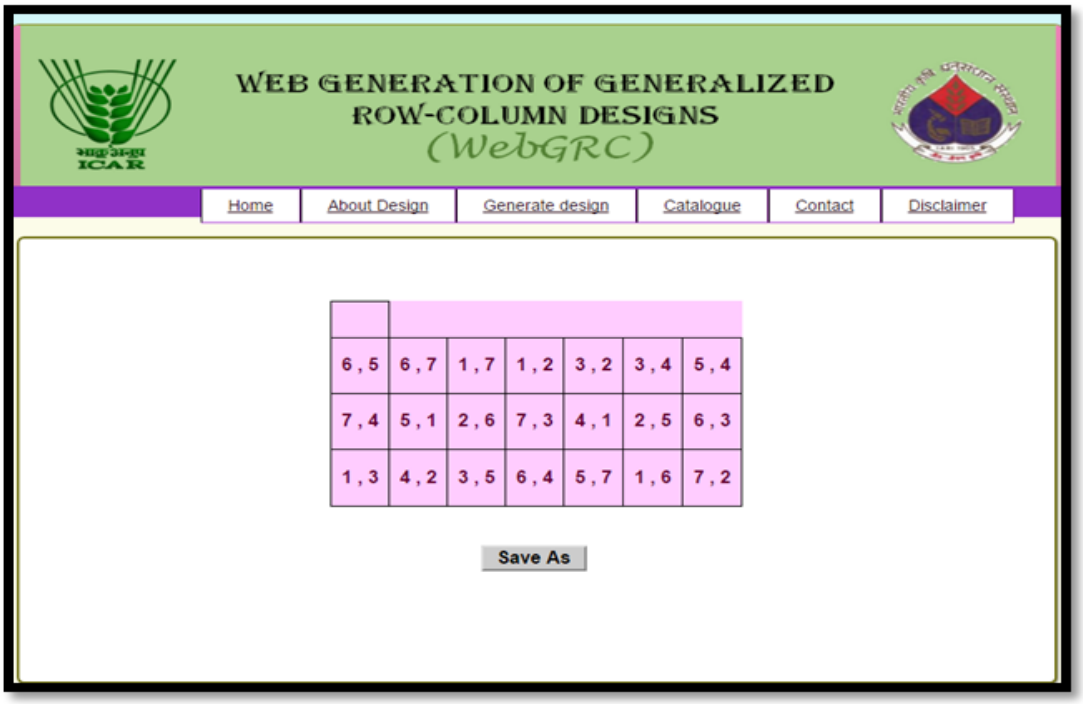

Fig. 4.8: Randomized layout of design for v = 7

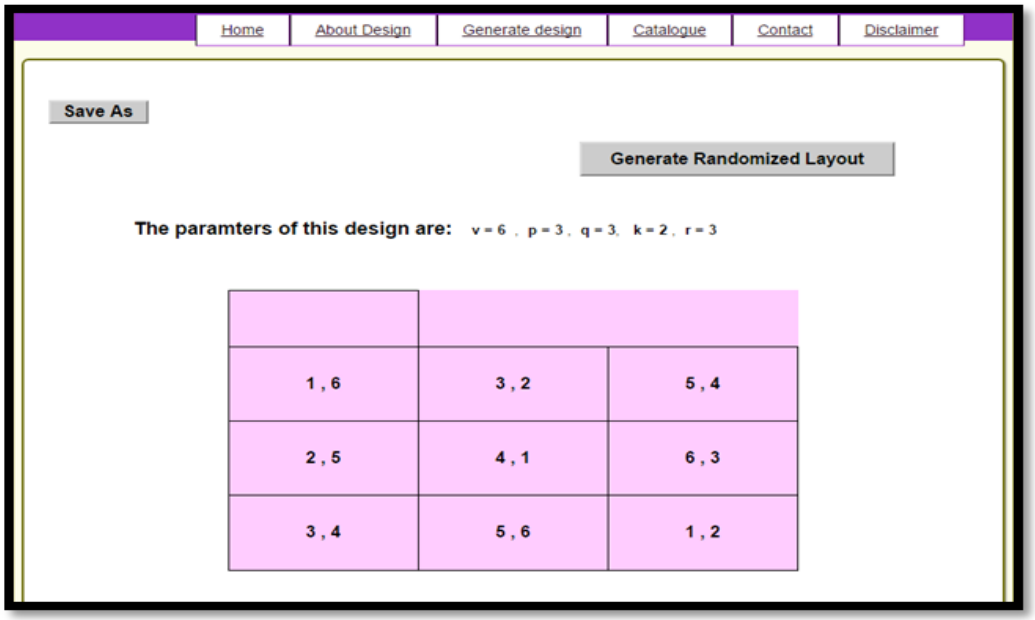

Fig. 4.9: GRC design for v = 6

- Catalogue of GRC designs has been developed and is included in the software as shown in Fig. 4.10. The user can select the design seeing the parameters of the design. By clicking on the design, the layout of the design is generated.

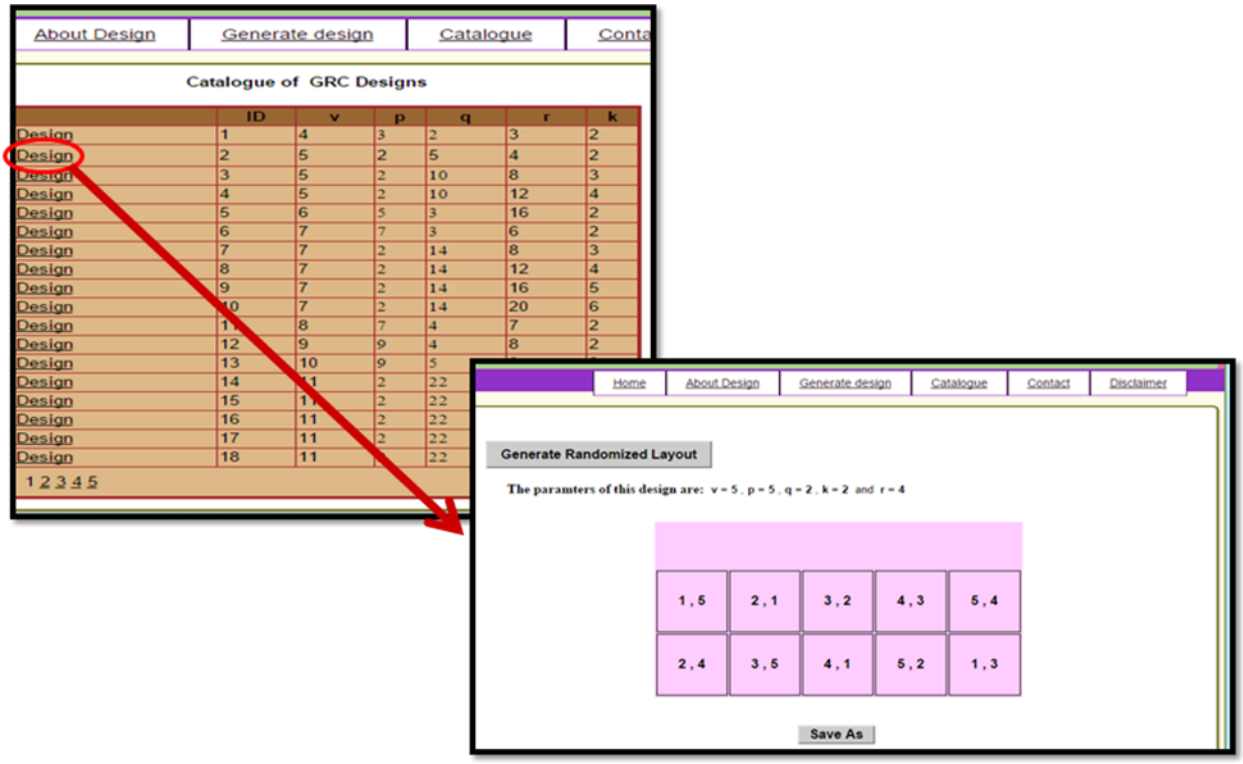

Fig. 4.10: Catalogue and generation of GRC Design

- The methods of construction of GRC designs are given in the "about design" section as shown in Fig. 4.11.

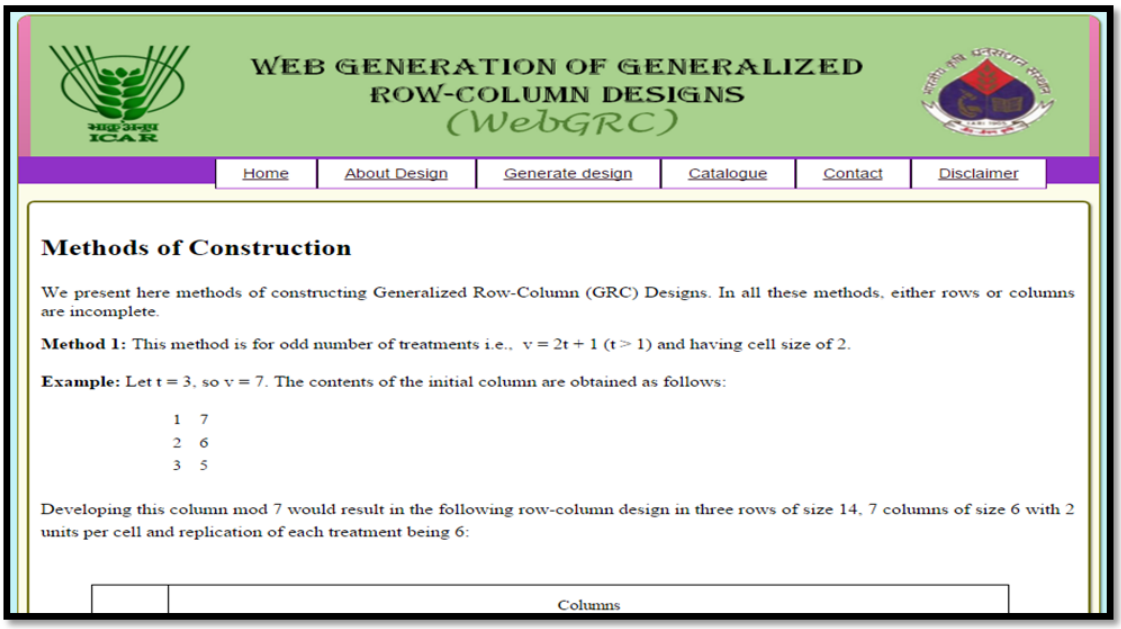

Fig. 4.11: About GRC design

CHAPTER V

SUMMARY AND CONCLUSIONS

Row-column design is used when there are two cross classified sources of variation in experimental units that influence the response variable. These designs are used to control variability in field and animal experiments. Most of the row-column designs developed in the literature have one unit corresponding to the intersection of row and column. However, there may be instances when the number of treatments is substantially large with limited number of replicates. A more general class of row-column designs is required where there is more than one unit in each row-column intersection. These designs may be called as Generalized Row-Column (GRC) designs. GRC design is an arrangement of v treatments in p rows and q columns such that the intersection of each row and column consist of more than one unit.

A good amount of literature is available on GRC designs for making all possible pair-wise comparison among treatments. In this present study, the GRC design for two sets of treatments that are disjoint with one set consisting of test treatments and the other of control treatments has been studied. The two sets are disjoint in the sense that there are no common treatments between the two. The interest here is to estimate the contrasts pertaining to treatments of first set having test treatments vs. treatments of second set having control treatments with as high precision as possible. Balanced Bipartite Generalized Row-Column (BBP-GRC) designs have been defined and series of BBP-GRC designs for comparing a set of test treatments to a set of control treatments have been developed. The first series of BBP-GRC design is developed from any GRC design by replacing a set of treatments. The designs obtained are variance balanced in the sense that all the contrasts among test treatments are estimated with same variance and all the contrasts pertaining to test vs. control are estimated with the same and less variance. The second series of BBP-GRC design is developed by superimposing mutually orthogonal latin squares (MOLS) and augmenting one new treatment to the final design. This series of design is for one single control treatment. The design obtained is partially balanced with respect to the first set of treatments following a group divisible association scheme. For this series there are four types of variances i.e variance of estimated elementary

treatment contrast of the treatments of first set that are 1^{st} associates, treatments of first sets that are second associates, treatments that are 1^{st} associates with the new treatment from second set and treatments that are 2^{nd} associates with the new treatment from second set. The contrasts pertaining to test vs. control are estimated with less variance. The third series of BBP-GRC design is developed from balanced incomplete block (BIB) design. The designs obtained are variance balanced in the sense that all the contrasts among test treatments are estimated with same variance and all the contrasts pertaining to test vs. control are estimated with the same and less variance. The design obtained through this method is also variance balanced if it is developed by using any partially balanced incomplete block (PBIB) design. The forth series of BBP-GRC design is developed from any two associate class association scheme. The design obtained by this method may have unequal cell sizes. These designs are also variance balanced.

Under the second objective, Robustness of different classes of GRC designs against missing of one or more observations within a cell as per the efficiency criteria has been investigated. A list of robust GRC designs has prepared giving the parameters and the efficiency of the designs. A design is considered to be highly robust against missing observation(s) if the loss in efficiency of the residual design is not more than 5% and robust if the loss in efficiency of the residual design is between 5% to 10%. The efficiency of the GRC designs in the absence of one or more observations has been studied and the efficiency is found to be quite high for most of the designs and thus the designs are robust. There is a decreasing trend in efficiency with increase in number of missing observations. It is further seen that smaller designs are more affected by the missing observations.

Under the third objective, GRC designs with factorial treatment structure are considered and confounded GRC designs for symmetrical factorial have been obtained. Methods of constructing generalized confounded row-column (GCRC) designs, generalized partially confounded row-column (GPCRC) designs and fractional GCRC designs have been developed. GCRC and GPCRC designs are incomplete row-wise and complete column-wise. In the GCRC design so constructed there is flexibility in row sizes i.e. design with lesser row sizes can be constructed by confounding more effects row wise. The series of GCRC design is also resolvable row-wise. The fractional GCRC design is incomplete

both row and column-wise. The GCRC designs obtained here ensure that all lower order interactions including main effects are estimable.

A web solution named WebGRC has been developed for the generation of GRC designs that would be highly useful to the experimenters. The webpage displays the layout plans along with the randomized layout for given number of treatments. The parameters of the design so generated are also displayed. An online catalogue of the GRC designs is also prepared and included in the software wherein the user can select the design by seeing all the parameters and then can get the randomized layout. The details regarding the method of obtaining these designs are also included. This software will provide freely available solution for the researchers and students working in this area.

ABSTRACT

In field and animal experiments, where there are two sources of variation in experimental units that may influence the response variable, row-column designs are used. Most of the row-column designs developed in the literature have only one unit corresponding to the intersection of row and column. However, for the instances when the number of treatments is large with limited experimental resources, Generalized Row-Column (GRC) designs are used where there is more than one unit in each row-column intersection. The GRC designs developed in the literature are to study all possible pair-wise treatment comparisons. There may arise experimental situations where it is desired to compare treatments belonging to two disjoint sets and the interest is to estimate the contrasts pertaining to treatments from different sets with as high precision as possible. Balanced Bipartite Generalized Row-Column (BBP-GRC) designs have been defined and series of BBP-GRC have been developed in which the contrast of first set versus second set of treatments is estimated more precisely. The presence of missing observations, outliers in the data, etc. are some of the disturbances that may occur during experimentation. These disturbances may lead to less precise comparisons among treatments. Robustness of different classes of GRC designs against missing of one or more observations has been investigated. It is found that the efficiency is quite high (more than 90%) for most of the designs and the designs are robust and there is a decreasing trend in efficiency with increase in number of missing observations. The GRC designs developed in the literature are mostly for single factor experiments. Situations may arise wherein the experiment consist of more than one factor with each factor having more than one levels. Generalized confounded row-column (GCRC) designs, generalized partially confounded row-column (GPCRC) designs and fractional GCRC designs have been developed which ensure that all lower order interactions including main effects are estimable. For easy accessibility of GRC designs, a web solution named WebGRC has been developed that provides the online generation of randomized layout of these designs along with an online catalogue within a permissible range.

सार

खेत एवं पशुओं से सम्बंधित परीक्षणों में जहाँ परीक्षण इकाइयों में परिवर्तन के दो ऐसे स्रोत हों जो परिणामी चर को प्रभावित करने की क्षमता रखते हों तो इस स्थिति में रो-कॉलम अभिकल्पनाओं का प्रयोग किया जाता है । पठन सामग्री में अभी तक उपलब्ध लगभग सभी रो-कॉलम अभिकल्पनाओं में रो-कॉलम प्रतिच्छेदन पर केवल एक ही इकाई होती है । ऐसी स्थिति में जहाँ ट्रीटमेंट की संख्या अधिक हो और परीक्षण संसाधनों की कमी हो तो रो-कॉलम प्रतिच्छेदन में एक से अधिक इकाइयां होने पर जनरलाईज्ड रो-कॉलम (GRC) अभिकल्पनाओं का प्रयोग किया जाता है । अभी तक उपलब्ध अभिकल्पनाओं से ट्रीटमेंटों के सभी संभव युग्मों की तुलनाओं का अध्ययन किया जाता है । परीक्षणों में कभी कभी ऐसी स्थिति उत्पन्न हो जाती है जब दो विच्छेदित समूहों से संबंधित ट्रीटमेंट की तुलना की आवश्यकता होती है और विभिन्न ट्रीटमेंटों से संबंधित कंट्रास्ट के अधिक शुद्धता के साथ आकलन करनी होती है । बैलेंस्ड बाइपारटाइट जनरलाईज्ड रो-कॉलम अभिकल्पनाओं को परिभाषित किया गया है तथा ट्रीटमेंटों के पहले व दूसरे समूहों के परस्पर कंट्रास्टों के अधिक शुद्ध आकलन के लिए बैलेंस्ड बाइपारटाइट जनरलाईज्ड रो-कॉलम (BBP–GRC) अभिकल्पनाओं की एक श्रेणी विकसित की गयी है । प्रेक्षणों की अनुपलब्धताए, ऑउटलायरों का पाया जाना आदि कुछ ऐसी बातें है जो परीक्षण के दौरान सामने आ सकती हैं । इनके कारण ट्रीटमेंटों की परस्पर तुलनाओं के आकलन की शुद्धता में कमी आ सकती है । एक या अधिक अनुपलब्ध प्रेक्षणों वाली जनरलाईज्ड रो-कॉलम अभिकल्पनाओं के विभिन्न वर्गों की प्रबलता की भी जाँच की गयी है । यह देखने में आया है कि अधिकांश अभिकल्पनाओं से अधिकतम उच्च स्तर (> 90%) की दक्षता पायी गयी है तथा अभिकल्पनयें प्रबल हैं । साथ ही यह भी देखा गया है कि अनुपलब्ध प्रेक्षणों की संख्या के बढ़ने के साथ साथ परीक्षण की दक्षता में गिरावट का ट्रेंड आ जाता है । अभी तक विकसित की गयीं अधिकांश जनरलाईज्ड रो-कॉलम अभिकल्पनायें एकल कारक परीक्षणों के लिए हैं । ऐसी स्थिति भी आ सकती है जहाँ परीक्षण में एक से अधिक कारक हों और एक कारक के एक से अधिक स्तर हों । जनरलाईज्ड कनफाउंडेड रो-कॉलम (GCRC) अभिकल्पनायें, जनरलाईज्ड पार्शियलि कनफाउंडेड रो-कॉलम (GPCRC) अभिकल्पनायें तथा फ्रैक्शनल GCRC अभिकल्पनायें विकसित की गयी हैं जिनसे मुख्य प्रभाओं सहित निम्न स्तर के सभी प्रतिक्रियाओं (interactions) का अनुमान सुनिश्चित किया जा सकता है । जनरलाईज्ड रो-कॉलम अभिकल्पनाओं की उपलब्धता को आसान बनाने के लिए WebGRC के नाम से एक वैब सोल्युशन (Web solution) विकसित किया गया है जिससे इन अभिकल्पनाओं के यादृच्छिक लेआउट (lay out) ऑनलाइन प्राप्त किये जा सकते हैं ।

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APPENDIX

SAS CODE FOR OBTAINING THE C-MATRIX AND THE HARMONIC MEAN OF NON-ZERO EIGEN-VALUES OF C-MATRIX OF ORIGINAL DESIGN AND THE RESIDUAL DESIGN FOR GRC DESIGN

```
proc iml;

/*design [put non-zero values]*/

a={

1    6    2    7    3    8    4    9    5    0    ,
2    8    3    9    4    10   5    6    1    0    ,
3    10   4    6    5    7    1    8    2    0    ,
4    7    5    8    1    9    2    10   3    0    ,
5    9    1    10   2    6    3    7    0    0

};

/*define cell sizes*/

b={

2    2    2    2    1    ,
2    2    2    2    1    ,
2    2    2    2    1    ,
2    2    2    2    1    ,
2    2    2    2    0

};

cc=b[+, ];
dd=b[ ,+];
bb=j(nrow(b)*ncol(b),1,0);
k=1;
do i=1 to nrow(b);
do j=1 to ncol(b);
bb[k, ]= b[i,j];
k=k+1;
```

```
end;
end;
b1=bb[loc(bb>0),];
*print b1;
aa=j(nrow(a)*ncol(a),1,0);
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
aa[k, ]= a[i,j];
k=k+1;
end;
end;
m1=j(nrow(a)*ncol(a),1,1);/*mean vector*/
dir=j(nrow(a)*ncol(a),max(a),0);/*design matrix
                    obseravation VS treatment*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
if a[i,j]>0 then
          do;
          dir[k,a[i,j]]=1;
          k=k+1;
          end;
end;
end;
r=j(nrow(a)*ncol(a),nrow(dd),0);/*design  matrix  observation
VS row*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
r[k,i]=1;
k=k+1;
```

```
end;
end;
c=j(nrow(a)*ncol(a),ncol(b),0);/*design  matrix  observation
VS column*/
k=1;
do i=1 to nrow(b);
do j=1 to ncol(b);
do l=1 to b[i,j];
c[k,j]=1;
k=k+1;
end;
end;
end;
cell=j((nrow(a)*ncol(a)),nrow(b1),0);/*design matrix
                                observation VS cell*/
kk=1;
z=0;
do k=1 to nrow(b1);
do j=1 to b1[k];
if aa[z+j, ]>0 then
     do;
     cell[kk,k]=1;
     kk=kk+1;
     end;
end;
z=z+b1[k];
end;
x=m1||dir||r||c;/*design matrix*/
*print x[format=3.0];
x1=dir;
x2=m1||r||c;
c_mat=(x1`*x1)-(x1`*x2*(ginv(x2`*x2))*x2`*x1)/*C matrix*/;
```

```
print c_mat;
eig=eigval(c_mat);
eig1=eig[loc(eig>0.005),];/*positive eigen values*/
eig2=1/eig1;
HM1=nrow(eig2)/sum(eig2);
print HM1;
quit;
```

SAS CODE FOR OBTAINING THE C-MATRIX OF BBP-GRC DESIGN AND VARIANCE OF ESTIMATE OF ELEMENTARY TREATMENT CONTRAST FOR COMPARING THE TREATMENTS FROM FIRST SET WITH THE TREATMENTS FROM SAME SET, THE TREATMENTS FROM FIRST SET WITH THE TREATMENTS FROM SECOND SET AND AVERAGE VARIANCE

```
%let t=4;/*number of treatments in first set*/
%let cc=2;/*number of treatments in second set*/
proc iml;
/*design [put zeroes for unequal row(column) sizes]*/
a={1 2    5    6    1    3    5    6    1    4    5    6
     1    5    5    6    1    6    5    6    2    3    5
     6    2    4    5    6    2    5    5    6    2    6
     5    6    3    4    5    6    3    5    5    6    3
     6    5    6    4    5    5    6    4    6    5    6
     5    6    5    6    ,
3    4    5    6    2    4    5    6    2    3    5    6
     2    3    4    6    2    3    4    5    1    4    5
     6    1    3    5    6    1    3    4    6    1    3
     4    5    1    2    5    6    1    2    4    6    1
     2    4    5    1    2    3    6    1    2    3    5
     1    2    3    4
```

```
};
/*define cell sizes*/
b={4 4    4    4    4    4    4    4    4    4    4    4
     4    4    4    ,
4    4    4    4    4    4    4    4    4    4    4    4
     4    4    4

};
cc=b[+, ];
dd=b[ ,+];
bb=j(nrow(b)*ncol(b),1,0);
k=1;
do i=1 to nrow(b);
do j=1 to ncol(b);
bb[k, ]= b[i,j];
k=k+1;
end;
end;
b1=bb[loc(bb>0),];
aa=j(nrow(a)*ncol(a),1,0);
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
aa[k, ]= a[i,j];
k=k+1;
end;
end;
m1=j(sum(b),1,1);/*mean vector*/
dir=j(sum(b),max(a),0);/*design matrix observation VS
treatment*/
k=1;
do i=1 to nrow(a);
```

```
do j=1 to ncol(a);
if a[i,j]>0 then
          do;
          dir[k,a[i,j]]=1;
          k=k+1;
          end;
end;
end;
r=j(sum(b),nrow(dd),0);/*design matrix observation VS row*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
if a[i,j]>0 then
          do;
          r[k,i]=1;
          k=k+1;
          end;
end;
end;
c=j(sum(b),ncol(b),0);/*design matrix observation VS
column*/
k=1;
do i=1 to nrow(b);
do j=1 to ncol(b);
if b[i,j]>0 then do;
do l=1 to b[i,j];
          c[k,j]=1;
          k=k+1;
          end;
end;
end;
end;
```

```
cell=j(sum(b),nrow(b1),0);/*design matrix observation VS
cell*/
kk=1;
z=0;
do k=1 to nrow(b1);
do j=1 to b1[k];
     cell[kk,k]=1;
     kk=kk+1;
     end;
end;
x=m1||dir||r||c; /*design matrix*/
*print x[format=3.0];
x1=dir;
x2=m1||r||c;
c_mat=(x1`*x1)-(x1`*x2*(ginv(x2`*x2))*x2`*x1) /*C matrix*/;
print c_mat;
g_invc=ginv(c_mat);
k=1;
tcont=j(comb(&t,2),(&t+&cc),0);
do i=1 to &t;
do j=i+1 to &t;
tcont[k,i]=1;
tcont[k,j]=-1;
k=k+1;
end;
end;
k=1;
if &cc>1 then do;
cccont=j(comb(&cc,2),(&t+&cc),0);
do i=&t+1 to (&t+&cc);
do j=i+1 to (&t+&cc);
cccont[k,i]=1;
```

```
cccont[k,j]=-1;
k=k+1;
end;
end;
end;
else do;
cccont=j(1,(&t+&cc),0);
end;
k=1;
totcont=j(comb((&t+&cc),2),(&t+&cc),0);
do i=1 to (&t+&cc);
do j=i+1 to (&t+&cc);
totcont[k,i]=1;
totcont[k,j]=-1;
k=k+1;
end;
end;
*print tcont cccont totcont;
var_t=vecdiag(tcont*g_invc*tcont`);
if &cc>1 then do;
var_c=vecdiag(cccont*g_invc*cccont`);
end;
else do;
var_c=0;
end;
var_tot=vecdiag(totcont*g_invc*totcont`);
print var_t var_c var_tot;
avar_t=sum(var_t)/nrow(var_t);
avar_c=sum(var_c)/nrow(var_c);
avar_tot=sum(var_tot)/nrow(var_tot);
if &cc>1 then do;
```

```
avar_tvsc=(sum(var_tot)-
(sum(var_t)+sum(var_c)))/(nrow(var_tot)-
(nrow(var_t)+nrow(var_c)));
end;
else do;
avar_tvsc=(sum(var_tot)-sum(var_t))/(nrow(var_tot)-
nrow(var_t));
end;
print avar_t avar_tvsc avar_c avar_tot;
quit;
```