ESTIMATION OF POPULATION MEAN THROUGH IMPROVED RATIO AND PRODUCT TYPE ESTIMATORS USING AUXILIARY INFORMATION

By

Banti Kumar
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Thesis submitted to Faculty of Basic Sciences
in partial fulfillment of requirements
for the degree of

DOCTOR OF PHILOSOPHY

IN

STATISTICS

Division of Statistics and Computer Science
Faculty of Basic Sciences
Sher-e-Kashmir University of Agricultural Sciences & Technology of Jammu
Main Campus, Chatha, Jammu-180009

2017

CERTIFICATE-I

This is to certify that the thesis entitled **"Estimation of Population Mean through Improved Ratio and Product Type Estimators using Auxiliary Information"** submitted in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Statistics** to the Faculty of Basic Sciences, Sher-e-Kashmir University of Agricultural Sciences and Technology of Jammu is a record of bonafide research carried out by **Mr. Banti Kumar,** Registration No. **J-13-D-203-A** under my supervision and guidance. No part of the thesis has been submitted for any other degree or diploma. It is further certified that such help and assistance received during the course of investigation have been duly acknowledged.

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Acknowledgement

ACKNOWLEDGEMENT

Completion of this doctoral dissertation was possible with the support of several people. I would like to express my sincere gratitude to all of them.

First of all, I am extremely grateful to my research guide, Dr. Manish Kumar Sharma, Associate Professor (Statistics), Division of Statistics and Computer Science, for his valuable guidance, scholarly inputs and consistent encouragement I received throughout the research work. This feat was possible only because of the unconditional support provided by Sir. A person with an amicable and positive disposition, has always made himself available to clarify my doubts despite his busy schedules and I consider it as a great opportunity to do my doctoral programme under his guidance and to learn from his research expertise. Thank you Sir, for all your help and support.

I thank Prof. S. A. Mallick, Dean, Faculty of Basic Sciences and Prof. D.P. Abrol, Dean, Faculty of Agriculture, for the academic support, time to time moral support and the facilities provided to carry out the research work at the Institute.

My sincere thanks to Prof. S.E.H. Rizvi, Head, Division of Statistics and Computer Science, for his constant encouragement and concern for the completion of the thesis.

Members of my advisory committee have been very kind enough to extend their help at various phases of this research, whenever I approached them, and I do hereby acknowledge all of them. I thank Prof. Jyoti Kachroo, Head, Division of Agricultural Economics & ABM and Dr. Bupesh Kumar and Assistant Professor (Plant Breeding and Genetics) for their valuable suggestions.

I would like to thank Dr. M.Iqbal Jeelani Bhat, Assistant Professor, Division of Statistics and Computer Science for his kind assistance and giving wise advice, helping with various applications, and so on.

The thesis would not have come to a successful completion, without the help I received from the non-teaching staff of the Computer Centre. I would like to thank Ms. Bandana, Mr. Sajad and supporting staff of Faculty of Basic Sciences who were always ready to help me as and when required.

No research is possible without the Library, the centre of learning resources. I take this time to express my gratitude to all the library staff for their services.

My colleagues and friends Archana, Sunali Mahajan, Faizan Danish, Yash Pal, Jebi Sudan, Meenakshi Raina, Nancy Gupta, Ravi, Rakesh, Sohail, Sonika, Raman and Nishant have all extended their support in a very special way, and I gained a lot from them, through their personal and scholarly interactions, their suggestions at various points of my research programme. Your friendship makes my life a wonderful experience.

I especially thank my mom (Smt Raj Rani), dad (Shri Darshan Lal) and brother (Vikramjeet). They were always supporting me and encouraging me with their best wishes. I love them so much, and I would not have made it this far without them. My brother has been my best friend all my life and I love him dearly and thank him for all his advice and support.

I would like to thank my Uncle Shri Mohan lal, his family and my cousins Harsha, Sakshi and Varsha for your understanding and encouragement many moments of crisis in my life.

I seize this opportunity to express my utmost regards to Hon'ble Vice Chancellor and Director Education of SKUAST-J, who have been kind and helpful.

The DST INSPIRE Fellowship awarded to me to carry out research is gratefully acknowledged here. My sincere thanks are extended to DST for the timely disbursal of the grant.

Above all, I thanks to the God, the Almighty, for his showers of blessings and having made everything possible by giving me strength and courage to complete the research successfully.

All can't be mentioned but none has been forgotten.

Words once printed have life of their own

Place: Jammu

Date: July, 2017.

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ABSTRACT

Title of the Thesis	**:**	Estimation of Population Mean through Improved Ratio and Product Type Estimators using Auxiliary Information
Name of the student	**:**	Banti Kumar
Major Subject	**:**	Statistics
Name and Designation of the Major Advisor	**:**	Dr. Manish Kumar Associate Professor (Statistics) Division of Statistics and Computer Science
Degree to be awarded	**:**	Ph.D. (Statistics)
Name of the University	**:**	Sher-e-Kashmir University of Agricultural Sciences and Technology, Jammu

The auxiliary information can be efficiently used in survey sampling at pre-selection stage, selection stage and estimation stage. The study entitled **"Estimation of Population Mean through Improved Ratio and Product Type Estimators using Auxiliary Information**" has been conducted in order to develop improved ratio and product type estimators when auxiliary information is available for the estimation of population mean. It is a well established fact that if the auxiliary information is used at estimation stage, the estimates of the population mean of characteristic under study can be obtained with greater precision. In sampling theory and statistical inference, there is variety of procedures which can led to development of improved estimators for estimation the population mean under study. In this study, an attempt has been made to develop a general class of improved ratio and product type estimators (may be biased or unbiased) for estimation of population mean by modifying conventional estimators whose large sample properties are compared with the conventional estimator and some existing estimators. The expressions for the relative bias and relative mean squared error of the proposed estimators have been derived up to first order and second order approximation respectively. The theoretical and empirical comparisons of efficiency of the proposed estimators have been done with existing estimators. It has been observed that the proposed class of ratio type estimators performed better than conventional ratio estimator and estimators proposed by Sharma *et al.* (2010) on the basis of mean squared error criterion. Likewise, general improved class of product type estimators have performed better than conventional product type estimators and estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007) on the basis of relative mean squared error criterion.

The empirical study has been done through simulation data by using R and SAS softwares. Two populations /datasets have been generated for ratio and product type estimators each *i.e.,* populations P_1 and P_2 for ratio estimators and population P_3 and P_4

for product type estimators. The empirical comparisons showed that the proposed estimators were more efficient than the conventional and existing estimators.

In this regard, five types of general class of ratio type estimators have been developed. Further, by setting different values of scalars p and q, various estimators have been proposed. The proposed biased ratio estimators *i.e.*, $t_{1(-3,1)}$, $t_{2(-3,1)}$, $t_{3(-3,1)}$, $t_{4(-3,1)}$ and $t_{5(-3,1)}$ and proposed unbiased ratio estimators $t_{1(-1,-1)}$, $t_{2(-1,-1)}$, $t_{3(-1,-1)}$ and $t_{4(-1,-1)}$ were more efficient than the conventional ratio type estimators and the estimators proposed by Sharma *et al.* (2010) which is empirically analyzed as per population datasets P_1 and P_2. The proposed ratio estimator $t_{1(-3,1)}$ was the best among all other proposed ratio type estimators for small sample sizes.

In case of product type estimator, five types of general class of product type estimators have been developed. The proposed product type estimators $t^{p}_{1(1)}$, $t^{p}_{2(3)}$, $t^{p}_{3(2)}$, $t^{p}_{4(-2,1)}$ and $t^{p}_{5(3,1)}$ were found to be more efficient than the conventional product type estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007) in both the populations i.e., P_3 and P_4. The proposed product estimator $t^{p}_{2(3)}$ was the best among all other proposed product type estimators for small sample sizes.

Keywords: *Auxiliary information, ratio type estimator, product type estimator, relative mean squared error, relative bias and simulation.*

Signature of Major Advisor

Banti Kumar

Signature of the Student

CONTENTS

LIST OF TABLES

SYMBOLS USED

$\bar{y}_r$	Conventional Ratio estimator
$\bar{y}_p$	Conventional Product estimator
t_R^p	Estimator Proposed by Robson(1957)
t^*	Estimator Proposed by Singh(1989)
t_D^p	Estimator Proposed by Dubey (1993)
t_s^p	Estimator Proposed by Sharma *et al.*, (2007)
t_s	Estimator proposed by Sharma *et al.*, (2010)

Introduction

CHAPTER 1

INTRODUCTION

Getting the information about population mean through sampling methods has been used at early stages of the mankind. Statistical methodology have been widely used in sampling in order to estimate the population mean or population total of the characteristic under study with greater precision or least cost or both. Sample survey or survey sampling is a method of drawing an inference about the characteristic of a population or universe by observing only a part of the population. Use of sampling as an objective method to obtain statistical data was discernible towards the end of last century itself. Mahalanobis, Neyman and Sukhamate during the 1930's gave remarkable contribution in the history of sampling and upon numerous new avenues for fruitful research in the theory and philosophy of sample surveys. The earlier procedure in the history of sampling for selection of sample is simple random sampling. The concept of random sampling was introduced by Kiaer (1895) in order to study the socio-economic problems to replace the usual approach of complete enumeration of the total population. Later on, Bowley (1906) introduced the concept of probability sampling.

In survey sampling, we are often concerned with the estimation of population mean ($\bar{Y}$) using auxiliary information, which may be available (or may be made available by diverting a part of the resources) in one form or the other. Auxiliary information is the supplementary information supplied by auxiliary variables which are closely correlated with the study variable and are used to devise methods of estimation. The information on the auxiliary variable is assumed to be complete *i.e.,* the information on auxiliary variable is available for all the population units. Also, this auxiliary information is required before making the sample selection. The auxiliary information can be gathered more easily than the study variable and can be efficiently utilized for developing the estimators with greater precision. The utilization of the auxiliary information in sample surveys has been done by the researchers since the origin of sampling which resulted in the increased precision in the estimation procedure. Generally, in survey data, the auxiliary information is available in one or other form and at the moderate cost. Auxiliary information available in any form and can be utilized to devise sampling strategies which are better than those in which no auxiliary information is used. The utilization of auxiliary information in accordance to the form in which it is available.

A large number of literature is available in sampling in case when the population mean of the auxiliary variable is known. Bowley (1906), Neyman (1934), Neyman and Pearson (1938) made the initial efforts to utilize the auxiliary information in sampling theory. However, Watson (1937) and Cochran (1940, 1942) utilized auxiliary information in dividing estimation procedures for improving the precision of estimation. The first use of auxiliary information in selecting the units with varying probabilities has been done by Hansen and Hurwitz (1943). If used properly, this information may provide better estimates than those where such information is not used. The precision of the estimator ($\bar{y}$) of population mean can be increased by utilizing advance knowledge about auxiliary information (X) correlated with the variable under study e.g., ratio and product method of estimation. Since, sample mean $\bar{y}$ has been found minimum variance unbiased estimator while estimating the population mean $\bar{Y}$, if one is prepared to sacrifice the property, improved estimators according to the mean squared error criterion can be obtained as proposed by Searls (1964).

In sample surveys the auxiliary information on one or more variables may be utilized in three basic ways:

i. At the pre-selection stage or the designing stage i.e. the information may be used in stratifying the population. Neyman and Pearson (1938) gave the method of two phase sampling for stratification. Hansen *et al.* (1953), Kish (1965), Raj (1968), Cochran (1977) and Dayal (1979) have also done tremendous work in this line. Hansen *et al.*(1946), Rao (1968), Srinath (1971) proposed the theory of stratification using double sampling for the estimation of population mean of study variable in the presence of non-response. Sedransk (1965) carried out empirical studies on the basis of double sampling for stratification. The information collected on the first sample has been used in stratifying the sample and also at the estimation stage for increasing the precision of the estimator (Ige and Tripathi, 1987).

ii. At the selection stage i.e., in selecting the units for sample with or without replacement and with varying probabilities proportional to some suitable measure of size. Stratified sampling is not applicable in case when the sampling frames for strata is not available but instead one can use the probability proportional to size (pps) provided strata weights (W_h) are exactly known. Simple random sampling has not been proven effective in cases when the sampling units are not of same size, as it does not consider larger units in the population which may be very useful. In such cases, auxiliary information about the size of the units has been found to be more efficiently utilized for selecting sample in order to obtain the more precised estimators of the population parameters. In such cases, assigning unequal probabilities to different units of the population for selection of the sampling units has been found more attractive. When the units are unequal in size and study variable is directly related to the size of the unit, the probabilities may be assigned proportional to the size of the unit. Such a sampling procedure is known as PPS Sampling. The technique of PPS sampling was introduced by Mahalanobis (1938) while sampling of plots for a crop survey and was discussed in detail by Hansen and Hurwitz (1943) and made use of auxiliary information for the first time in selecting the units with probabilities proportional to size (PPS). Lahiri (1951) have shown that the ratio estimator $\hat{\bar{Y}}_R = \left(\bar{y}/\bar{x}\right)\bar{X}$ becomes unbiased in case of pps sampling for estimation of population mean which was biased in simple random sampling scheme. Horvitz and Thompson (1952) gave the generalization of this theory to pps sampling without replacement (wor). Also, Narain (1951) has compared wor and wr methods of pps sampling independently.

iii. At the post-selection stage or at the estimation stage i.e. through defining ratio, regression, difference and product estimators based on the auxiliary information.

In the present study, the auxiliary information has been used at the estimation stage for developing the general improved ratio and product type estimators. The use of ratio and product type estimators has been discussed in details in case of simple random sampling without replacement and stratified sampling in books by Cochran (1977), Sukhatme and Sukhatme (1976), Raj (1968), Murthy (1967), Kish (1965) and others. Reddy (1974), Agarwal and Kumar (1980), Gupta (1978), Srivenkataramana and Srinath (1976), Srivenkataramana and Tracy (1979), Kaur (1983), Chaubey *et al.* (1984) defined estimators for using knowledge on population mean of auxiliary variable. Hartley and Ross (1954) used the bias reduction technique whereas an almost unbiased estimator in SRSWOR has been considered by Murthy and Nanjamm (1959). Gray and Schucany (1972) generalized the well known Jackknife technique for obtaining an unbiased or almost unbiased estimators.

Ratio type estimator was proposed by Cochran (1940) which is quite effective when there is a positive correlation between the characteristics Y under study and X the auxiliary character study whereas, if the correlation between the study variable and auxiliary variable is negative, product estimator proposed by Robson (1957) is effective.

Rao (1966), Sahoo and Swain (1989), Pandey and Dubey (1989) and Singh and Narain (1989) considered almost unbiased ratio estimators. Naik and Gupta (1991) proposed a general class of estimators for estimating the populations mean using auxiliary information. Olkin (1958) proposed improved ratio estimator for estimation of population mean under study by taking into consideration the linear combination of ratio estimator based on each auxiliary variable separately making use of auxiliary information having positive correlation with the variable under consideration. He further recommended the use of information based on more than one supplementary characteristic that were positively correlated with study variable, considering a linear combination of ratio estimators based on each auxiliary variable independently. One can use ratio type estimator for estimating the population mean when there is high correlation between the study variate and the auxiliary variate or it can be used when the auxiliary variate satisfy the condition (i) if $C_x/2C_y < \rho \leq +1$ and both Y and X are positive or negative (ii) if $-C_x/2C_y < \rho \leq +1$ and either Y or X is negative (Singh and Chaudhary, 1995).

For estimation of the population mean of the study variable, product estimators were proposed by Robson (1957) and Murthy (1964) based on the information provided by the mostly correlated variable are quite well known in sampling theory. Kushwaha and

Singh (1988) proposed product type estimators which were the particular case of the unbiased class of product estimator as considered by Tripathi and Singh (1988). Further, the product type estimators can be used for the estimation of population mean when the study variable and the auxiliary variables are highly negatively correlated to each other or it can be used when the auxiliary variate satisfy the condition (i) if $-1 < \rho < -C_x/2C_y$ and both Y and X are positive or negative or (ii) if $C_x/2C_y < \rho \leq +1$ and either Y or X is negative (Singh and Chaudhary, 1995).

Further, in order to support the efficiency of the modified ratio and product estimators, secondary data or a simulation study can be used. The term simulation have been derived from the Latin word ‘‘simulare’’ means to pretend. The model pretends to be the real system when simulated. Fritzson (2004) defined simulation as ‘‘an experiment performed on a model’’ which supports the fact that simulation is always used to achieve some goal. Generally, simulation is defined as the application of a model in deriving strategies which help to solve a problem or answer a question pertaining to a system. Simulation is used to compare the results by running model a large number of times which gives an insight to the behavior of the actual system. Simulation should be used when the consequences of a proposed action, plan or design cannot be directly and immediately observed (i.e., the consequences are delayed in time and/or dispersed in space) and/or it is simply impractical or prohibitively expensive to test the alternatives directly. Simulation is particularly valuable when there is significant uncertainty regarding the outcome or consequences of a particular alternative under consideration. Probabilistic simulation allows us to deal with this uncertainty in a quantifiable way. Simulation should always be preferred over performing the experiments on real system, since it minimizes the cost and risk in executing the experiment. Some variable may not be accessible which can be easily studied and controlled through simulation. The parameters of a system model can be easily manipulated. Simulation software helps in evaluating, comparing and optimizing alternative designs, plans and policies. It further provides tools for explaining and defending decisions to various stakeholders.

For a model whose parameters are known or fixed, then each time the model will yield same results. But in case the model is stochastic (represented as statistical distributions or some random pattern), then it will yield different results each time. The simulation study used some empirical measures like relative bias (RB) and the relative mean squared error (RMSE) for comparing the precision of different estimators. The values of RB provide information about empirical bias of the different estimators, whereas the values of RMSE reveal the efficiencies of the estimators.

Keeping in view the role of auxiliary information in developing the improved and precise estimators at estimation stage, the following objectives have been undertaken in the present study.

- To find out the general class of improved ratio type estimators of population mean
- To find out the general class of improved product type estimators of population mean
- To identify the interval in which the proposed classes of estimators will perform better than conventional estimators of population mean according to relative mean squared error criterion
- To examine the efficiencies of these classes of estimators empirically.

In the present study, performances of the proposed estimators (may be biased or unbiased) have been compared with respect to conventional ratio and product estimators and some existing estimators like Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007) and Sharma *et al.* (2010) by conducting a simulation study. Therefore, it is proposed to carry out some investigations that are chiefly concerned with the improved estimation by utilizing the auxiliary information at the estimation stage through ratio and product type estimators.

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Review of Literature

CHAPTER 2

REVIEW OF LITERATURE

2.1 Ratio estimators of population

Cochran (1940) proposed ratio method of estimation of population mean when the variable under study and auxiliary variable are positively correlated as $\hat{\bar{Y}}_R = \frac{\bar{y}}{\bar{x}}\bar{X}$, where, $\bar{y}$ and $\bar{x}$ are the unbiased estimators of the population mean of study variable and auxiliary variable respectively. The proposed estimator was more efficient than the $\bar{y}$ for estimating the $\bar{Y}$.

Koop (1951) gave an expression for the bias of the ratio estimator and explored the possibilities to reduce it. The role of auxiliary information in sample surveys for designing efficient estimators of population parameters is well recognized. The conventional ratio and product estimators are consistent but biased for population parameters as referred by Sukhatme (1954), Cochran (1977) and Murthy (1977).

Assuming that the population variance (σ^2) is known, Upadhyaya and Srivastava (1976) proposed a class of estimators of population mean. Large sample properties were studied and compared with conventional estimator of population mean.

Hansen and Hurwitz (1943) suggested that the bias of the ratio estimator is usually negligible. Hartley and Ross (1954) suggested an unbiased ratio estimator for population mean ($\bar{Y}$) as $t_{HR} = \bar{r}_s\bar{x} + \frac{n(N-1)}{(n-1)N}(\bar{y}_s - \bar{r}_s\bar{x}_s)$, where, $\bar{y}_s$ and $\bar{x}_s$ are the sample means for the respective variables and sample s is drawn using simple random sampling without replacenment (SRSWOR) design.

Robson (1957) found its exact variance in case of finite population whereas Goodman and Hartley (1958) studied its precision.

Beale (1962) and Tin (1965) corrected the ratio estimators for its bias and defined modified ratio estimators which are almost unbiased.

Searls (1964) had made an important contribution in the development of estimator of population mean. He considered the product of a scalar and sample mean as an estimator of population mean and chose the value of scalar such that it minimized the mean squared error. This optimum estimator required the knowledge of coefficient of variation. Rao (1966) developed unbiased ratio type estimator of population mean by combining two biased ratio estimators.

John (1969) considered an alternative multivariate generalization of the ratio estimators for an arbitrary sampling design and showed that it has the same variance as Olkin's (1958) estimator upto a first order of approximation. The multivariate ratio estimators were compared with the multivariate product and regression estimators with respect to variance and ease of computation. For the calculation of the multivariate ratio and product estimates an iterative procedure was suggested and demonstrated by working out an example.

Tripathi (1978) gave an alternative weight vector in which coefficient of variations and correlation coefficients were assumed to be known. It was also demonstrated by an empirical study that, by using proposed weights there was no loss in precision in the ratio estimators as compared to optimum weights. Efficiency of modified ratio estimators were also compared.

Kulkarni (1978) compared the absolute biases of both modified and usual ratio estimators and obtained the regions in which the modified ratio estimator had less absolute bias than the usual ratio estimator.

Chakrabarty (1979) developed ratio type estimators which were more efficient than conventional ratio estimator as $t_1 = (1-\alpha)\bar{y} + \alpha\bar{y}\frac{\bar{X}}{\bar{x}}$ where, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Two auxiliary characters were used in different ways by Agarwal and Kumar (1980), one auxiliary character was used for the purpose of selection of sample and the other for the purpose of estimation. The PPS and ratio estimators were combined suitably to estimate the population mean, in order to minimize the mean squared error. They proposed an estimator following this approach and proved that the proposed estimator would always be more precise than that of either PPS estimator or ratio estimator under PPS sampling scheme.

Srivenkataramana and Tracy (1979) proposed an alternative to ratio method of estimation, using simple transformation which enabled the use of product method in place of ratio method. The product method proposed by Murthy (1977) had an edge over ratio method i.e. expressions for bias and mean squared error could be exactly evaluated. The optimum situations were indicated in the minimum mean squared error sense. A good guess of a certain parameters, which does not seem very restrictive for practice, was required in a procedure. For dealing with the bias of the estimator, two methods were used**.** An extension to the use of multiauxiliary information was also outlined.

Singh *et al*. (1980) modified the usual ratio method of estimation when regression of y on x in the population is of the general form y = f(x), where f(x) is some function of x. They found that the usefulness of taking f(x) as an auxiliary character needs no emphasis, if f(x) is known completely. In case, the function f(x) is not defined completely but its form is known, it can be utilized for ratio method of estimation.

Das (1980) proposed ratio type estimator by taking the ratio of approximate minimum mean squared error estimators of the characteristics under study and the auxiliary character. The efficiency of the ratio estimator under size stratification had been worked out by Kumar (1985) and efficiencies were also compared with some of the well-known sampling strategies. On comparing the efficiencies, it had been established that the stratified ratio sampling strategy performs better. Rao (1981) studied the unbiasedness of ratio estimator.

A comparison of four ratio estimators based on interpenetrating subsamples and with or without Jack-knifing had been done both theoretically and empirically by Gosh and Gomez (1986), with respect to bias, variance and mean squared error.

Kushwaha and Singh (1988) proposed a general class of almost unbiased ratio and product type estimators.

For the estimation of mean of a finite population Srivastava *et al*. (1990) proposed a generalized chain ratio estimator. They made a comparison of suitable estimator and derived the expressions for bias and mean squared error and also studied the efficiency of proposed estimator.

A general class of ratio estimators alternative to product estimators had been proposed by Kushwaha (1991). By using Jackknife technique, efforts were made to reduce / eliminate its bias to make the class of estimators to be almost unbiased. For the proposed class of estimators, the expressions for the bias and mean squared error were derived to the first order of approximation under simple random sampling without replacement (SRSWOR) strategy and optimum estimator had also been identified.

Birader and Singh (1995) defined a class of unbiased ratio type estimators for population mean ($\bar{Y}$) based on a linear combination of three estimators *viz,* $\bar{y}$, $\left(\frac{\bar{X}}{n}\right)\sum_{i=1}^{n}\left({y_i}/{x_i}\right)$ and $\left(\frac{\bar{x}}{n}\right)\sum_{i=1}^{n}\left({y_i}/{x_i}\right)$ as $\hat{\hat{Y}} = \theta_1\bar{y} + \theta_2\bar{r}\bar{X} + \theta_3\bar{r}\bar{x}$, Where $\theta'_i s$ are chosen constants such as $\sum_{i=1}^{3}\theta_i = 1, r = {\bar{y}}/{\bar{x}}$ and $\bar{r} = \frac{1}{n}\sum_{i=1}^{n}\left({y_i}/{x_i}\right)$.

Singh and Singh (1995) proposed another class of unbiased ratio type estimators for population mean ($\bar{Y}$) and shown that Hartley and Ross (1954) estimator is a particular member of this class.

The predictive approach was used by Sahoo and Sahoo (1995) which was advocated by Basu (1971), to develop an almost unbiased ratio estimator of a finite population mean which was found to be more efficient than its competitors.

Mohanty and Sahoo (1995) considered simple linear transformations using the known minimum and maximum values of the auxiliary information for the estimation of finite population mean under ratio method of estimation. They also showed the suggested transformations provide efficient ratio estimators even when the conventional ratio estimator is less efficient than simple mean per unit estimator.

Amdekar (1996) used the ratio method of estimation in case of overlapping clusters. From empirical study it was observed that the overlapping cluster sampling with ratio estimator was generally better than simple random sampling with ratio estimator. It was shown that overlapping cluster sampling besides being operationally convenient was likely to be better from variance point of view, when used with ratio estimator.

Singh and Singh (1999) proposed a class of unbiased ratio type estimators for population mean ($\bar{Y}$). Kumar (2002) proposed three ratio type estimators, which were though biased, but more efficient than the conventional ratio estimators of population mean.

Singh and Espejo (2003) considered a class of ratio–product estimators for estimating a finite population mean. The asymptotically optimum estimator in the class was identified, along with its approximate mean-squared error. This estimator requires prior knowledge of the parameter $C = \rho C_y / C_x$, where ρ is the correlation coefficient between the study variate y and the auxiliary variate x, and C_y and C_x are coefficients of variation of y and x respectively. If C is unknown in advance, then it can be replaced by its consistent estimate $\hat{C}$, with the resulting estimator known as an 'estimator based on the estimating optimum'. It was shown that, to the first order of approximation, both estimators have the same mean-squared error, and were generally more efficient than the usual ratio and product estimators.

Khan *et al.* (2007) attempted to define a class of almost unbiased estimators by combination of simple mean, ratio estimator and jack knife version of ratio estimator. They also identified the optimum estimator in the class and obtained the interval for optimum weights.

Singh *et al.* (2008) proposed an almost unbiased estimator using known value of some population parameter(s) and derived the expressions for bias and mean squared error (MSE) under simple random sampling scheme. They extended their study to two phase sampling.

Gupta and Shabbir (2008) proposed an alternative form of ratio-type estimator which was better than the competing ratio, regression, and other ratio-type estimators proposed by Kadilar and Cingi (2004, 2006). The results were also supported by the analysis of three real data sets that were considered by Kadilar and Cingi (2004, 2006).

Sharma *et al*. (2008) proposed an improved ratio type estimator whose large sample properties were compared with the existing ratio and Tin (1965) estimator. They found that the proposed estimators performed better than the conventional ratio estimator and Tin (1965) estimator, according to the mean squared error criterion.

Koyuncu and Kadilar (2009) proposed a family of estimators using the results of Prasad (1989) and derived the expressions for the bias and MSE of the proposed family. Besides, considering the minimum cases of these MSE equations, a comparison of the efficiency conditions between the Khoshnevisan and proposed families are obtained. The proposed family of estimators was found to be more efficient than Khoshnevisan's family of estimators under certain conditions.

Solanki *et al.* (2012) suggested class of estimators for estimating the finite population mean $\bar{Y}$of the study variable *y* using known population mean $\bar{X}$of the auxiliary variable x as $t = \bar{y}\left[2 - \left\{\frac{p}{P}\right\}^{\lambda} exp\,\frac{\delta(p-P)}{(p+P)}\right]$, where, λ is a constant suitably chosen by minimizing mean squared error of the estimator. They obtained asymptotic expressions of bias and variance of the proposed class of estimators. Further, they showed that the proposed class of estimators were more efficient than conventional ratio, conventional product, Bahl and Tuteja (1991) and Kadilar and Cingi (2003) estimators. Also an empirical study has been conducted in support of their results.

Subramani and Kumarapandiyan (2012) proposed a class of modified ratio estimators for estimation of population mean of the study variable using the linear combination of the known values of the Co-efficient of Variation and the Median of the auxiliary variable. The biases and the mean squared errors of the proposed estimators were derived and were compared with that of existing modified ratio estimators. Further, they derived the conditions for which the proposed estimators performed better than the existing modified ratio estimators. The performances of the proposed estimators are also assessed with that of the existing estimators for certain natural populations. From the numerical study it was observed that the proposed modified ratio estimators performed better than the existing modified ratio estimators.

Olufadi (2013) developed a following estimator under SRSWOR, $\bar{y}_{PR} = \bar{y}\left[\theta\left(\frac{\bar{x}^*}{\bar{X}}\frac{Z}{\bar{z}^*}\right) + (1-\theta)\left[\frac{X}{\bar{x}^*}\frac{\bar{z}^*}{\bar{Z}}\right]\right]$, Where, θ is a constant, chosen to minimize the variance of $\bar{y}_{PR}$ and extended it to stratified random sampling design which were found to be more efficient than the usual unbiased estimator, the traditional ratio (Cochran, 1940) and product estimators and the estimators proposed by Singh (1969), Srivenkataramana (1980), Bandyopadhyay (1980), Singh *et al*. (2005), Sharma and Tailor (2010) and Tailor *et al*. (2012) under simple and stratified random sampling. Further, he made an empirical study using four natural data in support of the proposed estimator.

Yadav and Kadilar (2013) proposed an improved family of estimators for the population mean. The proposed family of estimators provided significant improvement over the estimators of the families proposed by Khoshnevisan *et al.* (2007), Koyuncu and Kadilar (2009) and Adewara *et al.* (2012) and also lead to the better perspective of application in various applied areas. The numerical demonstration showed that the proposed family of estimators were the most efficient estimators.

Sharma *et al.* (2013) tried to found out the second order biases and mean squared errors of some estimators using information on auxiliary attribute and compared the performance of the estimators with the help of a numerical illustration.

Sharma and Singh (2014) proposed new ratio type estimator using auxiliary information on two auxiliary variables based on simple random sampling without replacement (SRSWOR). The proposed estimator was found to be more efficient than the estimators constructed by Olkin (1958), Singh (1965), Singh and Kumar (2012) in terms of second order mean squared error.

Malik *et al.* (2014) proposed a new estimator for population mean of the study variable y in the case of stratified random sampling using the information based on auxiliary variable x. They derived the expression of mean squared error (MSE) of the proposed estimator up to the first order of approximation. They also verified the theoretical conditions by a numerical example. An empirical study demonstrates the efficiency of the suggested estimator over sample mean estimator, usual separate ratio, separate product estimator and other proposed estimators.

Lu and Yan (2014) proposed a class of ratio estimators of a finite population mean using two auxiliary variables and obtained mean squared error (MSE) equations for the class of proposed estimators. They identified theoretical conditions that make proposed family estimators more efficient than the traditional ratio estimator and the estimator proposed by Abu-Dayeh *et al.* (2003) using two auxiliary variables.

Rao (2015) reviewed some ratio estimators with two auxiliary variables available in literature and compared their efficiencies by simulation for different distributions with known correlation coefficients and showed that the simulation method is more appropriate when there is no closed expression for the bias and mean squared error of the estimators.

Sharma and Singh (2015) dealt with the problem which involves the estimation of the qualitative characteristics. They proposed three estimators when study variable is itself an attribute using the quantitative auxiliary information. They further determined the bias and mean squared error of the proposed estimators to the first order of approximation. In theoretical and empirical efficiency comparisons, it has been shown that the proposed estimator was more efficient than the other estimators under consideration.

Khan and Hussain (2015) suggested an improved class of ratio-type estimators for finite population mean, under maximum and minimum values using the auxiliary variable in case of simple random sampling. They found that some theoretical conditions under which the suggested class of ratio-type estimators have always efficient than the usual unbiased, Sarndal (1972), the classical ratio estimator, Singh and Tailor (2003), Sisodia and Dwivedi (1981) and Kadilar and Cingi (2006) estimators. Theoretical results were also verified with the help of real data set which clearly indicated that the proposed class of ratio-type estimators has smaller mean squared error and higher percent relative efficiency.

Singh (2015) proposed a class of ratio type estimators to estimate population mean of the characteristic under study. He obtained particular cases of proposed estimator which had improvement over existing estimator. The expression for bias and mean squared error of the proposed estimator including its particular cases have been derived up to the first order of approximation and compared theoretically with the other improved ratio-type estimators and conditions found for which the proposed estimator is better than improved ratio-type estimators. An empirical study has also been carried out to demonstrate the efficiencies of proposed estimator.

Subramani and Ajith (2016) proposed a modified ratio-cum-product estimator for the estimation of finite population mean of a study variable Y using the known coefficient of variation of the auxiliary variable X, which was correlated with the study variable. The bias and the mean squared error of the proposed estimator were obtained. Both analytical and numerical comparisons with some existing estimators have been done. As a result, it was found that the proposed estimator was more efficient than the existing estimators.

Clement (2016) suggested an improved ratio estimator in stratified random sampling as $t_{st}(\alpha_h, \delta_h) = \lambda^* \bar{y}_h$ where, the coefficient $\lambda = \left\{2 - \left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha} exp\left[\delta(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}})\right]\right\}$ and obtained the expression for the bias and mean squared error (MSE) analytically. He further made an empirical illustration to support the analytical study.

Misra *et al.* (2017) attempted to develop an improved ratio type estimator of population mean using predictive method of estimation by using linear combination of coefficient of skewness and the quartile deviation of auxiliary variable using the idea of motivated by Jeelani *et al.* (2013). They derived the mathematical expressions for the bias and mean squared error (MSE) of the proposed estimator up to the first order approximation. They also theoretical comparison of efficiency of proposed estimator with the usual ratio estimator, usual product estimator, and Singh *et al.* (2014) estimators. An empirical study had also been carried out.

2.2 Product type estimators

When the auxiliary variable is negatively correlated with the variable under study, Robson (1957) proposed the product estimator of population mean as $\hat{\hat{Y}} = \frac{\overline{yx}}{\bar{X}}$

Murthy (1964) proposed the product type estimator and also developed an unbiased estimator as $t^* = \frac{\overline{yx}}{\bar{X}} - \frac{(1-f)}{n}\frac{S_{xy}}{\bar{X}}$.

Following the predictive approach advocated by Basu (1971), Sahoo and Sahoo (1999) developed two almost unbiased product estimators of a finite population mean. These estimators were compared empirically with some traditional product estimators.

Shukla (1976) obtained unbiased product estimators (to the first degree of approximation) with the help of the technique developed by Quenouille (1956) and has established that the new estimator was better than the other product estimator in the sense of mean squared error.

Sahai and Ray (1980) proposed a transformed estimator for a wide range of the value of the correlation coefficient between the main and auxiliary variables. Alternative estimators had been proposed by Sahai and Ray (1980) which have more practical utility as t$= \bar{y}\left[2 - \left\{\frac{p}{P}\right\}^W\right]$, where, W is a constat.

Srivastava *et al*. (1981) analyzed the properties of product estimator. When there is no prior knowledge about population mean of auxiliary character, Srivastava and Bhatnagar (1981) proposed an estimator as $t_\alpha = \left[1 - \frac{(1-f)^2 s_x^2}{n\bar{x}^2 + \alpha(1-f)^2 s_x^2}\right]\bar{y}$, where, α is the characterizing scalar, $f = \frac{n}{N}$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Srivastava (1983) adopted the predictive approach advocated by Basu (1971) for estimating the population mean and found that the use of mean per unit estimator, regression estimator and ratio estimator as a predictor for the mean of unobserved units in the population resulted in the corresponding customary estimators of the mean of the whole population. The new estimator so obtained was compared with the customary product estimator.

Upadhyaya and Singh (1984) proposed a class of estimators of population mean which is a special case of the family of estimators proposed by Srivastava and Bhatnagar (1983). Its large sample properties were compared with the estimators proposed by Upadhyaya and Srivastava (1976). The estimator proposed by Upadhyaya and Singh (1984) has less bias than the estimator of Upadhyaya and Srivastava (1976) whereas both the estimators have identical mean squared error upto second order.

Singh (1987) proposed a family of estimators for population mean and proved that the estimators defined by Upadhyaya and Srivastava (1976), Srivastava and Bhatnagar (1983), Upadhyaya and Singh (1984) are the special cases. The large sample properties of the estimator has been worked out and compared with the special cases of the estimators.

Rao (1987) discussed a method and obtained a simpler derivation of the class of unbiased product estimators for simple random sampling without replacement as well as for interpenetrating sub -samples design.

A general class of almost unbiased ratio and product type estimators was proposed by Kushwaha and Singh (1988), for estimating population mean $\overline{Y}$ of the study characteristic Y employing Jack-knife technique introduced by Quenouille (1956). They confined there study to systematic sampling and obtained the explicit expressions for the variance of class of estimators to the first order of approximation. In the class of unbiased estimators they also identified the minimum variance unbiased estimator.

Kataria and Singh (1989) considered a class of estimators for population mean and found an optimum estimator whose large sample properties have been studied. It has smaller bias than the estimators proposed by Upadhyaya and Srivastava (1976) and Upadhyaya and Singh (1984). All the three estimators have same mean squared error upto second order.

Sampath (1989) studied the optimal estimators of unknown population parameters in the several classes of ratio and product type estimators. He showed that, in a regular class of estimators, the replacement of some or all of the parameters by their unbiased estimate did not alter the efficiency of the optimal estimators in terms of its mean squared error.

Singh (1989) proposed a class of unbiased product estimator of population mean as $t_1^* = \frac{\overline{yx}}{\bar{X}}\left[1 + \frac{(1-f)}{n}\frac{s_{xy}}{\bar{x}\bar{y}}\right]^{-1}$ where, $s_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$.

Bhatnagar (1990) proposed an estimator which is a special case of the family of the estimators of population mean proposed by Bhatnagar (1981, 1986). The proposed estimator has smaller relative mean squared error than the estimators considered by Upadhayaya and Singh (1984) and Singh (1987).

Dubey (1993) considered product estimator as $t_2^* = \frac{\overline{yx}}{\bar{X}} - \frac{(1-f)}{n}\frac{s_{xy}}{\bar{x}}$ was almost unbiased and also more efficient under certain conditions than the usual product estimator and the estimator proposed by Robson (1957).

Bhatnagar (1996) made a comparative study for product type estimators proposed by Robson (1957) and Singh (1989).

For estimating the population means of character of interest, Naik and Gupta (1996) investigated the feasibility of using the prior knowledge regarding the proportion of unit in the population and defined the ratio, product and regression estimators. Expressions for their bias and mean squared error to the first order approximation were obtained. Comparison was made among the estimators as well as with the sample proportion.

Bhatnagar and Kumar (1998) compared the estimators proposed by Bhatnagar (1990), Upadhyaya and Singh (1984), Singh (1987) and Kataria and Singh (1989). The estimator proposed by Bhatnagar (1990) was found to be more efficient than the others.

Goyel *et al.* (2000) proposed a product type estimator of population mean for symmetric or skewed populations using auxiliary information.

Sharma *et al.* (2007) proposed a general class of product type estimators whose large sample properties were compared with the conventional product type estimators as well as estimators proposed by Robson (1957), Singh (1989) and Dubey (1993) according to mean squared error criterion.

Sharma and Bhatnagar (2008) proposed new class of product type estimators for the estimation of population mean and also compared their large sample properties with the conventional product type estimators as well as estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.*(2007). They also showed that the proposed estimators showed an improvement over these conventional estimators according to mean squared error criterion.

Malik *et al.* (2014) proposed a new estimator for population mean of the study variable y in the case of stratified random sampling using the information based on auxiliary variable x. They derived the expression of mean squared error (MSE) of the proposed estimator up to the first order of approximation. They also verified the theoretical conditions by a numerical example. An empirical study demonstrates the efficiency of the suggested estimator over sample mean estimator, usual separate ratio, separate product estimator and other proposed estimators.

Singh and Audu (2015) proposed an exponential ratio-product type estimator for population under simple random sampling scheme and derived expressions for the bias and MSE of the proposed estimator up to first order approximation. They obtained optimum MSE of the proposed estimator. The efficiency of the proposed estimator was compared theoretically and empirically with existing estimators. The empirical comparison showed that the proposed estimator was more efficient than others for both when the correlations between the study and auxiliary variables were positive and negative.

Yadav and Mishra (2015) gave a method for the estimation of population mean using predictive method of estimation utilizing auxiliary information. They proposed improved ratio-cum-product type predictive estimators to estimate the population mean and derived the expressions for the bias and mean squared error (MSE) up to the first order of approximation. The minimum value of MSE of proposed estimator was also obtained for the optimum value of the characterizing scalar. A comparison has been made with the ratio and product type estimators and the conditions under which the proposed estimator was more efficient.

Ekpenyong and Enang (2015) proposed a modified class of ratio estimators and suggested a new class of product estimators to estimate the population mean. The efficiencies of these estimators were compared and the most efficient unbiased estimator was identified among these estimators. Numerical illustration was employed to validate their claim.

Vishwakarma *et al*., (2016) developed improved ratio and product type estimators for estimating the finite population mean of the study variable using auxiliary information in simple random sampling. The expressions for the bias and mean squared error of the proposed estimators have been derived upto first order of approximation. They further demonstrated the efficiencies of the proposed estimators over other well known estimators through theoretical and empirical studies.

Materials and Methods

CHAPTER 3

MATERIALS AND METHODS

It is well known fact that the suitable use of auxiliary information in probability sampling results in considerable reduction in variance of the estimator of population mean/total. For increasing the precision of the estimator of the population characteristics under study, the auxiliary information may be used either at the stage of designing or at the stage of estimation, depending upon the form in which such information is available to use. The problem of estimating the population mean of the variable under study using auxiliary variable has received considerable attention in sampling. Ratio and product estimators and their generalizations, which utilize the auxiliary information, have been widely used in practice. The quest for adaptive as well as improved estimators led to the development of a variety of procedures in sampling theory and statistical inference. A most widely used simple way is to modify the conventional estimators in such a way that they become more efficient than the existing estimators while the other way is to use the auxiliary information to increase precision of the estimators. For simple random sampling the classes of estimators have been developed using auxiliary information. These are the classes of estimators for mean, ratio and product type estimators. The auxiliary variable X is correlated with Y the study variable.

3.1 Notations

Let a random sample of size n is drawn from a population of size N and observations on auxiliary variable X and study variables Y are obtained. Further, the sample means $\bar{x}$ and $\bar{y}$ are unbiased estimators of population means $\bar{X}$ and $\bar{Y}$ respectively while s_x^2 and s_y^2 are unbiased estimators of population variances σ_x^2 and σ_y^2 respectively. Similarly, let s_{xy} be an unbiased estimator of population covariance σ_{xy}.

We follow the convention that the lower case letters y_i and x_i stand for i^{th} unit in the sample (i=1, 2, …….., n) and upper case letters Y_i and X_i stand for the i^{th} unit in the population (i=1, 2, ……., N). Let,

$$\left.\begin{aligned}
&\bar{y} = \frac{1}{n}\sum^{n} y_i \quad ; \quad \bar{x} = \frac{1}{n}\sum^{n} x_i \\
&\bar{Y} = \frac{1}{N}\sum^{N} Y_i \quad ; \quad \bar{X} = \frac{1}{N}\sum^{N} X_i \\
&s_y^{\,2} = \frac{1}{n-1}\sum^{n}(y_i - \bar{y})^2 \quad ; \quad s_x^{\,2} = \frac{1}{n-1}\sum^{n}(x_i - \bar{x})^2 \\
&S_y^{\,2} = \frac{1}{N-1}\sum^{N}(Y_i - \bar{Y})^2 \quad ; \quad S_x^{\,2} = \frac{1}{N-1}\sum^{N}(X_i - \bar{X})^2 \\
&\sigma_y^{\,2} = \frac{1}{N}\sum^{N}(y_i - \bar{Y})^2 \quad ; \quad \sigma_x^{\,2} = \frac{1}{N}\sum^{N}(x_i - \bar{X})^2 \\
&s_{xy} = \frac{1}{n-1}\sum^{n}(x_i - \bar{x})(y_i - \bar{y}); \quad \sigma_{xy} = \frac{1}{N}\sum^{N}(X_i - \bar{X})(Y_i - \bar{Y}).
\end{aligned}\right\} \tag{3.1}$$

Let E (T), RB (T) and RM (T) denote expected value, relative bias and relative mean squared error of an estimator T, respectively.

Let,

$$\left.\begin{aligned}
&U_x = \frac{\bar{x} - \bar{X}}{\bar{X}}, \qquad V_x = \frac{s_x^2 - \sigma_x^2}{\bar{X}^2} \\
&U_y = \frac{\bar{y} - \bar{Y}}{\bar{Y}}, \qquad V_y = \frac{s_y^2 - \sigma_y^2}{\bar{Y}^2} \\
&\text{and,} \quad W = \frac{s_{xy} - \sigma_{xy}}{\bar{X}\,\bar{Y}}.
\end{aligned}\right\} \tag{3.2}$$

There are several techniques to evaluate the moments and cross moments of U_x, U_y, V_x, V_y and W; see e.g. Kendall and Stuart (1952), Sukhatme and Sukhatme (1970) and Bhatnagar (1981).

Further, we write

$$C_{ab} = \frac{1}{N-1}\sum^{N}\left(\frac{x_i-\overline{X}}{\overline{X}}\right)^a\left(\frac{y_i-\overline{Y}}{\overline{Y}}\right)^b \tag{3.3}$$

where, a and b are non – negative integers.

Also,

$$\theta = \left(\frac{C_{02}}{C_{20}}\right)^{1/2} \text{ and} \tag{3.4}$$

$$\rho = \frac{C_{11}}{(C_{02})^{1/2}(C_{20})^{1/2}} . \tag{3.5}$$

The present study confines itself in making the use of auxiliary information at estimation stage for constructing the estimator for population parameter on the basis of nature of sampling units of the population parameter under simple random sampling scheme.

Simulation: The comparison of the efficiencies of the modified ratio and product type estimators with respect to different estimators have been done through simulation data which is generated by using SAS and R softwares. The commands used are as:

R commands:

```
library(MASS)
N=200# sample size
rho=;mu1=;s1=;mu2=;s2=; #parameter  values
mu=c(mu1,mu2)# par of bivariate normal the mean vector
sigma=matrix(c(s1^2,s1*s2*rho,s1*s2*rho,s2^2),2) # covariance matrix
bvn1=mvrnorm(N,mu=mu,Sigma=sigma)# bivariate normal sample
bvn1 #
```

SAS macros:

```
/* Generate data */
data;
mean1=; *mean for y1;
mean2=; *mean for y2;
sig1=; *SD for y1;
sig2=; *SD for y2;
rho=; *Correlation between y1 and y2;
do i = 1 to N;
r1 = rannor(1245);
r2 = rannor(2923);
y1 = mean1 + sig1*r1;
y2 = mean2 + rho*sig2*r1+sqrt(sig2**2-sig2**2*rho**2)*r2;
output;
end;
keep y1 y2;
procprint;
Run;
```

Sharma (2011) derived the following results which were found be useful in the study of exact moments of estimators in the context of a normal distribution as:

(i) $E(U_x^{\,2}) = \dfrac{C_{20}}{n}$ (ii) $E(U_y^{\,2}) = \dfrac{C_{02}}{n}$

(iii) $E(U_x^{\,3}) = \dfrac{C_{30}}{n^2}$ (iv) $E(U_y^{\,3}) = \dfrac{C_{03}}{n^2}$

(v) $E(U_x^{\,4}) = \dfrac{C_{40} - 3C_{20}^{\,2}}{n^3} + \dfrac{3C_{20}^{\,2}}{n^2}$

(vi) $E(U_x U_y) = \dfrac{C_{11}}{n}$ (vii) $E(U_x^{\,2} U_y) = \dfrac{C_{21}}{n^2}$

(viii) $$E(U_x{}^3U_y)=\frac{C_{31}-3C_{20}C_{11}}{n^3}+\frac{3C_{20}C_{11}}{n^2}$$

(ix) $$E(U_x{}^2U_y{}^2)=\frac{C_{22}-2C_{11}{}^2-C_{20}C_{02}}{n^3}+\frac{2C_{11}{}^2+C_{20}C_{02}}{n^2}$$

(x) $$E(U_xV_x)=\frac{C_{30}}{n}$$ (xi) $$E(U_yV_x)=\frac{C_{21}}{n}$$

(xii) $$E(U_y{}^2V_x)=\frac{C_{22}-2C_{11}{}^2-C_{20}C_{02}}{n^2}$$

(xiii) $$E(U_xU_yV_x)=\frac{C_{31}-3C_{20}C_{11}}{n^2}$$

(xiv) $$E(U_x{}^2U_yV_x)=\frac{C_{41}-4C_{30}C_{11}-6C_{20}C_{21}}{n^3}+\frac{2C_{30}C_{11}+C_{20}C_{21}}{n^2}$$

(xv) $$E(V_x{}^2)=\frac{C_{40}-3C_{20}{}^2}{n}+\frac{2C_{20}{}^2}{n-1}$$

(xvi) $$E(U_xU_y{}^2V_x)=\frac{C_{32}-4C_{21}C_{11}-3C_{12}C_{20}-2C_{21}C_{02}+C_{30}C_{02}}{n^3}+\frac{C_{30}C}{n^2}$$

(xvii) $E(U_yV_y)=\dfrac{\phi C_{02}{}^2}{n}$ and $\phi=\dfrac{\vartheta_1}{\sqrt{C_{02}}}$ where, $\vartheta_1=\dfrac{C_{03}}{C_{02}{}^{\frac{3}{2}}}$ is the Pearson's measure of skewness of the population.

(xviii) $$E(U_y{}^3V_y)=\frac{C_{02}{}^3}{n^2}\left[\frac{3C_{03}}{C_{02}}-\frac{2}{n}\right]$$

(xix) $$E(U_y{}^4)=\frac{C_{04}-3C_{02}{}^2}{n^3}+\frac{3C_{02}{}^2}{n^2}$$

(xx) $$E\left(U_y{}^2V_y\right)=\delta_2\,\frac{C_{02}{}^2}{n^2}$$

(xxi) $E\left(U_x V_y^{\ 2}\right) = \dfrac{C_{12}}{n^2}$

(xxii) $E\left(U_x W\right) = \dfrac{C_{21}}{n}$ (xxiii) $E\left(U_y W\right) = \dfrac{C_{12}}{n}$

Sukhatme *et al*. (1997), given the following relations for the fourth order central moments and product moments under normal population,

(xiv) $C_{40} = 3C_{20}^{\ 2}$

(xv) $C_{31} = 3C_{20}C_{11}$

(xvi) $C_{22} = C_{20}C_{02} + 2C_{11}^{\ 2}$.

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Improved Class of Ratio Type Estimators

CHAPTER 4

IMPROVED CLASS OF RATIO TYPE ESTIMATORS

4.1 Introduction

The auxiliary information can be efficiently utilized at the estimation stage of the survey sampling for improving the ratio and product estimators in order to estimate the population mean irrespective of the sampling design used. Ratio estimator is useful for estimation of population mean $\bar{Y}$ when there exists positive correlation between variable under study and auxiliary variable and also if conditions like (i) if $C_x/2C_y < \rho \leq +1$ and both Y and X are positive or negative (ii) if $-C_x/2C_y < \rho \leq +1$ and either Y or X is negative (Singh and Chaudhary, 1995) are satisfied.

Several modifications have been made in the conventional ratio estimator to achieve higher precision. The main contributions available in the literature in this regard has been made by Sukhatme (1954), Cochran (1977), Hartley and Ross (1954), Beale (1962), Tin (1965), Chakrabarty (1979), Birader and Singh (1995), Sharma *et al.* (2010) etc.

In this chapter, some ratio type estimators have been developed for estimation of population mean which were more efficient than the conventional estimator and the estimator proposed by Sharma *et al*., (2010). Further, the proposed estimators have been compared with the conventional, Sharma *et al.* (2010) and with each other as well. The relative bias and relative mean squared error of the proposed estimators have been obtained theoretically upto order O (n^{-1}) and O (n^{-2}) respectively.

4.2 CONVENTIONAL RATIO TYPE ESTIMATOR

The conventional ratio type estimator for estimating the population mean is given as

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{4.1}$$

where, $\bar{y}$ and $\bar{x}$ are unbiased estimators of $\bar{Y}$ and $\bar{X}$, the population means of the characteristics under study and auxiliary characteristics respectively.

Theorem 4.1: The relative bias and relative mean squared error of the ratio type estimator ($\bar{y}_r$) upto O (n^{-1}) and O (n^{-2}) respectively are as

$$RB(\bar{y}_r) = \frac{1}{n}(C_{20}-C_{11}) \tag{4.2}$$

$$RM(\bar{y}_r) = \frac{1}{n}(C_{02} - 2C_{11} + C_{20}) + \frac{1}{n^2}[(2(2C_{21} - C_{12} - C_{30}) + 3(3C_{20}^2 - 6C_{20}C_{11} + 2C_{11}^2 + C_{20}C_{02})]. \tag{4.3}$$

Proof: Expressing $\bar{y}$, $\bar{x}$ and s_{xy} in terms of U_y, U_x and W respectively, in the estimator $\bar{y}_r$, we obtain

$$\frac{\bar{y}_r - \bar{Y}}{\bar{Y}} = \left(1 + U_y\right)(1 + U_x)^{-1} - 1. \tag{4.4}$$

Expanding the expression (4.4) and retaining the terms upto order $O(n^{-3/2})$, we have

$$\frac{\bar{y}_r - \bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + d^*_{-1} + d^*_{-\frac{3}{2}}, \tag{4.5}$$

where,

$$\left.\begin{aligned} d^*_{-\frac{1}{2}} &= U_y - U_x, \\ d^*_{-1} &= U_x^2 - U_x U_y, \\ d^*_{-\frac{3}{2}} &= U_x^2 U_y - U_x^3. \end{aligned}\right\} \tag{4.6}$$

Here, the suffixes indicate the order of their magnitude.

Taking expectations of the expression (4.5) upto O (n^{-1}), we have

$$E\left[\frac{\bar{y}_r - \bar{Y}}{\bar{Y}}\right] = E\left(d^*_{-\frac{1}{2}}\right) + E(d^*_{-1}), \tag{4.7}$$

where,

$$\left.\begin{aligned} E\left(d^*_{-\frac{1}{2}}\right) &= 0, \\ E(d^*_{-1}) &= \frac{1}{n}(C_{20} - C_{11}). \end{aligned}\right\} \tag{4.8}$$

We get the expression (4.2) for relative bias, to the order of our approximation after substituting the expression (4.8) in (4.7), of $\bar{y}_r$.

For relative mean squared error of $\bar{y}_r$,

$$E\left[\frac{\bar{y}_r-\bar{Y}}{\bar{Y}}\right]^2 = E\left({d^*_{-\frac{1}{2}}}^2\right) + 2E\left(d^*_{-\frac{1}{2}}d^*_{-1}\right) + E\left({d^*_{-1}}^2 + 2d^*_{-\frac{1}{2}}d^*_{-\frac{3}{2}}\right). \tag{4.9}$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned}
E\left({d^*_{-\frac{1}{2}}}^2\right) &= E\left(U_y^2 + U_x^2 - 2U_yU_x\right)\\
&= \frac{1}{n}(C_{02} + C_{20} - 2C_{11}),\\
E\left(d^*_{-\frac{1}{2}}d^*_{-1}\right) &= E\left(U_x^2U_y - U_xU_y^2 - U_x^3 + U_x^2U_y\right)\\
&= \frac{1}{n^2}(2C_{21} - C_{12} - C_{30}),\\
E\left({d^*_{-1}}^2\right) &= E\left(U_x^4 + U_x^2U_y^2 - 2U_x^3U_x\right)\\
&= \frac{1}{n^2}(3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}),\\
E\left(d^*_{-\frac{1}{2}}d^*_{-\frac{3}{2}}\right) &= E\left(U_x^4 + U_x^2U_y^2 - 2U_x^3U_y\right)\\
&= \frac{1}{n^2}(2C_{11}^2 + 3C_{20}^2 + C_{20}C_{02} - 6C_{20}C_{11}).
\end{aligned}\right\} \tag{4.10}$$

where, terms of higher order than O (n^{-2}) have been dropped. Substituting the expression (4.10) in (4.9) and after a little algebraic simplification, we obtained (4.3) of theorem 4.1.

4.3 The classes of improved ratio type estimators

4.3.1 The following improved class of ratio type estimators for $\bar{Y}$ have been proposed as

$$\boldsymbol{t_1 = \bar{y}_r + \frac{p}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}\bar{X}} + q\,\frac{s_{xy}}{\bar{y}\bar{X}}\right]}, \quad (4.11)$$

where, p and q are scalars specifying the estimator.

Theorem 4.2: The relative bias and relative mean squared error of the estimator t_1 upto order O (n^{-1}) and O (n^{-2}) respectively are as

$$RB(t_1) = RB(\bar{y}_r) + \frac{p}{n}(C_{20} + \ qC_{11}), \quad (4.12)$$

$$RM(t_1) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{20}(C_{20} - 3C_{11} + 2C_{02}) + 2C_{21} - C_{30}] + \frac{pq}{n^2}[C_{11}(C_{20} - C_{11}) + 2C_{12} - 2C_{21}] + \frac{p^2}{n^2}[C_{20} + qC_{11}]^2. \quad (4.13)$$

Proof: Expressing $\bar{y}$, $\bar{x}$, s_x^2 and s_{xy} in terms of U_y, U_x, V_x and W respectively in the estimator t_1, we have

$$\frac{t_1 - \bar{Y}}{\bar{Y}} = \left(1 + U_y\right)(1 + U_x)^{-1} + \frac{p}{n}\left(1 + U_y\right)(1 + U_x)^{-1}(V_x + C_{20}) + \frac{pq}{n}(W + C_{11}). \quad (4.14)$$

Expanding the terms on the right hand side of (4.14) and retaining the terms to order O ($n^{-3/2}$), we have

$$\frac{t_1 - \bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + e^*_{-1} + e^*_{-\frac{3}{2}}, \quad (4.15)$$

where,

$$\left.\begin{aligned}
d^*_{-\frac{1}{2}} &= U_y - U_x;\\
e^*_{-1} &= U_x^2 - U_xU_y + \frac{p}{n}C_{20} + \frac{pq}{n}C_{11};\\
e^*_{-\frac{3}{2}} &= -U_x^3 + U_x^2U_y + \frac{p}{n}\left[V_x - U_xC_{20} + U_yC_{02}\right] + \frac{pq}{n}W.
\end{aligned}\right\} \quad (4.16)$$

Taking expectations of the (4.15) upto O (n^{-1}), we get

$$E\left[\frac{t_1-\bar{Y}}{\bar{Y}}\right] = E\left[d^*_{-\frac{1}{2}}\right] + E[e^*_{-1}], \tag{4.17}$$

where,

$$\left.\begin{aligned} &E\left[d^*_{-\frac{1}{2}}\right] = 0; \\ &E[e^*_{-1}] = \frac{1}{n}(C_{20} - C_{11}) + \frac{p}{n}(C_{20} + qC_{11}). \end{aligned}\right\} \tag{4.18}$$

We get the expression (4.12) for relative bias, to the order of our approximation after substituting the expression (4.18) in (4.17), of t_1

For relative mean squared error of t_1,

$$E\left[\frac{t_1-\bar{Y}}{\bar{Y}}\right]^2 = E\left({d^*_{-\frac{1}{2}}}^2\right) + E\left({e^*_{-1}}^2\right) + 2E\left({d^*_{-\frac{1}{2}}}^2 e^*_{-1}\right) + 2E\left(d^*_{-\frac{1}{2}} e^*_{-\frac{3}{2}}\right). \tag{4.19}$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned} E\left({d^*_{-\frac{1}{2}}}^2\right) &= E\left(U_y^2 + U_x^2 - 2U_yU_x\right) \\ &= \frac{1}{n}(C_{02} + C_{20} - 2C_{11}), \\ E\left({e^*_{-1}}^2\right) &= \frac{1}{n^2}[3C_{20}^2 + 2C_{11} + C_{20}C_{02} - 6C_{20}C_{11}] + \frac{2}{n^2}[pC_{20} + pqC_{11}][C_{20} - C_{11}] \\ &\quad + \frac{1}{n^2}[pC_{20} + pqC_{11}]^2, \\ E\left(d^*_{-\frac{1}{2}} e^*_{-1}\right) &= \frac{1}{n^2}[2C_{21} - C_{12} - C_{30}], \\ E\left(d^*_{-\frac{1}{2}} e^*_{-\frac{3}{2}}\right) &= \frac{1}{n^2}[3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}] \\ &\quad + \frac{p}{n^2}[C_{21} - 2C_{11}C_{20} + 2C_{02}C_{20} - C_{30}] + \frac{pq}{n^2}[C_{12} - C_{21}]. \end{aligned}\right\} \tag{4.20}$$

The terms of order higher than O (n^{-2}) have been dropped. Substituting the results of (4.20) in (4.19) and after algebraic simplification we obtained (4.13) of theorem 4.2.

From (4.12), it is observed that estimator t_1 has smaller bias than conventional ratio type estimator $\bar{y}_r$, if $p < 0$ and $C_{20} + qC_{11} > 0$.

From (4.13) and (4.3), it is observed that the relative mean squared error of both estimators *i.e.*, t_1 and $\bar{y}_r$, are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_1 has smaller relative mean squared error than that of $\bar{y}_r$, if

$$p[2C_{20}(C_{20} - 3C_{11} + 2C_{02}) + 2C_{21} - C_{30}] + pq[C_{11}(C_{20} - C_{11}) + 2C_{12} - 2C_{21}] + [pC_{20} + pqC_{11}]^2 < 0. \tag{4.21}$$

Under bivariate normal distribution, the expression (4.21) reduces to

$$p[2C_{20}(C_{20} - 3C_{11} + 2C_{02})] + p[C_{11}(C_{20} - C_{11})] + \frac{p^2}{n^2}[C_{20} + C_{11}]^2 < 0. \tag{4.22}$$

Sharma (2010) proposed the following ratio type estimator

$$t_s = \bar{y}_r + \frac{1}{n}\bar{y}_p\left[p\frac{s_y^2}{\bar{y}^2} + q\frac{s_{xy}}{\bar{x}\bar{y}}\right]. \tag{4.23}$$

The relative bias and relative mean squared error of t_s are as

$$RB(t_s)=RB(\bar{y}_r) + \frac{1}{n}(pC_{02} + qC_{11}), \tag{4.24}$$

$$RM(t_s) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{03} - 2C_{12} + 2C_{02}C_{11} - 2C_{02}^2] + \frac{q}{n^2}[2C_{20}C_{11} - 2C_{11}^2 + C_{12} - C_{21}] + \frac{1}{n^2}[pC_{02} + qC_{11}]^2. \tag{4.25}$$

The comparison of relative biases of t_1 and t_s showed that t_1 has smaller bias than t_s, if

$$pC_{20} + pqC_{11} < pC_{02} + qC_{11}. \quad (4.26)$$

From (4.13) and (4.25), it is observed that the t_1 has smaller relative mean squared error than t_s, if

$$p[2C_{20}(C_{20} - 3C_{11} + 2C_{02}) + 2C_{21} - C_{30} - 2C_{03}^2 + 2C_{12} - 2C_{02}C_{11} + 2C_{02}^2] + pq[C_{11}(C_{20} - C_{11}) + 2C_{12} - 2C_{21}] + p^2[C_{20} + qC_{11}]^2 + q[2C_{11}^2 - 2C_{20}C_{11} + C_{12} - C_{21}] + [pC_{02} + qC_{11}]^2 < 0. \quad (4.27)$$

Under bivariate normal distribution, expression (4.27) reduces to

$$p[2C_{20}(C_{20} - 3C_{11} + 2C_{02}) + 2C_{21} - C_{30} - 2C_{03}^2 + 2C_{12} - 2C_{02}C_{11} + 2C_{02}^2] + pq[C_{11}(C_{20} - C_{11}) + 2C_{12} - 2C_{21}] + p^2[C_{20} + qC_{11}]^2 + q[2C_{11}^2 - 2C_{20}C_{11} + C_{12} - C_{21}] + [pC_{02} + qC_{11}]^2 < 0. \quad (4.28)$$

4.3.1.1 Proposed Improved Ratio Type Estimators: On the basis of proposed modified class of ratio type estimators, some special cases have been considered:

Case I: Consider p=0 in (4.11), the general class of improved ratio type estimator t_1 will reduce to conventional ratio type estimator. Hence, $t_{1(0,q)} = \bar{y}_r$, is the particular member of proposed class of ratio type estimator t_1.

Case II: Consider q=0 in (4.11), the estimator t_1 will become

$$t_{1(p,0)} = \boldsymbol{t_1} = \boldsymbol{\bar{y}_r} + \boldsymbol{\frac{p}{n}\bar{y}\frac{s_x^2}{\bar{x}\bar{X}}} \quad (4.29)$$

The relative bias and relative mean squared error of the estimator $t_{1(p,0)}$ are as

$$RB(t_{1(p,0)}) = RB(\bar{y}_r) + \frac{p}{n}C_{20}, \quad (4.30)$$

$$RM(t_{1(p,0)}) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{20}^2 - 6C_{20}C_{11} + 4C_{20}C_{02} + 2C_{21} - C_{30} + pC_{20}^2]. \quad (4.31)$$

From (4.30), it can be observed that for $p < 0$, the estimator $t_{1(p,0)}$ has smaller bias than $\bar{y}_r$ upto order O (n^{-1}).

From (4.3) and (4.31), the relative mean squared error of the estimator $t_{1(p,0)}$ and $\bar{y}_r$ are identical upto order O (n^{-1}). Thus,

$$RM(t_{1(p,0)}) - RM(_{\bar{y}_r}) = p[2C_{20}^2 - 6C_{20}C_{11} + 4C_{20}C_{02} + 2C_{21} - C_{30} + pC_{20}^2]. \qquad (4.32)$$

From (4.32), the proposed estimator $t_{1(p,0)}$ is more efficient than $\bar{y}_r$, if

$$2C_{20}^2 - 6C_{20}C_{11} + 4C_{20}C_{02} + 2C_{21} - C_{30} + pC_{20}^2 > 0 \text{ and } p < 0. \qquad (4.33)$$

For bivariate normal population, the expression (4.33) gives

$$\rho < \frac{4\theta^2+1}{6\theta} \text{ and } 0.200 < \theta < 0.500. \qquad (4.34)$$

Similarly, the estimator (4.23) reduces to $t_{s(p,0)}$.

It is observed that the relative bias of $t_{1(p,0)}$ is smaller than estimator $t_{s(p,0)}$ upto order O (n^{-1}), if $C_{20} > C_{02}$ and $p < 0$.

Further, estimator $t_{1(p,0\,)}$ will be more efficient than $t_{s(p,o)}$, if

$$p < 0 \text{ and } \quad \rho < \frac{3\theta^4+4\theta^2+1}{2\theta^3+6\theta} \text{ which results to } \theta \text{ lies between } 0.462 < \theta < 0.999. \quad (4.35)$$

From (4.34) and (4.35), it has been concluded that the proposed class of estimators will be more efficient than $\bar{y}_r$ and $t_{s(p,0)}$, if $0.462 < \theta < 0.500$.

Case III: For $p < 0$ and $q < 0$, consider the values of p and q as p= -1 and q= -1, the estimator t_1 will be as

$$t_{1(-1,-1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}\bar{X}} - \frac{s_{xy}}{\bar{X}\bar{y}}\right]. \qquad (4.36)$$

The proposed estimator is unbiased.

The relative variance of the estimator $t_{1(-1,-1)}$ will be

$$RV\left[t_{1(-1,-1)}\right] = RM(\bar{y}_r) + \frac{1}{n^2}\left[5C_{20}C_{11} - C_{20}^2 - 4C_{20}C_{02} + 2C_{11}^2 + C_{30} - 2C_{12}\right]. \quad (4.37)$$

Under bivariate normal distribution, the expression (4.37) reduces to

$$RV\left[t_{1(-1,-1)}\right] = RM(\bar{y}_r) + \frac{1}{n^2}\left[5C_{20}C_{11} - C_{20}^2 - 4C_{20}C_{02} + 2C_{11}^2\right]. \quad (4.38)$$

From (4.38) and (4.3), it can be observed that the estimator $t_{1(-1,-1)}$ has smaller variance than $\bar{y}_r$, if $\rho < \frac{2\theta^2+1}{5\theta}$ which results to θ lies between $0.711 < \theta < 2.2801$. (4.39)

Further, the estimator $t_{1(-1,-1)}$ performed better than $t_{s(-1,-1)}$ upto order O (n^{-2}), if

$$7C_{20}C_{11} - 4C_{20}C_{02} - C_{20}^2 - 3C_{02}^2 - C_{11}^2 + C_{30} + 2C_{03} - 2C_{12} - 2C_{21} < 0. \quad (4.40)$$

Under bivariate normal distribution, the expression (4.40) becomes

$$7C_{20}C_{11} - 4C_{20}C_{02} - C_{20}^2 - 3C_{02}^2 - C_{11}^2 < 0. \quad (4.41)$$

Thus, the estimator $t_{1(-1,-1)}$ will be more efficient than $t_{s(-1,-1)}$, if

$\rho < \frac{2\theta^4+5\theta^2+1}{7\theta}$ which results in θ lies between $0.407 < \theta < 0.891$.

Thus, it can be concluded than the proposed estimator $t_{1(-1,-1)}$ will be more efficient than $\bar{y}_r$ and $t_{1(-1,-1)}$, if $0.711 < \theta < 0.891$.

Case IV: For $p < 0$ and $q > 0$, consider the values of p and q as p=-1 and q=1**,** the estimator t_1 will become

$$t_{1(-1,1)} = \boldsymbol{\bar{y}_r} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\boldsymbol{\bar{y}}\left[\frac{\boldsymbol{s_x^2}}{\boldsymbol{\bar{x}\bar{X}}} + \frac{\boldsymbol{s_{xy}}}{\boldsymbol{\bar{y}\bar{X}}}\right]. \quad (4.42)$$

The relative bias and relative mean squared error of the estimator $t_{1(-1,1)}$ are as

$$RB\left(t_{1(-1,1)}\right) = RB(\bar{y}_r) - \frac{1}{n}(C_{20} + C_{11}), \tag{4.43}$$

$$RM\left(t_{1(-1,1)}\right) = RM(\bar{y}_r) + \frac{1}{n^2}[7C_{20}C_{11} - C_{20}^2 - 4C_{20}C_{02} + 2C_{11}^2 + C_{30} - 2C_{12}]. \tag{4.44}$$

From (4.2) and (4.43), it is observed that the relative bias of $t_{1(-1,1)}$ is smaller bias than that of estimator $\bar{y}_r$ upto order O (n^{-1}), if $C_{20} > -C_{11}$.

From (4.3) and (4.44), it is observed that the relative mean squared error of the estimator $\bar{y}_r$ is identical to the estimator $t_{1(-1,1)}$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

$$\text{Thus, } 7C_{20}C_{11} - C_{20}^2 - 4C_{20}C_{02} + 2C_{11}^2 + C_{30} - 2C_{12} < 0. \tag{4.45}$$

For bivariate normal distribution, the expression (4.45) reduces to

$$7C_{20}C_{11} - C_{20}^2 - 4C_{20}C_{02} + 2C_{11}^2 < 0. \tag{4.46}$$

From (4.45), the proposed estimator $t_{1(-1,1)}$ is more efficient than $\bar{y}_r$, if

$$\rho > \frac{2\theta^2+1}{7\theta} \text{ which results to } 0.70 < \theta < 3.35. \tag{4.47}$$

For p= -1 and q=1, the estimator t_s becomes

$$t_{s(-1,1)} = \bar{y}_r + \frac{1}{n}\bar{y}_p\left[\frac{s_{xy}}{\bar{x}\bar{y}} - \frac{s_y^2}{\bar{y}^2}\right]. \tag{4.48}$$

The relative bias and relative mean squared error of the estimator $t_{s(-1,1)}$ are as

$$RB\left(t_{s(-1,1)}\right)=RM(\bar{y}_r) + \frac{1}{n}(C_{11} - C_{02}), \tag{4.49}$$

$$RM\left(t_{s(-1,1)}\right) = RM(\bar{y}_r) - 4C_{02}C_{11} + 3C_{02}^2 + 2C_{20}C_{11} - C_{11}^2 - 2C_{03} + 4C_{12} - C_{21}. \tag{4.50}$$

From (4.43) and (4.49), it is observed that the relative bias of estimator $t_{1(-1,1)}$ is smaller than $t_{s(-1,1)}$, if $\rho < \frac{\theta^2-1}{2\theta}$ which results to θ lies between $1.001 < \theta < 2.414$. (4.51)

From (4.44) and (4.50), it is found that estimator $t_{1(-1,1)}$ is more efficient than $t_{s(-1,1)}$, if

$\rho < \frac{3\theta^4+\theta^2+1}{4\theta^3+5\theta}$ which results to $0.63 < \theta < 1.65$. (4.52)

Thus, it can be observed from (4.51) and (4.52) that the estimator $t_{1(-1,1)}$ will perform better than $t_{s(-1,1)}$ and $\bar{y}_r$, if θ lies between $0.70 < \theta < 1.65$. (4.53)

4.3.2 The following improved class of ratio type estimators for estimating the population mean $\bar{Y}$ have been proposed as

$$\boldsymbol{t_2 = \bar{y}_r + \frac{p}{n}\bar{y}\left[\frac{\bar{x}s_x^2}{\bar{X}^3} + q\,\frac{s_{xy}}{\bar{y}\bar{X}}\right]}, \tag{4.54}$$

where, p and q are scalars specifying the estimators.

Theorem 4.3: The relative bias and relative mean squared error of the estimator t_2 upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB(t_2) = RB(\bar{y}_r) + \frac{p}{n}(C_{20} + qC_{11}), \tag{4.55}$$

$$RM(t_2) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{21} + 2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{30}] + \frac{pq}{n^2}[2C_{20}C_{11} - 2C_{11}^2 + 2C_{12} - 2C_{21}] + \frac{1}{n^2}[pC_{20} + pqC_{11}]^2. \tag{4.56}$$

Proof: Expressing $\bar{y}$, $\bar{x}$, s_x^2 and s_{xy} in terms of U_y, U_x, V_x and W respectively in the estimator t_2, we have

$$\frac{t_2-\bar{Y}}{\bar{Y}} = \left(1+U_y\right)(1+U_x)^{-1} + \frac{p}{n}\left(1+U_y\right)(1+U_x)(V_x+C_{20}) + \frac{pq}{n}(W+C_{11}) - 1. \tag{4.57}$$

Expanding the terms on the right hand side of (4.57) and retaining the terms to order O ($n^{-3/2}$), we have $\frac{t_2-\bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + e^*_{-1} + f^*_{-\frac{3}{2}}$, (4.58)

where,

$$\left.\begin{aligned} d^*_{-\frac{1}{2}} &= U_y - U_x, \\ e^*_{-1} &= U_x^2 - U_xU_y + \frac{p}{n}C_{20} + \frac{pq}{n}C_{11}, \\ f^*_{-\frac{3}{2}} &= -U_x^3 + U_x^2U_y + \frac{p}{n}\left(V_x + U_yC_{20}\right) + \frac{pq}{n}W. \end{aligned}\right\} \tag{4.59}$$

Taking expectation of (4.58) to both sides upto order O (n^{-1}), we obtain the relative bias of estimator t_2 as in expression (4.55).

For relative mean squared error to order O (n^{-2}) can be obtained by taking the expectations of square on both sides of expression (4.58), we find that

$$E\left(\frac{t_2-\bar{Y}}{\bar{Y}}\right)^2 = E\left[{d^*_{-\frac{1}{2}}}^2 + {e^*_{-1}}^2 + 2d_{-\frac{1}{2}}{}^*e_{-1}{}^* + 2d^*_{-\frac{1}{2}}f^*_{-\frac{3}{2}}\right], \tag{4.60}$$

where,

$$\left.\begin{aligned} E\left({d^*_{-\frac{1}{2}}}^2\right) &= \frac{1}{n^2}[C_{02} + C_{20} - 2C_{11}], \\ E\left({e^*_{-1}}^2\right) &= \frac{1}{n^2}[3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}] + \frac{2}{n^2}[pC_{20} + pqC_{11}][C_{20} - C_{11}] \end{aligned}\right\}$$

$$\left.\begin{aligned}
&\qquad\qquad + \frac{1}{n^2}[pC_{20} + pqC_{11}]^2, \\
&E\left(d^*_{-\frac{1}{2}} e^*_{-1}\right) = \frac{1}{n^2}[2C_{21} - C_{12} - C_{30}], \\
&E\left(d^*_{-\frac{1}{2}} f^*_{-\frac{3}{2}}\right) = \frac{1}{n^2}[6C_{20}^2 + 4C_{11}^2 + 2C_{20}C_{02} - 12C_{20}C_{11}] \\
&\qquad\qquad + \frac{p}{n^2}[2C_{12} + 2C_{02}C_{20} - 2C_{30} - 2C_{11}C_{20}] + \frac{pq}{n^2}[C_{12} - C_{21}].
\end{aligned}\right\} \tag{4.61}$$

By using (4.61) in (4.60) and upon simple algebraic calculations, the relative mean squared error of the estimator t_2 can be obtained.

From (4.55) and (4.2), it can be observed that the estimator t_2 has smaller bias than conventional ratio type estimator $\bar{y}_r$, if the following conditions are satisfied

$p < 0$ and $C_{20} + qC_{11} > 0$.

From (4.56) and (4.3), it is seen that the estimator t_2 and $\bar{y}_r$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of O (n^{-2}).

Now, comparison of relative mean squared error of t_2 and $\bar{y}_r$, upto order $O(n^{-2})$ showed that estimator t_2 will be more efficient than $\bar{y}_r$, if

$$p[2C_{21} + 2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{30}] + pq[2C_{20}C_{11} - 2C_{11}^2 + 2C_{12} - 2C_{21}] + p^2[C_{20} + qC_{11}]^2 < 0. \tag{4.62}$$

Under bivariate normal distribution, expression (4.62) becomes

$$p[2C_{02}C_{20} - 2C_{20}C_{11}] + pq[2C_{20}C_{11} - 2C_{11}^2] + p^2[C_{20} + qC_{11}]^2 < 0. \tag{4.63}$$

From (4.55) and (4.12), it is observed that both the estimators t_2 and t_1 have identical relative biases upto order O (n^{-1}).

From (4.56) and (4.13), it is seen that the estimator t_2 and t_1 have identical relative mean squared error upto order O (n^{-1}). They differ in terms of O (n^{-2}).

Further, the comparison of relative mean squared error of t_2 and t_1 showed that t_2 is more efficient than $t_{1,}$ if

$$p[4C_{20}C_{11} - 2C_{20}C_{02} - 2C_{20}^2 - C_{30}] + pq[C_{20}C_{11} - C_{11}^2] < 0. \qquad (4.64)$$

For bivariate normal population, the expression (4.64) reduces to

$$p[4C_{20}C_{11} - 2C_{20}C_{02} - 2C_{20}^2] + pq[C_{20}C_{11} - C_{11}^2] < 0. \qquad (4.65)$$

The expressions (4.55) and (4.24) showed that estimator t_2 has smaller bias than estimator t_s , if $pC_{20} + pqC_{11} < pC_{02} + qC_{11}$.

From (4.56) and (4.25), it is seen that the estimator t_2 and t_s have identical relative mean squared error upto order O (n^{-1}). They differ in terms of O (n^{-2}).

Further, the comparison of relative mean squared error of t_2 and t_s showed that t_2 is more efficient than t_s, if

$$p[2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{30} + 2C_{21} - C_{03}^2 + 2C_{12} - 2C_{02}C_{11} + C_{02}^2] + q[2C_{20}C_{11} - 2C_{11}^2 + 2C_{12} - 2C_{21}] + pq[2C_{11}^2 - 2C_{20}C_{11} - C_{12} + C_{21}] + p^2[C_{20} + qC_{11}]^2 - [pC_{02} + qC_{11}]^2 < 0. \qquad (4.66)$$

Under bivariate normal distribution, expression (4.66) reduces to

$$p[2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{02}C_{11} + C_{02}^2] + q[2C_{20}C_{11} - 2C_{11}^2] + pq[2C_{11}^2 - 2C_{20}C_{11}] + p^2[C_{20} + qC_{11}]^2 - [pC_{02} + qC_{11}]^2 < 0. \qquad (4.67)$$

4.3.2.1 Under different conditions of p and q scalars of proposed ratio type estimators, special cases are as:

Case I: If $p = 0$, the proposed ratio type estimator is $\bar{y}_r$. Thus, $\bar{y}_r$ is a particular member of the proposed class of ratio estimators.

Case II: Consider q=0 in (4.54), the estimator t_2 reduces to $t_{2(p,0)} = \boldsymbol{\bar{y}_r + \frac{p}{n}\bar{y}\frac{\bar{x}s_x^2}{\bar{X}^3}}$. (4.68)

The relative bias and relative mean squared error of the estimator t_2 upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB\left(t_{2(p,0)}\right) = RB(\bar{y}_r) + \frac{p}{n}C_{20}, \tag{4.69}$$

$$RM\left(t_{2(p,0)}\right) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{21} + 2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{30} + pC_{20}^2]. \tag{4.70}$$

From (4.69) and (4.2), it is observed that the relative bias of $t_{2(p,0)}$ is smaller than that of estimator $\bar{y}_r$ upto order O (n^{-1}), if $p < 0$.

From (4.70) and (4.3), the relative mean squared error of the estimator $\bar{y}_r$ is identical to the estimator $t_{2(p,0)}$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(p,0)}$ is more efficient than $\bar{y}_r$ upto order O(n^{-2}), if

$$p[2C_{21} + 2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{30} + pC_{20}^2] < 0. \tag{4.71}$$

Under bivariate normal distribution, expression (4.71) reduces to

$$p[2C_{02}C_{20} - 2C_{20}C_{11} + pC_{20}^2] < 0. \tag{4.72}$$

Thus, the estimator $t_{2(p,\,o)}$ will be more efficient than $\bar{y}_r$, if $p < 0$ and $\rho < \frac{2\theta^2-1}{2\theta}$ which results to θ lies between $0.710 < \theta < 1.366$.

From (4.69) and (4.30), it is found that the estimator $t_{2(p,0)}$ and $t_{1(p,0)}$ have identical bias upto order O (n^{-1}).

Further, the comparison of (4.70) and (4.31) showed that the estimator $t_{2(p,0)}$ and $t_{1(p,0)}$ are equally efficient upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Thus, estimator $t_{2(p,0)}$ is more efficient than $t_{1(p,0)}$, if

$$p[4C_{20}C_{11} - 2C_{20}C_{02} - 2C_{20}^2 - C_{30}] < 0. \quad (4.73)$$

Under bivariate normal distribution, expression (4.73) reduces to

$[4C_{20}C_{11} - 2C_{20}C_{02} - 2C_{20}^2] < 0$ which results to $p < 0$ and $\theta^2 > 1$. (4.74)

It has been found that the estimator $t_{2(p,0)}$ have smaller bias than $t_{s(p,0)}$, if $p < 0$ and $C_{20} > C_{02}$.

The comparison of the estimators $t_{2(p,0)}$ and $t_{s(p,0)}$ showed that both are equally efficient upto order O (n^{-1}) but for order O (n^{-2}), estimator $t_{2(p,0)}$ is more efficient than $t_{s(p,0)}$, if

$$p[2C_{21} + 2C_{02}C_{20} - 2C_{20}C_{11} - 2C_{30} + pC_{20}^2 - 2C_{03}^2 + 2C_{12} - 2C_{02}C_{11} + 2C_{02}^2 - pC_{02}^2] < 0. \quad (4.75)$$

For bivariate normal population, expression (4.75) reduces to

$$p[2C_{02}C_{20} - 2C_{20}C_{11} + pC_{20}^2 - 2C_{02}C_{11} + 2C_{02}^2 - pC_{02}^2] < 0. \quad (4.76)$$

which gives $p < 0$ and $0.581 < \theta < 1.00$.

Thus, it can be concluded that the proposed estimator $t_{2(p,0)}$ will be better than $\bar{y}_r$, $t_{s(p,0)}$ and $t_{1(p,0)}$ upto order O (n^{-2}), if $1.000 < \theta < 1.366$.

Case III: For $p < 0$ and $q < 0$, consider the scalars p and q as p= -1 and q= -1, the estimator t_2 will reduce to $t_{2(-1,-1)} = \boldsymbol{\bar{y}_r} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\boldsymbol{\bar{y}}\left[\frac{\bar{x}s_x^2}{\bar{X}^3} - \frac{s_{xy}}{\bar{y}\bar{X}}\right]$. (4.77)

The estimator is unbiased and the relative variance of the estimator $t_{2(-1,-1)}$ will be

$$RV\left(t_{2(-1,-1)}\right) = RM(\bar{y}_r) - 2C_{02}C_{20} + 2C_{20}C_{11} - C_{11}^2 + C_{20}^2 + 2C_{30} - 4C_{12} + 2C_{21}. \tag{4.78}$$

Under bivariate normal population, the above expression (4.78) reduces to

$$RV\left(t_{2(-1,-1)}\right) = RM(\bar{y}_r) - 2C_{02}C_{20} + 2C_{20}C_{11} - C_{11}^2 + C_{20}^2. \tag{4.79}$$

From (4.79) and (4.3), it can be seen that the estimator $t_{2(-1,-1)}$ is more efficient than $\bar{y}_r$, if $\rho < \frac{3\theta^2-1}{2\theta}$ and the value of θ should lie between $0.578 < \theta < 1.000$.

Further, comparison of (4.78) and (4.37) showed that estimator $t_{2(-1,-1)}$ and $t_{1(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(-1,-1)}$ is more efficient than $t_{1(-1,-1)}$, if

$$2C_{20}C_{02} - 3C_{20}C_{11} - 3C_{11}^2 + 2C_{20}^2 + C_{30} - 2C_{12} + 2C_{21} < 0. \tag{4.80}$$

Under bivariate normal distribution, expression (4.80) reduces to

$$2C_{20}C_{02} - 3C_{20}C_{11} - 3C_{11}^2 + 2C_{20}^2 < 0. \tag{4.81}$$

Thus, the estimator $t_{2(-1,-1)}$ will perform better than $t_{1(-1,-1)}$, if

$\rho > \frac{2-\theta^2}{3\theta}$ which results to θ lies between $0.562 < \theta < 1.414$.

Further, it has been be observed that estimator $t_{2(-1,-1)}$ and $t_{s(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(-1,-1)}$ is more efficient than $t_{s(-1,-1)}$, if

$$4C_{20}C_{11} - 2C_{20}C_{02} - 4C_{11}^2 + C_{20}^2 - 3C_{02}^2 + 2C_{30} + 2C_{03} - 4C_{12} < 0. \quad (4.82)$$

Under bivariate normal population, the expression (4.82) reduces to

$$4C_{20}C_{11} - 2C_{20}C_{02} - 4C_{11}^2 + C_{20}^2 < 0. \quad (4.83)$$

Thus, the estimator $t_{2(-1,-1)}$ will perform better than $t_{s(-1,-1)}$, if

$\rho > \frac{3\theta^4 - 2\theta^2 - 1}{4\theta^3}$ and the value of θ lies between $1.0001 < \theta < 1.77$.

Thus, it can be concluded that the estimator $t_{2(-1,-1)}$ will be more efficient than $\bar{y}_r$ and $t_{1(-1,-1)}$, if $0.578 < \theta < 1.000$.

Case IV: For $p < 0$ and $q > 0$, consider p=-1 and q=1, the estimator t_2 will become

$$t_{2(-1,1)} = \bar{\boldsymbol{y}}_{\boldsymbol{r}} - \frac{\boldsymbol{1}}{\boldsymbol{n}}\bar{\boldsymbol{y}}\left[\frac{\bar{\boldsymbol{x}}\boldsymbol{s}_{\boldsymbol{x}}^{\boldsymbol{2}}}{\bar{\boldsymbol{X}}^{\boldsymbol{3}}} + \frac{\boldsymbol{s}_{\boldsymbol{xy}}}{\bar{\boldsymbol{y}}\bar{\boldsymbol{X}}}\right] \quad (4.84)$$

The relative bias and relative mean squared error of the estimator $t_{2(-1,1)}$ are as

$$RB\left(t_{2(-1,1)}\right) = RB(\bar{y}_r) - \frac{1}{n}(C_{20} + C_{11}), \quad (4.85)$$

$$RM\left(t_{2(-1,1)}\right) = RM(\bar{y}_r) + \frac{1}{n^2}[3C_{11}^2 + C_{20}^2 + 2C_{20}C_{11} - 2C_{20}C_{02} + 2C_{30} - 2C_{12}]. \quad (4.86)$$

From (4.85) and (4.2), it is observed that the relative bias of $t_{2(-1,1)}$ is smaller than that of estimator $\bar{y}_r$ upto order O (n^{-1}), if $C_{20} > -C_{11}$.

From (4.86) and (4.3), the relative mean squared error of the estimator $\bar{y}_r$ is identical to the estimator $t_{2(-1,1)}$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(-1,1)}$ is more efficient than $\bar{y}_r$, if

$$3C_{11}^2 + C_{20}^2 + 2C_{20}C_{11} - 2C_{20}C_{02} + 2C_{30} - 2C_{12} < 0. \tag{4.87}$$

For bivariate normal distribution, the expression (4.87) becomes

$$3C_{11}^2 + C_{20}^2 + 2C_{20}C_{11} - 2C_{20}C_{02} < 0 \tag{4.88}$$

which gives $\theta^2 + 1 > 2\rho\theta$ which is true for $\theta > 0$.

From (4.85) and (4.43), it is observed that estimator $t_{2(-1,1)}$ and $t_{1(-1,1)}$ have identical relative bias upto order O (n^{-1}).

Further, comparison of (4.86) and (4.44) showed that estimator $t_{2(-1,1)}$ and $t_{1(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(-1,1)}$ is more efficient than $t_{1(-1,1)}$, if

$$C_{11}^2 + 2C_{20}^2 - 5C_{20}C_{11} + 2C_{20}C_{02} + C_{30} < 0. \tag{4.89}$$

Under bivariate normal distribution, expression (4.89) yields the following results

$\rho > \frac{3\theta^2+2}{5\theta}$ which results to θ lies between $0.82 < \theta < 1.00$.

From (4.85) and (4.49), it is observed that that estimator $t_{2(-1,1)}$ have smaller relative bias upto order O (n^{-1}) than $t_{s(-1,1)}$, if $C_{20} + 2C_{11} > C_{02}$.

Further, from (4.86) and (4.50), it can be observed that estimator $t_{2(-1,1)}$ and $t_{s(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(-1,1)}$ is more efficient than $t_{s(-1,1)}$, if

$$4C_{11}^2 + C_{20}^2 - 3C_{02}^2 - 2C_{20}C_{02} + 4C_{02}C_{11} - 2C_{30} + 2C_{03} - 6C_{12} + C_{21} < 0 \tag{4.90}$$

For bivariate normal population, the expression (4.90) gives

$$\rho > \frac{3\theta^4-2\theta^2-1}{4\theta^3} \text{ and } \quad 1.0001 < \theta < 1.77. \tag{4.91}$$

Thus, it can be concluded that the estimator $t_{2(-1,1)}$ will be more efficient than $\bar{y}_r$, $t_{1(-1,1)}$ and $t_{s(-1,1)}$, if θ ≥1.000.

4.3.3 The next improved class of ratio type estimators for estimation of population mean $\bar{Y}$ have been proposed as

$$\boldsymbol{t_3 = \bar{y}_r + \frac{p}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}^2} + q\,\frac{s_{xy}}{\bar{x}\bar{y}}\right]}, \tag{4.92}$$

where, p and q are the scalars specifying the estimator.

Theorem 4.4: The relative bias and relative mean squared error of the estimator t_3 upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB(t_3) = RB(\bar{y}_r) + \frac{p}{n}[C_{20} + qC_{11}], \tag{4.93}$$

$$RM(t_3) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{20}(3C_{20} - 4C_{11} + C_{02}) + 2(C_{21} - C_{30}] + \frac{pq}{n^2}[4C_{11}(C_{20} - C_{11}) + 2(C_{12} - C_{21})] + \frac{1}{n^2}[pC_{20} + pqC_{11}]^2. \tag{4.94}$$

Proof: Expressing $\bar{y}$, $\bar{x}$, s_x^2 and s_{xy} in terms of U_y, U_x, V_x and W respectively, in the estimator $\bar{y}_r$, we have

$$\frac{t_3-\bar{Y}}{\bar{Y}} = (1+U_y)(1+U_x)^{-1} + \frac{p}{n}(1+U_y)(1+U_x)^{-2}(V_x + C_{20}) + \frac{pq}{n}(1+U_x)^{-1}(W + C_{11}) - 1. \tag{4.95}$$

Expanding the terms on the right hand side of (4.95) and retaining the terms to order O ($n^{-3/2}$), we have

$$\frac{t_3-\bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + e^*_{-1} + g^*_{-\frac{3}{2}}, \tag{4.96}$$

where,

$$\left.\begin{aligned} d^*_{-\frac{1}{2}} &= U_Y - U_X, \\ e^*_{-1} &= U_x^2 - U_x U_y + \tfrac{p}{n} C_{20} + \tfrac{pq}{n} C_{11}, \\ g^*_{-\frac{3}{2}} &= -U_x^3 + U_x^2 U_y + \frac{p}{n}\left(V_x - 2U_x C_{20} + U_y C_{20}\right) + \frac{pq}{n}(W - U_x). \end{aligned}\right\} \quad (4.97)$$

Taking expectation of (4.96) on both sides upto order O (n^{-1}), we obtain the relative bias of estimator t_3.

For relative mean squared error of t_3 to order O (n^{-2}), we find that

$$E\left(\frac{t_3-\bar{Y}}{\bar{Y}}\right)^2 = E\left[{d^*_{-\frac{1}{2}}}^2 + {e^*_{-1}}^2 + 2d_{-\frac{1}{2}}{}^* e_{-1}{}^* + 2d^*_{-\frac{1}{2}} f^*_{-\frac{3}{2}}\right], \quad (4.98)$$

where,

$$\left.\begin{aligned} E\left({d^*_{-\frac{1}{2}}}^2\right) &= \tfrac{1}{n^2}[C_{20} + C_{02} - 2C_{11}], \\ E\left({e^*_{-1}}^2\right) &= \tfrac{1}{n^2}[3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}] \\ &\quad + \tfrac{1}{n^2}[pC_{20} + pqC_{11}][2C_{20} - 2C_{11}] + \tfrac{1}{n^2}[pC_{20} + pqC_{11}]^2, \\ 2E\left(d^*_{-\frac{1}{2}} e^*_{-1}\right) &= \tfrac{1}{n^2}[4C_{21} - 2C_{12} - 2C_{30}], \\ 2E\left(d^*_{-\frac{1}{2}} f^*_{-\frac{3}{2}}\right) &= \tfrac{1}{n^2}[6C_{20}^2 + 4C_{11}^2 + 2C_{20}C_{02} - 12C_{20}C_{11}] \\ &\quad + \frac{p}{n^2}[4C_{20}^2 - 6C_{20}C_{11} + 2C_{02}C_{20} + 2C_{21} - 2C_{30}] \\ &\quad + \tfrac{pq}{n^2}[2C_{20}C_{11} - 2C_{11}^2 + 2C_{12} - 2C_{21}]. \end{aligned}\right\} \quad (4.99)$$

By substituting the expression (4.99) in (4.98), relative mean squared error of estimator t_3 can be obtained.

From (4.93), it is observed that the estimator t_3 has smaller bias than conventional ratio type estimator $\bar{y}_r$, if $p < 0$ and $C_{20} + qC_{11} > 0$.

From (4.94) and (4.3), it is observed that relative mean squared error of both the estimators t_3 and $\bar{y}_r$ are identical upto order O (n^{-1}).

Now comparing the relative mean squared error of both estimators upto order O (n^{-2}), we find that the estimator t_3 has smaller relative mean squared error than that of $\bar{y}_r$, if

$$p[2C_{20}(3C_{20} - 4C_{11} + C_{02}) + 2(C_{21} - C_{30}] + pq[4C_{11}(C_{20} - C_{11}) + 2(C_{12} - C_{21})] + [pC_{20} + pqC_{11}]^2 < 0. \quad (4.100)$$

Under bivariate normal distribution, expression (4.100) reduces to

$$p[2C_{20}(3C_{20} - 4C_{11} + C_{02})] + pq[4C_{11}(C_{20} - C_{11})] + [pC_{20} + pqC_{11}]^2 < 0. \quad (4.101)$$

From (4.93) and (4.12), it can be observed that estimator t_3 and t_1 has identical bias upto order O (n^{-1}).

Further, the comparison of (4.94) and (4.13) showed that the estimator t_3 has smaller relative mean squared error than t_1, if

$$p[4C_{20}^2 - 2C_{20}C_{11} - 2C_{20}C_{02}] + pq[3C_{20}C_{11} - 3C_{11}^2] < 0. \quad (4.102)$$

From (4.93) and (4.55), it can be seen that estimators t_3 and t_2 has identical bias upto order O (n^{-1}).

Further, (4.94) and (4.66) showed that estimator t_3 performs better than t_2 upto order O (n^{-2}), if $p[6C_{20}^2 - 6C_{20}C_{11}] + pq[2C_{20}C_{11} - 2C_{11}^2] < 0$. (4.103)

Further, the estimator t_3 has smaller bias than estimator t_s , if

$$pC_{20} + pqC_{11} - pC_{02} - qC_{11} < 0. \quad (4.104)$$

It has been observed that the estimator t_3 and t_s have identical relative mean squared error upto order O (n^{-1}). They differ in terms of O (n^{-2}).

Further, the comparison of relative mean squared error of t_3 and t_s showed that t_3 is more efficient than t_s, if following condition hold

$$p[6C_{20}^2 - 8C_{20}C_{11} + 2C_{20}C_{02} - 2C_{02}C_{11} + 2C_{02}^2 + 2C_{21} + 2C_{12} - 2C_{30} - 2C_{03}] + pq[4C_{20}C_{11} - 4C_{11}^2 + 2C_{12} - 2C_{21}] + q[2C_{11}^2 - 2C_{20}C_{11} - C_{12} + C_{21}] + [pC_{20} + pqC_{11}]^2 - [pC_{02} + qC_{11}]^2 < 0. \tag{4.105}$$

Under bivariate normal distribution, the expression (4.105) reduces to

$$p[6C_{20}^2 - 8C_{20}C_{11} + 2C_{20}C_{02} - 2C_{02}C_{11} + 2C_{02}^2] + pq[4C_{20}C_{11} - 4C_{11}^2] + q[2C_{11}^2 - 2C_{20}C_{11} - C_{12} + C_{21}] + [pC_{20} + pqC_{11}]^2 - [pC_{02} + qC_{11}]^2 < 0. \tag{4.106}$$

4.3.3.1: On the basis of proposed modified class of ratio type estimators, some special cases have been considered.

Case I: Consider p=0, $t_{(0,q)} = \bar{y}_r$ i.e., $\bar{y}_r$ is a particular member of the proposed modified class of ratio type estimator t_3.

Case II: Putting q=0 in **(**4.92**),** the estimator t_3 becomes

$$t_{3(p,0)} = \bar{y}_r + \frac{p}{n}\bar{y}\frac{s_x^2}{\bar{x}^2}. \tag{4.107}$$

The relative bias and relative mean squared error of the estimator $t_{3(p,0)}$ are as

$$RB\left(t_{3(p,0)}\right) = RB(\bar{y}_r) + \frac{p}{n}C_{20}, \tag{4.108}$$

$$RM\left(t_{3(p,0)}\right) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{20}(3C_{20} - 4C_{11} + C_{02}) + 2(C_{21} - C_{30}) + pC_{20}^2]. \tag{4.109}$$

From (4.108), it is observed that the relative bias of the estimator $t_{3(p,0)}$ is smaller than the estimator $\bar{y}_r$ if $p < 0$.

From (4.109), the relative mean squared error of the estimator $t_{3(p,0)}$ is smaller than the conventional ratio type estimator $\bar{y}_r$, if

$$pC_{20}^2 + 2C_{20}C_{02} + 6C_{20}^2 + 2C_{21} > 8C_{20}C_{11} + 2C_{30} \text{ and } p < 0. \qquad (4.110)$$

For bivariate normal population (4.110) reduces to

$$pC_{20}^2 + 2C_{20}C_{02} + 6C_{20}^2 > 8C_{20}C_{11} \text{and } p < 0, \qquad (4.111)$$

Thus, the proposed class of estimator $t_{3(p,0)}$ will be more efficient than $\bar{y}_r$, if $2\rho - 1.225 < \theta < 2\rho + 1.225$.

From (4.108), (4.30) and (4.69), it can be seen that the estimators $t_{3(p,0)}$, $t_{1(p,0)}$ and $t_{2(p,0)}$ have identical bias upto order O (n^{-1}).

From (4.109) and (4.31), it is observed that $t_{3(p,0)}$ is more efficient than $t_{1(p,0)}$ if $\theta > 0.5\rho - 1.5$.

From (4.109) and (4.68), it has been shown that the estimator $t_{3(p,0)}$ was more efficient than $t_{2(p,0)}$, if $p < 0$ and $C_{20} > C_{11}$.

Further, the comparison of relative bias of estimators $t_{3(p,0)}$ and $t_{s(p,0)}$ have showed that estimator $t_{3(p,0)}$ have smaller bias than $t_{s(p,0)}$, if $C_{20} > C_{02}$ and $p < 0$.

The comparison of relative mean squared error of $t_{3(p,0)}$ and $t_{s(p,0)}$ showed that $t_{3(p,0)}$ is more efficient than $t_{s(p,0)}$, if

$$p < 0 \text{ and } 6C_{20}^2 - 8C_{20}C_{11} + 2C_{20}C_{02} - 2C_{02}C_{11} + 2C_{02}^2 + pC_{20}^2 - pC_{02}^2 > 0. \qquad (4.112)$$

Thus, $t_{3(p,0)}$ is more efficient than $t_{s(p,0)}$, if $\rho < \frac{3\theta^4+2\theta^2+5}{2\theta^3+8\theta}$ which results to θ lies between $0.918 < \theta < 1.000$.

Case III: For $p < 0$ and $q < 0$, Consider the scalars p and q as p= -1 and q= -1, the estimator t_3 will reduce to $t_{3(-1,-1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}^2} - \frac{s_{xy}}{\bar{x}\bar{y}}\right]$. (4.113)

It has been observed that the bias of the estimator $t_{3(-1,-1)}$ is zero. Thus, it is unbiased.

The relative variance of the estimator $t_{3(-1,-1)}$ upto order O (n^{-2}) is as

$$RV\left(t_{3(-1,-1)}\right) = RM(\bar{y}_r) - 5C_{20}^2 - 3C_{11}^2 + 10C_{20}C_{11} - 2C_{20}C_{02} - 4C_{21} + 2C_{12} + 2C_{30}. \quad (4.114)$$

Both the estimators have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, the estimator $t_{3(-1,-1)}$ is more efficient than $\bar{y}_r$ upto order O (n^{-2}), if

$$10C_{20}C_{11} - 2C_{20}C_{02} - 5C_{20}^2 - 3C_{11}^2 < 0. \quad (4.115)$$

Thus, the proposed estimator $t_{3(-1,-1)}$will be more efficient than $\bar{y}_r$, if

$$\rho < \frac{\theta^2+1}{2\theta} \text{ and } \theta \text{ equals to unity.} \quad (4.116)$$

From (4.114) and (4.37), it is observed that estimator $t_{3(-1,-1)}$ is more efficient than $t_{1(-1,-1)}$ upto order O (n^{-2}), if

$$5C_{20}C_{11} + 2C_{20}C_{02} - 4C_{20}^2 - 5C_{11}^2 + 4C_{12} - 4C_{21} + C_{30} < 0. \quad (4.117)$$

Under bivariate normal distribution, the expression (4.117) reduces to

$$5C_{20}C_{11} + 2C_{20}C_{02} - 4C_{20}^2 - 5C_{11}^2 < 0. \quad (4.118)$$

Thus, the proposed estimator $t_{3(-1,-1)}$ will be more efficient than $t_{1(-1,-1)}$, if

$$3\theta^2 - 5\rho\theta + 4 > 0 \text{ which is true for } \theta > 0. \quad (4.119)$$

Expressions (4.114) and (4.78) showed that the estimators $t_{3(-1,-1)}$ and $t_{2(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Also, estimators $t_{3(-1,-1)}$ performed better than $t_{2(-1,-1)}$ upto order O (n^{-2}), if

$$8C_{20}C_{11} - 6C_{20}^2 - 2C_{11}^2 - 6C_{21} + 6C_{12} < 0. \tag{4.120}$$

Under bivariate normal distribution, the expression (4.120) reduces to

$$8C_{20}C_{11} - 6C_{20}^2 - 2C_{11}^2 < 0. \tag{4.121}$$

Thus, the estimator $t_{3(-1,-1)}$) will be more efficient than $t_{2(-1,-1)}$ under bivariate normal population, if $\rho < \frac{2\theta^2+6}{8\theta}$ and θ lies between $1.00 < \theta < 1.610$.

Further, it has been observed that estimator $t_{3(-1,-1)}$ and $t_{s(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{3(-1,-1)}$ is more efficient than $t_{s(-1,-1)}$, if

$$12C_{20}C_{11} - 2C_{20}C_{02} - 5C_{20}^2 - 3C_{02}^2 - 6C_{11}^2 - 6C_{21} + 2C_{12} + 2C_{03} + 2C_{30} < 0, \tag{4.122}$$

which under bivariate normal distribution gives

$$12C_{20}C_{11} - 2C_{20}C_{02} - 5C_{20}^2 - 3C_{02}^2 - 6C_{11}^2 < 0. \tag{4.123}$$

Thus, the estimator $t_{3(-1,-1)}$) will be more efficient than $t_{s(-1,-1)}$, if

$\rho < \frac{3\theta^4+8\theta^3+5}{12\theta}$ which results to θ lies between $0.553 < \theta < 0.681$.

Case IV: For $p < 0$ and $q > 0$**,** consider p=-1 and q=1, the estimator t_3 will become

$$t_{3(-1,1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}^2} + \frac{s_{xy}}{\bar{x}\bar{y}}\right]. \tag{4.124}$$

The relative bias and relative mean squared error the estimator $t_{3(-1,1)}$ are as

$$RB\left(t_{3(-1,1)}\right) = RB(\bar{y}_r) - \frac{1}{n}(C_{20} + C_{11}), \tag{4.125}$$

$$RM(t_{3(-1,1)}) = RM(\bar{y}_r) + 5C_{11}^2 + 6C_{20}C_{11} - 2C_{20}C_{02} - 5C_{20}^2 - 2C_{12} + 2C_{30}. \quad (4.126)$$

From (4.125), it is observed that the estimator $t_{3(-1,1)}$has smaller bias than conventional ratio estimator, if $C_{20} > -C_{11}$.

Both the estimators have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, the estimator $t_{3(-1,1)}$is more efficient than $\bar{y}_r$ upto order O (n^{-2}), if

$$5C_{11}^2 + 6C_{20}C_{11} - 2C_{20}C_{02} - 5C_{20}^2 - 2C_{12} + 2C_{30} < 0. \quad (4.127)$$

For bivariate normal population the expression (4.127) reduces to

$$C_{11}^2 + 6C_{20}C_{11} - 2C_{20}C_{02} - 5C_{20}^2 < 0. \quad (4.128)$$

Thus, the proposed estimator $t_{3\,(-1,\,1)}$ will be more efficient than $\bar{y}_r$, if

$\rho < \frac{5-3\theta^2}{6\theta}$ and $\theta < -\rho + 1.63$.

From (4.125), (4.43) and (4.85), it can be observed that the relative bias of $t_{3(-1,1)}$, $t_{1(-1,1)}$ and $t_{2(-1,1)}$ are identical.

Comparison on the basis of relative mean squared error showed that estimators $t_{3(-1,1)}$ and $t_{1(-1,1)}$ are equally efficient upto order O (n^{-1}).They differs in terms of O (n^{-2}).

From (4.126) and (4.44), it is observed that estimator $t_{3(-1,1)}$ is more efficient than $t_{1(-1,1)}$ under bivariate normal population, if $\rho > \frac{5\theta^2-4}{\theta}$ and $\theta < 0.1\rho - 0.9$.

Expressions (4.126) and (4.86) showed that the estimators $t_{3(-1,1)}$ and $t_{2(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Also, estimator $t_{3(-1,1)}$ performed better than $t_{2(-1,1)}$ upto order O (n^{-2}), if

$$4C_{20}C_{11} + 2C_{11}^2 - 6C_{20}^2 + 4C_{30} - 2C_{21} + 2C_{12} < 0 \tag{4.129}$$

Thus, the estimator $t_{3(-1,1)}$ will be more efficient than the $t_{2(-1,1)}$ under bivariate normal distribution, if $\theta < -\rho + 2$.

Further, it has been observed that that estimator $t_{3(-1,1)}$ have smaller relative bias upto order O (n^{-1}) than $t_{s(-1,1)}$, if $C_{20} + 2C_{11} > C_{02}$.

Further, it has been observed that estimator $t_{3(-1,1)}$ and $t_{s(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{3(-1,1)}$ is more efficient than $t_{s(-1,1)}$, if

$$4C_{20}C_{11} - 4C_{11}^2 - 2C_{20}C_{02} - 5C_{20}^2 + 4C_{02}C_{11} - 3C_{02}^2 + 2C_{30} + 2C_{03} - C_{21} - 4C_{12} < 0, \tag{4.130}$$

which under bivariate normal distribution gives $\rho < \frac{-3\theta^4 - 4\theta^2 + 5}{4\theta^3 + 4\theta}$ which results to $0.597 < \theta < 0.886$.

4.3.4. The following improved class of ratio type estimators for $\bar{Y}$ have been proposed as $\boldsymbol{t_4 = \bar{y}_r + \frac{p}{n}\bar{y}\left[\frac{s_x^2}{\bar{X}^2} + q\frac{s_{xy}}{\bar{x}\bar{y}}\right]}$ (4.131)

Theorem 4.5: The relative bias and relative mean squared error of the estimator t_4 upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB(t_4) = RB(\bar{y}_r) + \frac{p}{n}(C_{20} + qC_{11}), \tag{4.132}$$

$$RM(t_4) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{20}^2 - 4C_{20}C_{11} + 2C_{02}C_{20} + 2C_{12} - 2C_{30}] + \frac{pq}{n^2}[4C_{20}C_{11} - 4C_{11}^2 + 2C_{12} - 2C_{21}] + \frac{1}{n^2}[pC_{20} + pqC_{11}]^2. \tag{4.133}$$

Proof: Expressing $\bar{y}$, $\bar{x}$, s_x^2 and s_{xy} in terms of U_y, U_x, V_x and W respectively, in the estimator t_4, we obtain

$$\frac{t_4-\bar{Y}}{\bar{Y}} = \left(1 + U_y\right)(1 + U_x)^{-1} + \frac{p}{n}\left(1 + U_y\right)(V_x + C_{20}) + \frac{pq}{n}(1 + U_x)^{-1}(W + C_{11})\text{-1.} \tag{4.134}$$

Expanding the terms on the right hand side of (4.134) and retaining the terms to order O ($n^{-3/2}$), we have $\frac{t_4-\bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + e^*_{-1} + h^*_{-\frac{3}{2}}$, (4.135)

where,

$$\left.\begin{aligned} d^*_{-\frac{1}{2}} &= U_y - U_x, \\ e^*_{-1} &= U_x^2 - U_xU_y + \frac{p}{n}C_{20} + \frac{pq}{n}C_{11}, \\ h^*_{-\frac{3}{2}} &= -U_x^3 + U_x^2U_y + \frac{p}{n}\left(V_x + U_yC_{20}\right) + \frac{pq}{n}(W - U_xC_{11}). \end{aligned}\right\} \tag{4.136}$$

Taking expectation of (4.135) upto order O (n^{-1}), we get the relative bias of estimator t_4 as in expression (4.132) of theorem 4.5.

For relative mean squared error to order O (n^{-2}), we find that

$$E\left(\frac{t_4-\bar{Y}}{\bar{Y}}\right)^2 = E\left[{d^*_{-\frac{1}{2}}}^2 + {e^*_{-1}}^2 + 2{d_{-\frac{1}{2}}}^*{e_{-1}}^* + 2d^*_{-\frac{1}{2}}h^*_{-\frac{3}{2}}\right] \tag{4.137}$$

where,

$$E\left({d^*_{-\frac{1}{2}}}^2\right) = E\left(U_y^2 + U_x^2 - 2U_yU_x\right),$$

$$E\left({e^*_{-1}}^2\right) = \frac{1}{n^2}[3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}] + \frac{2}{n^2}[pC_{20} + pqC_{11}][C_{20} - C_{11}]$$

$$+\frac{1}{n^2}[pC_{20} + pqC_{11}]^2, \qquad (4.138)$$

$$E\left(d^*_{-\frac{1}{2}}e^*_{-1}\right) = \frac{1}{n^2}[2C_{21} - C_{12} - C_{30}],$$

$$E\left(d^*_{-\frac{1}{2}}h^*_{-\frac{3}{2}}\right) = \frac{1}{n^2}[2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11} + 3C_{20}^2] +$$

$$\frac{p}{n^2}[C_{21} + C_{02}C_{20} - C_{30} - C_{11}C_{20}] + \frac{pq}{n^2}[C_{20}C_{11} - C_{11}^2 + C_{12} - C_{21}].$$

where, terms of order higher than O (n^{-2}) have been dropped. Substituting the (4.138) expression in (4.137) and after a little algebraic simplification, we obtained (4.133) of theorem 4.5.

From (4.132), it is observed that estimator t_4 has smaller bias than conventional ratio type estimator $\bar{y}_r$, if $p < 0$ and $C_{20} + qC_{11} > 0$.

From (4.133) and (4.3), it is observed that the relative mean squared error of both estimators i.e., t_4 and $\bar{y}_r$, are identical upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_4 has smaller relative mean squared error than that of $\bar{y}_r$, if

$$p[2C_{20}^2 - 4C_{20}C_{11} + 2C_{02}C_{20} + 2C_{12} - 2C_{30}] + pq[4C_{20}C_{11} - 4C_{11}^2 + 2C_{12} - 2C_{21}] +$$

$$[pC_{20} + pqC_{11}]^2 < 0. \qquad (4.139)$$

From (4.12), (4.55), (4.93) and (4.132), it is observed that the estimators t_1, t_2, t_3 and t_4 have identical bias upto order O (n^{-1}).

From (4.13), (4.56), (4.94) and (4.133), it can be observed that the estimators t_1, t_2, t_3 and t_4 have identical relative mean squared error upto order O (n^{-1}).They differ in terms of order O (n^{-2}).

From (4.133) and (4.13), it can be seen that the estimator t_4 has smaller relative mean squared error than t_1 upto order O (n^{-2}), if

$$p[2C_{20}C_{11} - 2C_{02}C_{20} - C_{30}] + pq[3C_{20}C_{11} - 3C_{11}^2] < 0. \tag{4.140}$$

For bivariate normal population, the expression (4.140) reduces to

$$p[2C_{20}C_{11} - 2C_{02}C_{20}] + pq[3C_{20}C_{11} - 3C_{11}^2] < 0. \tag{4.141}$$

The comparison of t_4 and t_2 showed that estimator t_4 is more efficient than t_2, if

$$p[2C_{20}^2 - 2C_{20}C_{11}] + pq[2C_{20}C_{11} - 2C_{11}^2] < 0. \tag{4.142}$$

From, (4.133) and (4.94), it can be observed that the relative mean squared error of estimators t_4 is smaller than estimator t_3 upto O (n^{-2}), if $p < 0$ and $C_{11} > C_{20}$.

Comparison of relative mean squared error of estimator t_4 and t_s showed that t_4 is more efficient than t_s upto order O (n^{-2}), if

$$p[2C_{20}^2 + 2C_{02}^2 - 4C_{20}C_{11} - 2C_{02}C_{11} + 2C_{02}C_{20} + 2C_{12} + 2C_{21} - 2C_{30} - 2C_{03}] + pq[4C_{20}C_{11} - 4C_{11}^2 + 2C_{12} - 2C_{21}] + q[2C_{11}^2 - 2C_{20}C_{11} - C_{12} + C_{21}] + [pC_{20} + pqC_{11}]^2 - [pC_{02} + pqC_{11}]^2 < 0. \tag{4.143}$$

Under bivariate normal distribution, expression (4.143) reduces to

$$p[2C_{20}^2 + 2C_{02}^2 - 4C_{20}C_{11} - 2C_{02}C_{11} + 2C_{02}C_{20}] + pq[4C_{20}C_{11} - 4C_{11}^2] +$$
$$q[2C_{11}^2 - 2C_{20}C_{11}] + [pC_{20} + pqC_{11}]^2 - [pC_{02} + pqC_{11}]^2 < 0. \quad (4.144)$$

4.3.4.1: **Proposed ratio type estimators:** Consider the different values of p and q, the different cases proposed are as:

Case I: Consider p=0 in (4.131), estimator t_4 reduces to conventional ratio type estimator which is particular member of the proposed class t_4.

Case II: Consider q= 0, the estimator t_4 reduces to $t_{4(p,0)} = \bar{y}_r + \frac{p}{n}\bar{y}\frac{s_x^2}{\bar{X}^2}$. (4.145)

The relative bias and relative mean squared error of the estimator $t_{4(p,0)}$ respectively are as

$$RB(t_{4(p,0)}) = RB(\bar{y}_r) + \frac{p}{n}C_{20}, \quad (4.146)$$

$$RM(t_{4(p,0)}) = RM(\bar{y}_r) + \frac{p}{n^2}[2C_{20}^2 + 2C_{20}C_{02} - 4C_{20}C_{11} + 2C_{21} - 2C_{30} + pC_{20}^2]. \quad (4.147)$$

From (4.146), it is observed that the relative bias of the estimator $t_{4(p,0)}$ is smaller than the estimator $\bar{y}_r$ if $p < 0$.

From (4.147), the relative mean squared error of the estimator $t_{4(p,0)}$ is smaller than the conventional ratio type estimator $\bar{y}_r$, if

$$p[2C_{20}^2 + 2C_{20}C_{02} - 4C_{20}C_{11} + 2C_{21} - 2C_{30} + pC_{20}^2] < 0, \quad (4.148)$$

For bivariate normal population (4.148) reduces to

$$p[2C_{20}^2 + 2C_{20}C_{02} - 4C_{20}C_{11} + pC_{20}^2] < 0. \quad (4.149)$$

Thus, the estimator $t_{4(p,0)}$ will be more efficient than $\bar{y}_r$, if

$$p < 0 \text{ and } \rho - 0.707 < \theta < \rho + 0.707 \text{ which is true for } \theta > 0. \quad (4.150)$$

From (4.30), (4.69), (4.108) and (4.146), it can be seen that the estimators $t_{1(p,0)}, t_{2(p,0)}, t_{3(p,0)}$ and $t_{4(p,0)}$ have identical bias upto order O (n^{-1}).

From (4.147) and (4.31), it is observed that $t_{4(p,0)}$ is more efficient than $t_{1(p,0)}$ if $p < 0$ and $C_{02} < C_{11}$.

The comparison of relative mean squared error of $t_{4(p,0)}$ and $t_{2(p,0)}$ showed that $t_{4(p,0)}$ is more efficient than $t_{2(p,0)}$ if $C_{20} > C_{11}$ and $p < 0$.

Further, from (4.147) and (4.109), it can be observed that estimator $t_{4(p,0)}$ is more efficient than $t_{3(p,0)}$ upto order O (n^{-2}), if $p < 0$ and $C_{11} > C_{20}$.

The comparison of $t_{4(p,0)}$ and $t_{s(p,0)}$ showed that that $t_{4(p,0)}$ have smaller bias than $t_{s(p,0)}$ upto order O (n^{-1}), if $p < 0$ and $C_{11} > C_{20}$.

Thus, it can be concluded that the estimator $t_{4(p,0)}$ will be more efficient than $\bar{y}_r$, $t_{1(p,0)}$ and $t_{3(p,0)}$, if the value of C_{11} is greater than C_{20} and C_{02}.

The comparison of relative mean squared error of $t_{4(p,0)}$ and $t_{s(p,0)}$ showed that $t_{4(p,0)}$ was more efficient than $t_{s(p,0)}$, if

$$p[2C_{20}^2 - 4C_{20}C_{11} + 2C_{20}C_{02} - 2C_{02}C_{11} + 2C_{02}^2 + 2C_{21} - 2C_{30} - 2C_{03} + 2C_{12} + pC_{20}^2 - pC_{02}^2] < 0, \tag{4.151}$$

which under bivariate normal distribution reduces to

$$p[2C_{20}^2 - 4C_{20}C_{11} + 2C_{20}C_{02} - 2C_{02}C_{11} + 2C_{02}^2 + pC_{20}^2 - pC_{02}^2] < 0. \tag{4.152}$$

Thus, estimator $t_{4(p,0)}$ will be better than $t_{s(p,0)}$, if

$$p < 0,\ \rho < \frac{3\theta^4+2\theta^2+1}{2\theta^3+4\theta} \text{ and } \theta \text{ lies between } 0.567 < \theta < 1.000. \tag{4.153}$$

Case III: For $p < 0$ and $q < 0$**,** consider the values of scalars p and q as p=-1 and q=-1, the estimator t_4 reduces to $t_{4(-1,-1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{X}^2} - \frac{s_{xy}}{\bar{x}\bar{y}}\right]$ which is unbiased. (4.154)

The relative variance of the estimator $t_{4(-1,-1)}$ is as

$$RV\left(t_{4(-1,-1)}\right) = RM(\bar{y}_r) - C_{20}^2 - 3C_{11}^2 + 6C_{20}C_{11} - 2C_{02}C_{20} - 4C_{21} + 2C_{12} + 2C_{30}. \tag{4.155}$$

From (4.155) and (4.3), the relative mean squared error of the estimator $\bar{y}_r$ is identical to the estimator $t_{4(-1,-1)}$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(-1,-1)}$ is more efficient than $\bar{y}_r$, if

$$6C_{20}C_{11} - 2C_{20}C_{02} - C_{20}^2 - 3C_{11}^2 - 4C_{21} + 2C_{12} + 2C_{30} < 0. \tag{4.156}$$

For bivariate normal distribution, the expression (4.156) reduces to

$$6C_{20}C_{11} - 2C_{20}C_{02} - C_{20}^2 - 3C_{11}^2 < 0. \tag{4.157}$$

Thus, the estimator $t_{4(-1,-1)}$) will be more efficient than $\bar{y}_r$, if

$\rho < \frac{5\theta^2+1}{6\theta}$ and θ lies between $0.200 < \theta < 1.000$.

From (4.155) and (4.37), it can be shown that estimator $t_{4(-1,-1)}$ and $t_{1(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(-1,-1)}$ is more efficient than $t_{1(-1,-1)}$, if

$$C_{20}C_{11} + 2C_{20}C_{02} - 5C_{11}^2 + 4C_{12} - 4C_{21} + C_{30} < 0. \tag{4.158}$$

Under bivariate normal population, expression (4.158) becomes

$$C_{20}C_{11} + 2C_{20}C_{02} - 5C_{11}^2 < 0. \tag{4.159}$$

Thus, the estimator $t_{4(-1,-1)}$) will perform better than $t_{1(-1,-1)}$, if

$\rho < 3\theta$ and θ lies between $0.001 < \theta < 0.333$.

From (4.155) and (4.78), it can be shown that estimator $t_{4(-1,-1)}$ and $t_{2(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{4(-1,-1)}$ is more efficient than estimator $t_{2(-1,-1)}$ upto order O (n^{-2}), if

$$4C_{20}C_{11} - 2C_{20}^2 - 2C_{11}^2 - 6C_{21} + 6C_{12} < 0, \tag{4.160}$$

Under bivariate normal distribution, expression (4.160) reduces to

$$4C_{20}C_{11} - 2C_{20}^2 - 2C_{11}^2 < 0. \tag{4.161}$$

Thus, the estimator $t_{4(-1,-1)}$) will be more efficient than $t_{2(-1,-1)}$, if $\rho < \frac{\theta^2+1}{2\theta}$ and θ equals to unity.

From (4.155) and (4.114), it can be shown that estimator $t_{4(-1,-1)}$ and $t_{3(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{4(-1,-1)}$ is more efficient than estimator $t_{3(-1,-1)}$ upto order O (n^{-2}), if $C_{20} < C_{11}$ and $\theta > 1$.

Further, it has been observed that estimator $t_{4(-1,-1)}$ and $t_{s(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(-1,-1)}$ is more efficient than $t_{s(-1,-1)}$, if

$$8C_{20}C_{11} - 6C_{11}^2 - 2C_{20}C_{02} - C_{20}^2 - 3C_{02}^2 + 2C_{03} + 2C_{30} - 6C_{21} + 2C_{12} < 0, \tag{4.162}$$

which under bivariate normal distribution gives

$$8C_{20}C_{11} - 6C_{11}^2 - 2C_{20}C_{02} - C_{20}^2 - 3C_{02}^2 < 0. \tag{4.163}$$

Thus, the proposed estimator $t_{4(-1,-1)}$ will be better than $t_{s(-1,-1)}$ upto order O (n^{-2}), if $\rho > \frac{3\theta^4+8\theta^2+1}{8\theta}$ and θ lies between $0.333 < \theta < 0.694$. (4.164)

Thus, it has been observed that the estimator $t_{4(-1,-1)}$ will be more efficient than $\bar{y}_r$, $t_{1(-1,-1)}$, $t_{2(-1,-1)}$ and $t_{3(-1,-1)}$, if $0.200 < \theta < 0.333$.

Case IV: For $p < 0$ and $q > 0$, consider p= -1 and q= 1, the estimator t_4 becomes

$$t_{4(-1,1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{X}^2} + \frac{s_{xy}}{\bar{x}\bar{y}}\right]. \tag{4.165}$$

The relative bias and relative mean squared error of the estimator $t_{4(-1,1)}$ respectively are as

$$RB\left(t_{4(-1,1)}\right) = RB(\bar{y}_r) - \frac{1}{n}(C_{20} + C_{11}), \tag{4.166}$$

$$RM\left(t_{4(-1,1)}\right) = RM(\bar{y}_r) + \frac{1}{n^2}[5C_{11}^2 - C_{20}^2 - 2C_{20}C_{02} + 2C_{20}C_{11} + 2C_{30} - 2C_{12}]. \tag{4.167}$$

From (4.166) and (4.2), it is found that the relative bias of $t_{4(-1,1)}$ is smaller than that of estimator $\bar{y}_r$ upto order O (n^{-1}), if $C_{20} > -C_{11}$.

From (4.167) and (4.3), the relative mean squared error of the estimator $\bar{y}_r$ is identical to the estimator $t_{2(-1,1)}$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(-1,1)}$ is more efficient than $\bar{y}_r$, if

$$5C_{11}^2 - C_{20}^2 - 2C_{20}C_{02} + 2C_{20}C_{11} + 2C_{30} - 2C_{12} < 0 \tag{4.168}$$

For bivariate normal distribution, the expression (4.168) reduces to

$$5C_{11}^2 - C_{20}^2 - 2C_{20}C_{02} + 2C_{20}C_{11} < 0 \tag{4.169}$$

which gives $p < 0$ and $-0.33\rho - 0.67 < \theta < -0.33\rho + 0.67$ which is true for $\theta > 0$.

From (4.166) and (4.43), it is observed that estimator $t_{4(-1,1)}$ and $t_{1(-1,1)}$ have identical relative bias upto order O (n^{-1}).

From (4.167) and (4.44), it can be shown that estimator $t_{4(-1,1)}$ and $t_{1(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(-1,1)}$ is more efficient than $t_{1(-1,1)}$, if $\rho > \theta$.

From (4.166) and (4.85), it is observed that estimator $t_{4(-1,1)}$ and $t_{2(-1,1)}$ have identical relative bias upto order O (n^{-1}).

From (4.167) and (4.86), it can be shown that estimator $t_{4(-1,1)}$ and $t_{2(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{4(-1,1)}$ is more efficient than estimator $t_{2(-1,1)}$ upto order O (n^{-2}), if $C_{20} > C_{11}$.

From (4.166) and (4.125), it is observed that estimator $t_{4(-1,1)}$ and $t_{3(-1,1)}$ have identical relative bias upto order O (n^{-1}).

From (4.167) and (4.126), it can be shown that estimator $t_{4(-1,1)}$ and $t_{3(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{4(-1,1)}$ is more efficient than estimator $t_{3(-1,1)}$ upto order O (n^{-2}), if $C_{11} > C_{20}$.

It is observed that that estimator $t_{4(-1,1)}$ have smaller relative bias upto order O (n^{-1}) than $t_{s(-1,1)}$, if $C_{20} + 2C_{11} > C_{02}$.

The comparison of estimators $t_{4(-1,1)}$ and $t_{s(-1,1)}$ showed $t_{4(-1,1)}$ and $t_{s(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(-1,1)}$ is more efficient than $t_{s(-1,1)}$, if

$$6C_{11}^2 - C_{20}^2 - 3C_{02}^2 - 2C_{20}C_{02} + 4C_{02}C_{11} + 2C_{30} + 2C_{03} - 5C_{12} + C_{21} < 0, \qquad (4.170)$$

which under bivariate normal distribution gives

$$6C_{11}^2 - C_{20}^2 - 3C_{02}^2 - 2C_{20}C_{02} + 4C_{02}C_{11} < 0\ , \qquad (4.171)$$

which gives $\rho > \frac{3\theta^4 - 4\theta^2 + 1}{4\theta^3}$ and the value of θ should lies between $1.001 < \theta < 1.967$.

4.3.5: The following improved class of ratio type estimators for $\bar{Y}$ have been proposed as

$$\boldsymbol{t_5 = \bar{y}_r \left[1 + \frac{p}{n}\left(\frac{s_{xy}}{\bar{x}\bar{y}} + q\,\frac{s_x^2}{\bar{X}^2}\right)\right]}, \qquad (4.172)$$

where, p and q are scalars specifying the estimator.

Theorem 4.6: The relative bias and relative mean squared error of the estimator t_5 upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB(t_5) = RB(\bar{y}_r) + \frac{p}{n}(C_{11} + qC_{20}), \qquad (4.173)$$

$$RM(t_5) = RM(\bar{y}_r) + \frac{p}{n^2}[6C_{20}C_{11} - 6C_{11}^2 + 2C_{12} - 2C_{21}] + \frac{pq}{n^2}[4C_{20}^2 - 6C_{20}C_{11} +$$

$$2C_{02}C_{20} + 2C_{21} - 2C_{30}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2. \qquad (4.174)$$

Proof: Expressing $\bar{y}$, $\bar{x}$, s_x^2 and s_{xy} in terms of U_y, U_x , V_x and W respectively in the estimator t_5, we have

$$\frac{t_5-\bar{Y}}{\bar{Y}} = \left(1+U_y\right)(1+U_X)^{-1} + \frac{p}{n}(1+U_X)^{-2}(W+C_{11}) + \frac{pq}{n}(1+U_y)(1+U_X)^{-1}(V_x + C_{20}). \quad (4.175)$$

Expanding the terms on the right hand side of (4.175) and retaining the terms to order O ($n^{-3/2}$), we have

$$\frac{t_5-\bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + f^*_{-1} + i^*_{-\frac{3}{2}}, \quad (4.176)$$

where,

$$\left.\begin{aligned} d^*_{-\frac{1}{2}} &= U_y - U_x, \\ f^*_{-1} &= U_x^2 - U_x U_y + \frac{p}{n}C_{11} + \frac{pq}{n}C_{20}, \\ i^*_{-\frac{3}{2}} &= -U_x^3 + U_x^2 U_y + \frac{p}{n}(W - 2U_x C_{11}) + \frac{pq}{n}\left(V_x - U_x C_{20} + U_y C_{20}\right). \end{aligned}\right\} \quad (4.177)$$

Taking expectations of (4.176) to both sides upto order O (n^{-1}), we obtain the relative bias of estimator t_5 as in expression in (4.173) of theorem 4.6.

For relative mean squared error of estimator t_5 upto order O (n^{-2}), we find that

$$E\left(\frac{t_5-\bar{Y}}{\bar{Y}}\right)^2 = E\left[{d^*_{-\frac{1}{2}}}^2 + {f^*_{-1}}^2 + 2d^*_{-\frac{1}{2}}f^*_{-1} + 2d^*_{-\frac{1}{2}}i^*_{-\frac{3}{2}}\right]. \quad (4.178)$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned} E\left({d^*_{-\frac{1}{2}}}^2\right) &= E\left(U_y^2 + U_x^2 - 2U_y U_x\right) \\ &= \frac{1}{n}(C_{02} + C_{20} - 2C_{11}), \\ E\left({f^*_{-1}}^2\right) &= \frac{1}{n^2}[3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}] + \frac{p}{n^2}[2C_{20}C_{11} - 2C_{11}^2] \\ &\quad + \frac{pq}{n^2}[2C_{20}^2 - 2C_{20}C_{11}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2, \\ E\left(d^*_{-\frac{1}{2}}f^*_{-1}\right) &= \frac{1}{n^2}[2C_{21} - C_{12} - C_{30}], \\ E\left(d^*_{-\frac{1}{2}}i^*_{-\frac{3}{2}}\right) &= \frac{1}{n^2}[3C_{20}^2 + 2C_{11}^2 + C_{20}C_{02} - 6C_{20}C_{11}] + \frac{p}{n^2}[2C_{20}C_{11} - 2C_{11}^2 + C_{12} - C_{21}] \\ &\quad + \frac{pq}{n^2}[C_{20}^2 + C_{20}C_{02} - 2C_{20}C_{11} + C_{21} - C_{30}]. \end{aligned}\right\} \quad (4.179)$$

The terms of order higher than O (n^{-2}) have been dropped. Substituting the results of (4.179) in (4.178) and after algebraic simplification, we obtained (4.174) of theorem 4.6.

From (4.173), it is observed that estimator t_5 has smaller bias than conventional ratio type estimator $\bar{y}_r$, if $p < 0$ and $C_{11} + qC_{20} > 0$.

From (4.174) and (4.3), it is observed that the relative mean squared error of both estimators i.e., t_5 and $\bar{y}_r$, are identical upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_5 has smaller relative mean squared error than that of $\bar{y}_r$, if

$$\frac{p}{n^2}[6C_{20}C_{11} - 6C_{11}^2 + 2C_{12} - 2C_{21}] + \frac{pq}{n^2}[4C_{20}^2 - 6C_{20}C_{11} + 2C_{02}C_{20} + 2C_{21} - 2C_{30}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 < 0. \quad (4.180)$$

Under bivariate normal distribution, the expression (4.180) reduces to

$$\frac{p}{n^2}[6C_{20}C_{11} - 6C_{11}^2] + \frac{pq}{n^2}[4C_{20}^2 - 6C_{20}C_{11} + 2C_{02}C_{20}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 < 0. \quad (4.181)$$

From (4.173), (4.12), (4.55), (4.93) and (4.132), it is observed that the estimators t_5 has smaller bias than t_1, t_2, t_3 and $t_4$upto order O (n^{-1}), if $p < 0$ and $q > 1$.

From (4.174), (4.13), (4.56), (4.94) and (4.133), it can be observe that the estimators t_1, t_2, t_3 ,t_4 and t_5 have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

From (4.174) and (4.13), it can be seen that the estimator t_5 has smaller relative mean squared error than t_1 upto order O (n^{-2}), if

$$\frac{p}{n^2}[12C_{20}C_{11} - 4C_{02}C_{20} - 6C_{11}^2 - 2C_{20}^2 - 4C_{21} + 2C_{12} + C_{30}] + \frac{pq}{n^2}[4C_{20}^2 - 7C_{20}C_{11} + C_{11}^2 + 2C_{20}C_{02} + 2C_{21} - 2C_{30}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \quad (4.182)$$

For bivariate normal population, the expression (4.182) reduces to

$$\frac{p}{n^2}[12C_{20}C_{11} - 4C_{02}C_{20} - 6C_{11}^2 - 2C_{20}^2] + \frac{pq}{n^2}[4C_{20}^2 - 7C_{20}C_{11} + C_{11}^2 + 2C_{20}C_{02}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \quad (4.183)$$

The comparison of t_5 and t_2 showed that estimator t_5 is more efficient than t_2 under bivariate normal distribution, if

$$\frac{p}{n^2}[8C_{20}C_{11} - 2C_{02}C_{20} - 6C_{11}^2] + \frac{pq}{n^2}[4C_{20}^2 - 8C_{20}C_{11} + 2C_{11}^2 + 2C_{20}C_{02}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \quad (4.184)$$

From, (4.174) and (4.94), it can be observed that the relative mean squared error of estimators t_5 is smaller than estimator t_3 upto O (n^{-2}), if

$$\frac{p}{n^2}[14C_{20}C_{11} - 2C_{02}C_{20} - 6C_{11}^2 - 6C_{20}^2 - 2C_{21} + 2C_{30}] + \frac{pq}{n^2}[4C_{20}^2 + 4C_{11}^2 - 10C_{20}C_{11} + 2C_{20}C_{02} + 2C_{21} - 2C_{30}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \quad (4.185)$$

The expression (4.185) under bivariate normal distribution reduces to

$$\frac{p}{n^2}[14C_{20}C_{11} - 2C_{02}C_{20} - 6C_{11}^2 - 6C_{20}^2] + \frac{pq}{n^2}[4C_{20}^2 + 4C_{11}^2 - 10C_{20}C_{11} + 2C_{20}C_{02}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \quad (4.186)$$

The expression (4.174) and (4.133) showed that the estimator t_5 is better than t_4 upto order O (n^{-2}), if

$$\frac{p}{n^2}[10C_{20}C_{11} - 2C_{02}C_{20} - 6C_{11}^2 - 2C_{20}^2 - 2C_{21} + 2C_{30}] + \frac{pq}{n^2}[4C_{20}^2 + 4C_{11}^2 - 10C_{20}C_{11} + 2C_{20}C_{02} + 2C_{21} - 2C_{30}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \tag{4.187}$$

Under bivariate normal distribution, the expression (4.187) reduces to

$$\frac{p}{n^2}[10C_{20}C_{11} - 2C_{02}C_{20} - 6C_{11}^2 - 2C_{20}^2] + \frac{pq}{n^2}[4C_{20}^2 + 4C_{11}^2 - 10C_{20}C_{11} + 2C_{20}C_{02}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{20} + pqC_{11}]^2 < 0. \tag{4.188}$$

Comparison of relative mean squared error of estimator t_5 and t_s showed that t_5 is more efficient than t_s upto order O (n^{-2}), if

$$\frac{p}{n^2}[6C_{20}C_{11} - 2C_{02}C_{11} - 6C_{11}^2 + 2C_{02}^2 - 2C_{03} - 2C_{21} + 4C_{12}] + \frac{pq}{n^2}[4C_{20}^2 - 6C_{20}C_{11} + 2C_{02}C_{20} + 2C_{12} - 2C_{30}] + \frac{q}{n^2}[2C_{11}^2 - 2C_{20}C_{11} - C_{12} + C_{21}] + \frac{1}{n^2}[pC_{11} + pqC_{20}]^2 - \frac{1}{n^2}[pC_{02} + pqC_{11}]^2 < 0. \tag{4.189}$$

Some special cases

Case I: Consider p=0 in (4.172), estimator t_5 reduces to conventional ratio type estimator which is a particular member of modified class of ratio type estimator.

Case II: Consider q= 0, the estimator t_5 reduces to

$$t_{5(p,0)} = \bar{y}_r \left[1 + \frac{p}{n}\frac{s_{xy}}{\bar{x}\bar{y}}\right]. \tag{4.190}$$

The relative bias and relative mean squared error of the estimator $t_{5(p,0)}$ respectively are as

$$RB(t_{5(p,0)}) = RB(\bar{y}_r) + \frac{p}{n}C_{11}, \tag{4.191}$$

$$RM(t_{5(p,0)}) = RM(\bar{y}_r) + \frac{p}{n^2}[6C_{20}C_{11} - 6C_{11}^2 - 2C_{21} + 2C_{12} + pC_{11}^2]. \tag{4.192}$$

From (4.191), it is observed that the relative bias of the estimator $t_{5(p,0)}$is smaller than the estimator $\bar{y}_r$, if $p < 0$ and $C_{11} > 0$.

From (4.192), the relative mean squared error of the estimator $t_{5(p,0)}$ is smaller than the conventional ratio type estimator $\bar{y}_r$, if $C_{20} > \frac{7}{6}C_{11}$.

From (4.191), (4.30), (4.69), (4.108) and (4.146), it can be seen that the estimators $t_{5(p,0)}$ has smaller bias than $t_{1(p,0)}, t_{2(p,0)}$, $t_{3(p,0)}$ and $t_{4(p,0)}$ upto order O (n^{-1}) if $p < 0$ and $C_{11} > C_{20}$.

From (4.192) and (4.31), it is observed that $t_{5(p,0)}$ is more efficient than $t_{1(p,0)}$ if

$$\frac{p}{n^2}[12C_{20}C_{11} - 4C_{20}C_{02} - 6C_{11}^2 - 2C_{20}^2 + pC_{11}^2 - pC_{20}^2]<0 \tag{4.193}$$

Thus, the estimator $t_{5(p,0)}$ will be better than $t_{1(p,0)}$ if $p < 0$, $\rho > \frac{11\theta^2+1}{12\theta}$ which results to θ lies between $0.100 < \theta < 0.300$.

The comparison of relative mean squared error of $t_{5(p,0)}$ and $t_{2(p,0)}$ showed that $t_{5(p,0)}$ is more efficient than $t_{2(p,0)}$, if the following conditions are satisfied.

$$p[8C_{20}C_{11} - 2C_{20}C_{02} - 6C_{11}^2 + pC_{11}^2 - pC_{20}^2] < 0, \tag{4.194}$$

which gives $p < 0$, $\rho > \frac{9\theta^2-1}{8\theta}$ which results to θ lies between $0.340 < \theta < 1.000$.

Further, from (4.192) and (4.109), it can be observed that estimator $t_{5(p,0)}$ is more efficient than $t_{3(p,0)}$ upto order O (n^{-2}), if

$$p[14C_{20}C_{11} - 2C_{20}C_{02} - 6C_{11}^2 - 6C_{20}^2 + pC_{11}^2 - pC_{20}^2] < 0, \tag{4.195}$$

which gives $p < 0$ and $\rho > \frac{9\theta^2+5}{14\theta}$ which results to θ lies between $0.770 < \theta < 1.000$.

From (4.192) and (4.147), it can be observed that $t_{5(p,0)}$ is more efficient than $t_{4(p,0)}$ upto order O (n^{-2}), if

$$p\,[10C_{20}C_{11} - 2C_{20}C_{02} - 6C_{11}^2 - 2C_{20}^2 + pC_{11}^2 - pC_{20}^2] < 0, \quad (4.196)$$

which gives $p < 0$ and $\rho > \frac{9\theta^2+1}{10\theta}$ which results to θ lies between $0.334 < \theta < 1.000$.

The comparison of the estimators $t_{5(p,0)}$ and $t_{s(p,0)}$ showed that the $t_{5(p,0)}$ have smaller bias than $t_{s(p,0)}$ upto order O (n^{-1}), if $p < 0$ and $C_{11} > C_{02}$.

The comparison of relative mean squared error of $t_{5(p,0)}$ and $t_{s(p,0)}$ showed that $t_{5(p,0)}$ was more efficient than $t_{s(p,0)}$, if

$$p[6C_{20}C_{11} - 2C_{02}C_{11} - 6C_{11}^2 + 2C_{02}^2 + pC_{11}^2 - pC_{02}^2] < 0 \quad (4.197)$$

which gives $p < 0$ and $\rho < \frac{3\theta^3-7\theta}{2\theta^2-6}$ which results to θ lies between $0.001 < \theta < 1.000$.

Case III: For $p < 0$ and $q < 0$, Consider the scalars p and q as p=-1 and q=-1, the estimator t_5 reduces to $\boldsymbol{t_{5(-1,-1)} = \bar{y}_r\left[1 + \frac{1}{n}\left(\frac{s_x^2}{\bar{X}^2} - \frac{s_{xy}}{\bar{x}\bar{y}}\right)\right]}$. (4.198)

The relative bias of the estimator $t_{5(-1,-1)}$ is twice that of the conventional ratio type estimator. The relative mean squared error of the estimator $t_{5(-1,-1)}$ is as

$$RM\left(t_{5(-1,-1)}\right) = RM(\bar{y}_r) + 5C_{20}^2 + 7C_{11}^2 - 14C_{20}C_{11} + 2C_{02}C_{20} + 2C_{21} - 2C_{30}. \quad (4.199)$$

From (4.199), it can be observed that the estimator $t_{5(-1,-1)}$ performs better than the conventional ratio type estimator, if

$$5C_{20}^2 + 7C_{11}^2 - 14C_{20}C_{11} + 2C_{02}C_{20} + 2C_{21} - 2C_{30} < 0 \quad (4.200)$$

Which under bivariate normal distribution gives $\rho > \frac{9\theta^2+5}{14\theta}$ and $0.56 < \theta < 0.61$.

From (4.199) and (4.37), it can be shown that estimator $t_{5(-1,-1)}$ and $t_{1(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{5(-1,-1)}$ is more efficient than $t_{1(-1,-1)}$, if

$$6C_{20}^2 + 5C_{11}^2 - 19C_{20}C_{11} + 6C_{20}C_{02} + 2C_{12} + 2C_{21} - 3C_{30} < 0. \tag{4.201}$$

Under bivariate normal population, expression (4.201) becomes

$$6C_{20}^2 + 5C_{11}^2 - 19C_{20}C_{11} + 6C_{20}C_{02} < 0. \tag{4.202}$$

Thus, the estimator $t_{5(-1,-1)}$) will perform better than $t_{1(-1,-1)}$, if

$\rho < \frac{11\theta^2+6}{19\theta}$ and θ lies between $0.416 < \theta < 1.311$.

From (4.199) and (4.78), it can be shown that estimator $t_{5(-1,-1)}$ and $t_{2(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{5(-1,-1)}$ is more efficient than estimator $t_{2(-1,-1)}$ upto order O (n^{-2}), if

$$C_{20}^2 + 2C_{11}^2 - 4C_{20}C_{11} + C_{20}C_{1021} - C_{30} + C_{12} < 0, \tag{4.203}$$

Under bivariate normal distribution, expression (4.203) reduces to

$$C_{20}^2 + 2C_{11}^2 - 4C_{20}C_{11} + C_{20}C_{02} < 0, \tag{4.204}$$

Thus, the estimator $t_{5(-1,-1)}$) will be more efficient than $t_{2(-1,-1)}$, if

$\rho < \frac{3\theta^2+1}{4\theta}$ and θ should lie between $0.334 < \theta < 1.000$.

From (4.199) and (4.114), it can be shown that estimator $t_{5(-1,-1)}$ and $t_{3(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{5(-1,-1)}$ is more efficient than estimator $t_{3(-1,-1)}$ upto order O (n^{-2}), if

$$10C_{20}^2 + 10C_{11}^2 - 24C_{20}C_{11} + 4C_{20}C_{02} + 6C_{21} - 4C_{30} - 2C_{12} < 0, \tag{4.205}$$

Under bivariate normal distribution, expression (4.205) reduces to

$$10C_{20}^2 + 10C_{11}^2 - 24C_{20}C_{11} + 4C_{20}C_{02} < 0, \tag{4.206}$$

Thus, the estimator $t_{5(-1,-1)}$) will be more efficient than $t_{3(-1,-1)}$, if

$\rho < \frac{7\theta^2+5}{12\theta}$ and θ should lie between $0.715 < \theta < 1.000$.

From (4.199) and (4.155), it can be shown that estimator $t_{5(-1,-1)}$ and $t_{4(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{5(-1,-1)}$ is more efficient than estimator $t_{4(-1,-1)}$ upto order O (n^{-2}), if

$$6C_{20}^2 + 10C_{11}^2 - 20C_{20}C_{11} + 4C_{20}C_{02} + 6C_{21} - 4C_{30} - 2C_{12} < 0, \tag{4.207}$$

Under bivariate normal distribution, expression (4.207) reduces to

$$6C_{20}^2 + 10C_{11}^2 - 20C_{20}C_{11} + 4C_{20}C_{02} < 0, \tag{4.208}$$

Thus, the estimator $t_{5(-1,-1)}$) will be more efficient than $t_{4(-1,-1)}$, if

$\rho < \frac{7\theta^2+3}{10\theta}$ and θ should lie between $0.431 < \theta < 1.000$.

Further, it has been observed that estimator $t_{5(-1,-1)}$ and $t_{s(-1,-1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{5(-1,-1)}$ is more efficient than $t_{s(-1,-1)}$, if

$$5C_{20}^2 - 3C_{02}^2 + 4C_{11}^2 - 12C_{20}C_{11} + 2C_{20}C_{02} + 2C_{03} - 2C_{30} < 0, \tag{4.209}$$

which under bivariate normal distribution gives

$$5C_{20}^2 - 3C_{02}^2 + 4C_{11}^2 - 12C_{20}C_{11} + 2C_{20}C_{02} < 0, \tag{4.210}$$

Thus, the proposed estimator $t_{5(-1,-1)}$ will be better than $t_{s(-1,-1)}$ upto order O (n^{-2}), if $\rho > \frac{-3\theta^4+6\theta^2+5}{12\theta}$ and θ should lie between $0.542 < \theta < 1.622$.

Thus, it can be concluded that the estimator $t_{5(-1,-1)}$ will be more efficient than $\bar{y}_r$, $t_{1(-1,-1)}$, $t_{2(-1,-1)}$, $t_{3(-1,-1)}$, $t_{4(-1,-1)}$ and $t_{s(-1,-1)}$, if $0.715 < \theta < 1.000$.

Case IV: For $p < 0$ and $q > 0$, consider p= -1 and q= 1, the estimator t_5 becomes

$$t_{5(-1,1)} = \bar{y}_r \left[1 - \frac{1}{n}\left(\frac{s_x^2}{\bar{X}^2} + \frac{s_{xy}}{\bar{x}\bar{y}}\right)\right]. \tag{4.211}$$

The relative bias and relative mean squared error of the estimator $t_{5(-1,1)}$ respectively are as

$$RB\left(t_{5(-1,1)}\right) = RB(\bar{y}_r) - \frac{1}{n}(C_{20} + C_{11}), \tag{4.212}$$

$$RM\left(t_{5(-1,1)}\right) = RM(\bar{y}_r) + \frac{1}{n^2}[7C_{11}^2 - 3C_{20}^2 - 2C_{20}C_{02} + 2C_{20}C_{11} + 2C_{30} - 4C_{12} + 2C_{21}]. \tag{4.213}$$

From (4.212) and (4.2), it is found that the relative bias of $t_{5(-1,1)}$ can be reduced than that of estimator $\bar{y}_r$ upto order O (n^{-1}), if $C_{20} > -C_{11}$.

From (4.213) and (4.3), the relative mean squared error of the estimator $t_{5(-1,1)}$ is identical to the estimator $\bar{y}_r$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}). Further, it is found that estimator $t_{5(-1,1)}$ is more efficient than $\bar{y}_r$, if

$$7C_{11}^2 - 3C_{20}^2 - 2C_{20}C_{02} + 2C_{20}C_{11} + 2C_{30} - 4C_{12} + 2C_{21} < 0 \tag{4.214}$$

For bivariate normal distribution, the expression (4.214) reduces to

$$7C_{11}^2 - 3C_{20}^2 - 2C_{20}C_{02} + 2C_{20}C_{11} < 0 \tag{4.215}$$

Thus, the estimator $t_{5(-1,1)}$ will be more efficient than $\bar{y}_r$, if

$\rho < \frac{3-5\theta^2}{2\theta}$ which results to θ lies between $0.600 < \theta < 0.775$.

From (4.212) and (4.43), it is observed that estimator $t_{5(-1,1)}$ and $t_{1(-1,1)}$ have identical relative bias upto order O (n^{-1}).

From (4.213) and (4.44), it can be shown that estimator $t_{5(-1,1)}$ and $t_{1(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{5(-1,1)}$ is more efficient than $t_{1(-1,1)}$, if

$$2C_{20}C_{02} - 5C_{20}C_{11} + 5C_{11}^2 - 2C_{20}^2 < 0 \tag{4.216}$$

Thus, the estimator $t_{5(-1,1)}$ will be more efficient than $t_{1(-1,1)}$, if $\rho > \frac{7\theta^2-2}{5\theta}$ which results to θ lies between $0.541 < \theta < 1.000$.

From (4.212) and (4.85), it is observed that estimator $t_{5(-1,1)}$ and $t_{2(-1,1)}$ have identical relative bias upto order O (n^{-1}).

From (4.213) and (4.86), it can be shown that estimator $t_{5(-1,1)}$ and $t_{2(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}) and further, it is observed that estimator $t_{5(-1,1)}$ is more efficient than estimator $t_{2(-1,1)}$ upto order O (n^{-2}), if $C_{11} > C_{20}$.

From (4.212) and (4.125), it is observed that estimator $t_{5(-1,1)}$ and $t_{3(-1,1)}$ have identical relative bias upto order O (n^{-1}).

From (4.213) and (4.126), it can be shown that estimator $t_{5(-1,1)}$ and $t_{3(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{5(-1,1)}$ is more efficient than estimator $t_{3(-1,1)}$ upto order O (n^{-2}), if $\theta < \rho$.

Also, estimator $t_{5(-1,1)}$ has performed better than $t_{4(-1,1)}$ upto order O (n^{-2}), if $C_{11} < C_{20}$.

The comparison of $t_{5(-1,1)}$ and $t_{s(-1,1)}$ showed that estimator $t_{5(-1,1)}$ have smaller relative bias upto order O (n^{-1}) than $t_{s(-1,1)}$, if $C_{20} + 2C_{11} > C_{02}$.

Further, it has been observed that estimator $t_{5(-1,1)}$ and $t_{s(-1,1)}$ have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}). Further, it is found that estimator $t_{5(-1,1)}$ is more efficient than $t_{s(-1,1)}$, if

$\rho < \frac{3\theta^4-6\theta^2+3}{\theta^3}$ which results to θ lies between $1.003 < \theta < 1.397$.

Thus, it can be concluded that the estimator $t_{5(-1,1)}$ will be more efficient than $\bar{y}_r$, $t_{1(-1,1)}$, $t_{2(-1,1)}$, $t_{3(-1,1)}$ and $t_{4(-1,1)}$, if $0.600 < \theta < 0.775$.

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Improved Class of Product Type Estimators

CHAPTER 5

IMPROVED CLASS OF PRODUCT TYPE ESTIMATORS

5.1 Introduction

The presence of negative correlation between the study variable and the auxiliary variable urges the researcher to make use of product estimator for estimation of the population mean as proposed by Robson (1957) and was again defined by Murthy (1964). Later on, several modifications have been made in order to improve the conventional product type estimator. Robson (1957) made it unbiased by subtracting its unbiased estimate of its bias in order to estimate the population mean of the study variable and proposed an unbiased estimator. Further, Singh and Chaudhary (1995) defined the conditions in which product estimator can be efficiently used as :(i) if $-1 < \rho < -C_x/2C_y$ and both Y and X are positive or negative or (ii) if $C_x/2C_y < \rho \leq +1$ and either Y or X is negative. Later on, lot of work has been done in order to improve the product estimator. Basu(1971), Srivastava and Bhatnagar (1981), Kushwaha and Singh (1988), Tripathi and Singh (1988), Singh (1989), Dubey (1993), Bhatnagar (1996), Sahoo and Sahoo (1999), Goyal *et al.* (2000), Kumar (2002), Sharma *et al.* (2007), Sharma and Bhatnagar (2008) etc., have made their tremendous contribution in this regard.

5.2 PRODUCT ESTIMATOR

Robson (1957) proposed the conventional product type estimator for estimating the population mean is given as

$$\bar{y}_p = \frac{\bar{y}\bar{x}}{\bar{X}}, \tag{5.1}$$

where, $\bar{y}$ and $\bar{x}$ are unbiased estimators of $\bar{Y}$ and $\bar{X}$, the population means of the characteristics under study and auxiliary characteristics respectively.

Theorem 5.1: The relative bias and relative mean squared error of the product type estimator ($\bar{y}_p$) upto O (n^{-1}) and O (n^{-2}) respectively, as

$$RB(\bar{y}_p) = \frac{C_{11}}{n}, \tag{5.2}$$

$$RM(\bar{y}_p) = \frac{1}{n}(C_{20} + C_{02} + 2C_{11}) + \frac{1}{n^2}(C_{20}C_{02} + 2C_{11}^2 + 2C_{12} + 2C_{21}). \tag{5.3}$$

Proof: Expressing $\bar{y}$, $\bar{x}$ and s_{xy} in terms of U_y and U_x and W respectively, in the estimator $\bar{y}_p$, we obtain,

$$\frac{\bar{y}_p - \bar{Y}}{\bar{Y}} = d^*_{-\frac{1}{2}} + d^*_{-1}. \tag{5.4}$$

Where,

$$\left.\begin{aligned} d^*_{-\frac{1}{2}} &= U_x + U_y, \\ d^*_{-1} &= U_x U_y. \end{aligned}\right\} \tag{5.5}$$

By taking the expectation of (5.4) on both sides and using the moments, we obtain the relative bias of the conventional product type estimator upto order O (n^{-1}) as obtained in expression (5.2) of Theorem 5.1.

By taking the expectation of the square of the expression (5.4) on both sides, we have $E\left(\frac{\bar{y}_p - \bar{Y}}{\bar{Y}}\right)^2 = E\left({d^{**}_{-\frac{1}{2}}}^2\right) + E\left({d^{**}_{-1}}^2\right) + 2E(d^*_{-\frac{1}{2}} d^*_{-1})$, and using the results as discussed in chapter 3, the relative mean squared error of $\bar{y}_p$ upto order O (n^{-2}) can be obtained as in expression (5.3) of Theorem 5.1.

Since, the conventional product estimator proposed by Robson (1957) was biased; he tried to develop unbiased product estimator by subtracting unbiased estimate of the bias of $\bar{y}_p$ as follows:

$$\boldsymbol{t_R^p = \bar{y}_p - \frac{1}{n}\frac{s_{xy}}{\bar{X}}}. \tag{5.6}$$

The relative variance of t_R^p is as

$$RV\left(t_R^p\right) = RM\left(\bar{y}_p\right) - \frac{1}{n^2}\left[C_{11}^2 + 2(C_{12} + C_{21})\right]. \tag{5.7}$$

Singh (1989) proposed an almost unbiased product estimator of population mean of study variable as

$$t^* = \frac{\bar{y}\bar{x}}{\bar{X}}\left[1 + \frac{s_{xy}}{n\bar{x}\bar{y}}\right]^{-1} \tag{5.8}$$

whose relative bias and relative mean squared error of the estimator t^* upto order O (n^{-2}) are given by

$$RB(t^*) = \frac{C_{11}^2}{n^2} \text{and } RM(t^*) = RM\left(\bar{y}_p\right) - \frac{1}{n^2}\left[C_{11}^2 + 2(C_{12} + C_{21})\right]. \tag{5.9}$$

The estimator t^* is unbiased upto order O (n^{-1}) and has relative mean squared error equal to relative variance of t_R^p. Thus, $RM(t^*) = RV(t_R^p)$.

Dubey (1993) used modified the product estimator proposed by Robson (1957) by using the estimate of the population mean of the auxiliary variable instead of its population mean and proposed an almost unbiased product type estimator as

$$t_D^p = \bar{y}_p - \frac{s_{xy}}{n\bar{x}}. \tag{5.10}$$

The relative variance of the estimator t_D^p is as

$$RV(t_D^p) = RM\left(\bar{y}_p\right) + \frac{1}{n^2}\left[2C_{20}C_{11} + C_{11}^2 - 2C_{12} - 2C_{21}\right]. \tag{5.11}$$

Sharma *et al.* (2007) developed a general improved class of improved product estimators by taking the usual product estimator as linear combination of $\bar{y}$ alongwith $\frac{s_x^2}{\bar{X}^2}$ and worked out their relative biases and relative mean squared error upto the order O (n^{-1}) and O (n^{-2}) respectively as under:

$$t_s^p = \bar{y}\left[\frac{\bar{x}}{\bar{X}} + q\,\frac{s_x^2}{\bar{X}^2}\right]. \tag{5.12}$$

The relative bias and relative mean squared error of the estimator t_s^p upto order O (n^{-1}) and O (n^{-2}) are as

$$RB\left(t_s^p\right) = RB\left(\bar{y}_p\right) + q\,\frac{C_{20}}{n}, \tag{5.13}$$

$$RM(t_s^p) = RM\left(\bar{y}_p\right) + \frac{q}{n^2}\left[4C_{20}C_{11} + 2C_{20}C_{02} + 2C_{30} + 2C_{21} + qC_{20}^2\right]. \tag{5.14}$$

5.3 IMPROVED GENERAL CLASS OF PRODUCT TYPE ESTIMATORS

5.3.1 The following improved class of product type estimators for $\bar{Y}$ have been proposed as

$$\boldsymbol{t_1^p = \bar{y}\left[\frac{\bar{x}}{\bar{X}} + \frac{q}{n}\frac{s_x^2}{\bar{x}\bar{X}}\right]}, \tag{5.15}$$

where, q is the scalar specifying the estimator.

Theorem 5.2: The relative bias and relative mean squared error of the estimator t_1^p upto order O (n^{-1}) and O (n^{-2}) respectively as

$$RB\left(t_1^p\right) = RB\left(\bar{y}_p\right) + \frac{q}{n}C_{20}, \tag{5.16}$$

$$RM\left(t_1^p\right) = RM\left(\bar{y}_p\right) + \frac{q}{n^2}\left[qC_{20}^2 + 2C_{20}C_{02} + 2C_{20}C_{11} - 2C_{20}^2 + 2C_{21} + 2C_{30}\right]. \tag{5.17}$$

Proof: Expressing $\bar{y}$, $\bar{x}$ and s_x^2 in terms of U_y, U_x and V_x respectively in the expression (5.15), we have

$$\frac{t_1^p - \bar{Y}}{\bar{Y}} = \left(1 + U_y\right)(1 + U_x) + \frac{q}{n}(1 + U_x)^{-1}\left(1 + U_y\right)(V_x + C_{20}). \tag{5.18}$$

Expanding the terms on the right hand side of (5.18) and retaining the terms upto order O (n$^{-3/2}$), we have

$$\frac{t_1^p - \bar{Y}}{\bar{Y}} = d_{-\frac{1}{2}}^{**} + d_{-1}^{**} + d_{-\frac{3}{2}}^{**}, \tag{5.19}$$

where,

$$\left.\begin{aligned} d^{**}_{-\frac{1}{2}} &= U_x + U_y, \\ d^{**}_{-1} &= U_x U_y + \frac{q}{n} C_{20}, \\ d^{**}_{-\frac{3}{2}} &= \frac{q}{n}\left[V_x - U_x C_{20} + U_y C_{20}\right]. \end{aligned}\right\} \quad (5.20)$$

Taking expectations of the (5.19) upto O (n^{-1}), we get

$$E\left[\frac{t_1^p - \bar{Y}}{\bar{Y}}\right] = E\left[d^{**}_{-\frac{1}{2}}\right] + E[d^{**}_{-1}], \quad (5.21)$$

where,

$$\left.\begin{aligned} E\left[d^{**}_{-\frac{1}{2}}\right] &= 0, \\ E[d^{**}_{-1}] &= \frac{C_{11}}{n} + \frac{q}{n} C_{20}. \end{aligned}\right\} \quad (5.22)$$

We get the expression (5.16) for relative bias, to the order of our approximation after substituting the expression (5.22) in (5.21), of t_1^p.

For relative mean squared error of t_1^p,

$$E\left[\frac{t_1^p - \bar{Y}}{\bar{Y}}\right]^2 = E\left({d^{**}_{-\frac{1}{2}}}^2\right) + E\left({d^{**}_{-1}}^2\right) + 2E\left(d^{**}_{-\frac{1}{2}} d^{**}_{-1}\right) + 2E\left(d^{**}_{-\frac{1}{2}} d^{**}_{-\frac{3}{2}}\right). \quad (5.23)$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned} E\left({d^{**}_{-\frac{1}{2}}}^2\right) &= \frac{1}{n}(C_{02} + C_{20} + 2C_{11}), \\ E\left({d^{**}_{-1}}^2\right) &= \frac{1}{n^2}[2C_{11}^2 + C_{20}C_{02}] + \frac{2}{n^2} q C_{20} C_{11} + \frac{q^2}{n^2} C_{20}^2, \end{aligned}\right\} \quad (5.24)$$

$$\left.\begin{aligned} E\left(d^{**}_{-\frac{1}{2}}d^{**}_{-1}\right) &= \frac{1}{n^2}\left[C_{21}+C_{12}\right], \\ E\left(d^{**}_{-\frac{1}{2}}d^{**}_{-\frac{3}{2}}\right) &= \frac{q}{n^2}\left[C_{20}C_{02}-C_{20}^2+C_{30}+C_{21}\right]. \end{aligned}\right\}$$

The terms of order higher than O (n^{-2}) have been dropped. Substituting the results of (5.24) in (5.23) and after algebraic simplification we obtained (5.17) of Theorem 5.2.

From (5.16), it is observed that estimator t_1^p has smaller bias than conventional product type estimator $\bar{y}_p$, if $q < 0$.

From (5.17) and (5.3), it is observed that the relative mean squared error of both estimators i.e., t_1^p and $\bar{y}_p$, are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_1^p has smaller relative mean squared error than that of $\bar{y}_p$ upto order O (n^{-2}), if

$$q\left[qC_{20}^2+2C_{20}C_{02}+2C_{20}C_{11}-2C_{20}^2+2C_{21}+2C_{30}\right]<0, \tag{5.25}$$

For bivariate normal population, the expression (5.25) reduces to

$$q\left[qC_{20}^2+2C_{20}C_{02}+2C_{20}C_{11}-2C_{20}^2\right]<0, \tag{5.26}$$

which further results that t_1^p will be more efficient than $\bar{y}_p$, if

$$q<0 \text{ and } 2\theta^2+2\rho\theta>2-q. \tag{5.27}$$

The comparison of relative mean squared error of t_1^p and t_R^p showed that t_1^p is more efficient than t_R^p upto order O (n^{-2}), if

$$q\left[qC_{20}^2+2C_{20}C_{02}+2C_{20}C_{11}-2C_{20}^2+2C_{21}+2C_{30}\right]+\left[C_{11}^2+2C_{12}+2C_{21}\right]<0, \tag{5.28}$$

which under bivariate normal distribution reduces to the following expression

$$q[qC_{20}^2 + 2C_{20}C_{02} + 2C_{20}C_{11} - 2C_{20}^2] + [C_{11}^2] < 0. \tag{5.29}$$

From (5.17) and (5.11), it can be observed that the estimator t_1^p is more efficient than t_D^p upto order O (n^{-2}), if

$$q[qC_{20}^2 + 2C_{20}C_{02} + 2C_{20}C_{11} - 2C_{20}^2 + 2C_{21} + 2C_{30}] - [2C_{20}C_{11} + C_{11}^2 - 2C_{12} - 2C_{21}] < 0, \tag{5.30}$$

under bivariate normal population, the above expression (5.30) reduces to

$$q[qC_{20}^2 + 2C_{20}C_{02} + 2C_{20}C_{11} - 2C_{20}^2] - [2C_{20}C_{11} + C_{11}^2] < 0. \tag{5.31}$$

From (5.16) and (5.13), it can be observed that the estimator t_1^p and t_s^p have identical bias upto the order O (n^{-1}).

From (5.17) and (5.14), it can be observed that the estimator t_1^p has smaller relative mean squared error than t_s^p upto order O (n^{-2}), if

$$q[-C_{20}C_{11} - C_{20}^2] > 0. \tag{5.32}$$

Further, it can has been observed that t_1^p performs better than t_s^p, if

$q < 0$ and $\theta > 0$.

5.3.1.1. Proposed Product type estimators

Case I: Consider q=0 in (5.15), the general class of improved ratio type estimator t_1^p reduces to conventional product type estimator. Thus, $t_{1(0)}^p = \bar{y}_p$, is the particular member of proposed product type estimator t_1^p.

Case II: $q < 0$, consider the value of q as q=-1, the estimator t_1 reduces to

$$t_{1(-1)}^{p} = \bar{\boldsymbol{y}}\left[\frac{\bar{\boldsymbol{x}}}{\bar{\boldsymbol{X}}} - \frac{\boldsymbol{q}}{\boldsymbol{n}}\frac{\boldsymbol{s_x^2}}{\bar{\boldsymbol{x}}\bar{\boldsymbol{X}}}\right]. \tag{5.33}$$

The relative bias and relative mean squared error of the estimator $t_{1(-1)}^{p}$ upto order O (n^{-1}) and O (n^{-2}) are as

$$RB\left(t_{1(-1)}^{p}\right) = RB(\bar{y}_p) - \frac{1}{n}C_{20}, \tag{5.34}$$

$$RM\left(t_{1(-1)}^{p}\right) = RM(\bar{y}_p) + \frac{1}{n^2}[3C_{20}^2 - 2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{21} - 2C_{30}]. \tag{5.35}$$

The relative bias of the estimator $t_{1(-1)}^{p}$ will be smaller than conventional product type estimator $\bar{y}_p$ upto the order O (n^{-1}), if $C_{20} > 0$.

The comparison of relative mean squared error of $t_{1(-1)}^{p}$ and $\bar{y}_p$ showed that $t_{1(-1)}^{p}$ is more efficient than $\bar{y}_p$, if

$\rho > \frac{2\theta^2-3}{2\theta}$ and the value of θ should lie between $0.825 < \theta < 1.221$. (5.36)

The relative mean squared error of $t_{1(-1)}^{p}$ is smaller than t_R^p upto order O (n^{-2}), if

$$C_{11}^2 + 3C_{20}^2 - 2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{20}^2 - 2C_{30} + 2C_{12} < 0. \tag{5.37}$$

For bivariate normal population, the expression (5.37) reduces to

$$C_{11}^2 + 3C_{20}^2 - 2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{20}^2 < 0. \tag{5.38}$$

Thus, the proposed estimator $t_{1(-1)}^{p}$ will be efficient than estimator proposed by Robson (1957) upto order (n^{-2}), if

$\rho > \frac{3-\theta^2}{2\theta}$ and the value of θ should lie between $1.750 < \theta < 3.000$. (5.39)

From (5.35) and (5.11), it can be observed that the estimator $t^{p}_{1(-1)}$ is more efficient than t^{p}_{D}, if

$$3C_{20}^2 - C_{11}^2 - 2C_{20}C_{02} - 4C_{20}C_{11} - 2C_{20}^2 - 2C_{30} + 2C_{12} < 0, \tag{5.40}$$

Under bivariate normal population, the above expression (5.40) reduces to

$$3C_{20}^2 - C_{11}^2 - 2C_{20}C_{02} - 4C_{20}C_{11} - 2C_{20}^2 < 0. \tag{5.41}$$

Thus, the proposed estimator $t^{p}_{1(-1)}$ will be efficient than Estimator proposed by Dubey (1993) upto order (n^{-2}), if

$$\rho > \frac{3-3\theta^2}{4\theta} \text{and the value of } \theta \text{ should lie between } 1.000 < \theta < 1.868. \tag{5.42}$$

The estimators $t^{p}_{1(-1)}$ and $t^{p}_{s(-1)}$ have identical relative biases upto order O (n^{-1}). Further, it can be observed that the proposed estimator $t^{p}_{1(-1)}$has smaller relative mean squared error than $t^{p}_{s(-1)}$ upto order O (n^{-2}), if $C_{20}C_{11} + C_{20}^2 < 0$. (5.43)

The estimator $t^{p}_{1(-1)}$ performed better than $t^{p}_{s(-1)}$upto order O (n^{-2}), if $\theta > 0$.

Thus, it can be concluded that estimator $t^{p}_{1(-1)}$ will be more efficient than $\bar{y}_p$, t^{p}_{R}, t^{p}_{D} ,t^{*} and $t^{p}_{s(-1)}$ if , $1.000 < \theta < 1.221$.

Case III: For $q > 0$, consider the value of q as q=1, the estimator t^{p}_{1} will become

$$t^{p}_{1(1)} = \bar{y}_p + \frac{1}{n}\bar{y}\frac{s_x^2}{\bar{x}\bar{X}}. \tag{5.44}$$

The relative bias and relative mean squared error of the estimator $t^{p}_{1(1)}$ are as

$$RB\left(t^{p}_{1(1)}\right) = RB(\bar{y}_p) + \frac{1}{n}C_{20}, \tag{5.45}$$

$$RM\left(t_{1(1)}^{p}\right) = RM(\bar{y}_p) + \frac{1}{n^2}[2C_{20}C_{02} + 2C_{20}C_{11} - C_{20}^2 + 2C_{21} + 2C_{30}]. \quad (5.46)$$

From (5.45), it has been observed that the proposed estimator $t_{1(1)}^{p}$ has more bias than estimator $\bar{y}_p$ upto order O (n^{-1}).

The relative mean squared error of the proposed estimator $t_{1(1)}^{p}$ is smaller than estimator $\bar{y}_p$ upto order O (n^{-2}), if

$$2C_{20}C_{02} + 2C_{20}C_{11} - C_{20}^2 + 2C_{21} + 2C_{30} < 0. \quad (5.47)$$

Under bivariate normal distribution, the expression (5.47) reduces to

$$2C_{20}C_{02} + 2C_{20}C_{11} - C_{20}^2 < 0. \quad (5.48)$$

Thus, proposed estimator $t_{1(1)}^{p}$ will be more efficient than $\bar{y}_p$, if the value of θ should lie between $0.720 < \theta < 1.165$.

Further, it is found that proposed estimator $t_{1(1)}^{p}$ is more efficient than estimator t_R^p upto order O (n^{-2}), if

$$2C_{20}C_{02} + 2C_{20}C_{11} - C_{20}^2 + C_{11}^2 + 2C_{12} + 4C_{21} + 2C_{30} < 0, \quad (5.49)$$

Under bivariate normal distribution, expression (5.49) reduces to

$$2C_{20}C_{02} + 2C_{20}C_{11} - C_{20}^2 + C_{11}^2 < 0 \quad (5.50)$$

Thus, $t_{1(1)}^{p}$ will be more efficient than t_R^p, if the value of θ should lie between $0.578 < \theta < 1.000$.

Further, it is observed that estimator $t_{1(1)}^{p}$ is more efficient than estimator t_D^p upto order O (n^{-2}), if

$$2C_{20}C_{02} - C_{20}^2 - C_{11}^2 + 2C_{12} + 4C_{21} + 2C_{30} < 0, \tag{5.51}$$

under bivariate normal distribution, the above expression reduces to

$2C_{20}C_{02} - C_{20}^2 - C_{11}^2 < 0$, the value of θ should lie between $0.001 < \theta < 1.000$. (5.52)

Thus, the proposed estimator $t_{1(1)}^p$ will be more efficient than $\bar{y}_p$, t_R^p and t_D^p, if $0.578 < \theta < 1.000$.

It can be observed that the estimators $t_{1(1)}^p$ and $t_{s(1)}^p$ have identical bias upto order O (n^{-1}).

The estimator $t_{1(1)}^p$ has smaller relative mean squared error than $t_{s(1)}^p$ upto order O (n^{-2}), if

$\theta > 1$.

5.3.2 Another improved class of ratio type estimators for $\bar{Y}$ have been proposed as

$$\boldsymbol{t_2^p = \bar{y}_p + \frac{q}{n}\bar{y}_r\frac{s_{xy}}{\bar{x}\bar{y}}}, \tag{5.53}$$

where, q are scalars specifying the estimator.

Theorem 5.3: The relative bias and relative mean squared error of the estimator t_2^p upto order O (n^{-1}) and O (n^{-2}) respectively as

$$RB\left(t_2^p\right) = RB\left(\bar{y}_p\right) + \frac{q}{n}C_{11}, \tag{5.54}$$

$$RM\left(t_2^p\right) = RM\left(\bar{y}_p\right) + \frac{q}{n^2}\left[qC_{11}^2 - 2C_{11}^2 - 4C_{20}C_{11} + 2C_{12} + 2C_{21}\right]. \tag{5.55}$$

Proof: Expressing $\bar{y}$, $\bar{x}$ and s_{xy} in terms of U_y, U_x and W respectively in the expression (5.53), we have

$$\frac{t_2^p - \bar{Y}}{\bar{Y}} = \left(1 + U_y\right)(1 + U_x) + \frac{q}{n}(1 + U_x)^{-2}(W + C_{11}). \tag{5.56}$$

Expanding the terms on the right hand side of (5.56) and retaining the terms upto order O ($n^{-3/2}$), we have

$$\frac{t_2^p - \bar{Y}}{\bar{Y}} = d^{**}_{-\frac{1}{2}} + e^{**}_{-1} + e^{**}_{-\frac{3}{2}}, \tag{5.57}$$

where,

$$\left.\begin{aligned} d^{**}_{-\frac{1}{2}} &= U_y + U_x, \\ e^{**}_{-1} &= U_x U_y + \frac{q}{n} C_{11}, \\ e^{**}_{-\frac{3}{2}} &= \frac{q}{n}[W - 2U_x C_{11}]. \end{aligned}\right\} \tag{5.58}$$

Taking expectations of the (5.57) upto order O (n^{-1}), we get

$$E\left[\frac{t_2^p - \bar{Y}}{\bar{Y}}\right] = E\left[d^{**}_{-\frac{1}{2}}\right] + E[e^{**}_{-1}], \tag{5.59}$$

where,

$$\left.\begin{aligned} E\left[d^{**}_{-\frac{1}{2}}\right] &= 0, \\ E[e^{**}_{-1}] &= \frac{C_{11}}{n} + \frac{q}{n} C_{11}. \end{aligned}\right\} \tag{5.60}$$

We get the expression (5.54) for relative bias, to the order of our approximation after substituting the expression (5.60) in (5.59), of t_2^p.

For relative mean squared error of t_2^p ,

$$E\left[\frac{t_2^p - \bar{Y}}{\bar{Y}}\right]^2 = E\left({d^{**}_{-\frac{1}{2}}}^2\right) + E\left({e^{**}_{-1}}^2\right) + 2E\left(d^{**}_{-\frac{1}{2}} e^{**}_{-1}\right) + 2E\left(d^{**}_{-\frac{1}{2}} e^{**}_{-\frac{3}{2}}\right). \tag{5.61}$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned}
&E\left({d^{*}_{-\frac{1}{2}}}^{2}\right)=\frac{1}{n}\left(C_{02}+C_{20}+2C_{11}\right),\\
&E\left({e^{**}_{-1}}^{2}\right)=\frac{1}{n^2}\left[2C_{11}^2+C_{20}C_{02}\right]+\frac{2}{n^2}qC_{11}^2+\frac{q^2}{n^2}C_{11}^2,\\
&E\left(d^{**}_{-\frac{1}{2}}e^{**}_{-1}\right)=\frac{1}{n^2}\left[C_{21}+C_{12}\right],\\
&E\left(d^{**}_{-\frac{1}{2}}e^{**}_{-\frac{3}{2}}\right)=\frac{q}{n^2}\left[C_{12}+C_{21}-2C_{11}^2-2C_{20}C_{11}\right].
\end{aligned}\right\} \tag{5.62}$$

The terms of order higher than O (n^{-2}) have been dropped. Substituting the results of (5.62) in (5.61) and after algebraic simplification we obtained (5.55) of Theorem 5.3.

From (5.53), it is observed that estimator t_2^p is unbiased at q=-1.

From (5.55) and (5.3), it is observed that the relative mean squared error of both estimators i.e., t_2^p and $\bar{y}_p$, are identical upto order O (n^{-1}). Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_2^p has smaller relative mean squared error than that of $\bar{y}_p$, if

$$q\left[qC_{11}^2-2C_{11}^2-4C_{20}C_{11}+2C_{12}+2C_{21}\right]<0, \tag{5.63}$$

For bivariate normal population, the expression (5.63) reduces to

$$q\left[qC_{11}^2-2C_{11}^2-4C_{20}C_{11}\right]<0. \tag{5.64}$$

Further, it has been observed that estimator t_2^p performed better than $\bar{y}_p$, if

$$0<q<2\left(1-\frac{2\rho}{\theta}\right). \tag{5.65}$$

The comparison of relative mean squared error of t_2^p and t_R^p showed that t_2^p is more efficient than t_R^p, if

$$q\left[qC_{11}^2-2C_{11}^2-4C_{20}C_{11}+2C_{12}+2C_{21}\right]+\left[C_{11}^2+2C_{12}+2C_{21}\right]<0, \tag{5.66}$$

which under bivariate normal distribution reduces to the following expression

$$q[qC_{11}^2 - 2C_{11}^2 - 4C_{20}C_{11}] + [C_{11}^2] < 0. \tag{5.67}$$

From (5.55) and (5.11), it can be observed that the estimator t_2^p is more efficient than t_D^p, if

$$q[qC_{11}^2 - 2C_{11}^2 - 4C_{20}C_{11} + 2C_{12} + 2C_{21}] - [2C_{20}C_{11} + C_{11}^2 - 2C_{12} - 2C_{21}] < 0, \tag{5.68}$$

under bivariate normal population, the above expression (5.68) reduces to

$$q[qC_{11}^2 - 2C_{11}^2 - 4C_{20}C_{11}] - [2C_{20}C_{11} + C_{11}^2] < 0. \tag{5.69}$$

From (5.54) and (5.13), it can be observed that the estimator t_2^p has smaller bias than t_s^p, if $q > 0$ and $C_{11} < C_{20}$.

From (5.55) and (5.14), it can be observed that the estimators t_2^p has smaller relative mean squared error than t_s^p, if

$$q[qC_{11}^2 - qC_{20}^2 - 2C_{11}^2 - 8C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30}] < 0, \tag{5.70}$$

which under bivariate normal population gives

$$q > 0 \text{ and } qC_{11}^2 - qC_{20}^2 - 2C_{11}^2 - 8C_{20}C_{11} - 2C_{20}C_{02} < 0. \tag{5.71}$$

From (5.54) and (5.16), it can be observes that the estimator t_2^p has smaller bais than t_1^p, if $q > 0$ and $C_{20} > C_{11}$. (5.72)

From (5.55) and (5.17), it can be observed that the estimator t_2^p has smaller relative mean squared error than t_1^p, if

$$q[qC_{11}^2 - qC_{20}^2 - 2C_{11}^2 + 2C_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30}] < 0, \tag{5.73}$$

under bivariate population, expression (5.73) reduces to

$$q[qC_{11}^2 - qC_{20}^2 - 2C_{11}^2 + 2C_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02}] < 0, \tag{5.74}$$

which gives $q > 0$ and $qC_{11}^2 - qC_{20}^2 - 2C_{11}^2 + 2C_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02} < 0$. (5.75)

5.3.2.1. Proposed Product type estimators

Case I: Consider q=0 in (5.53), the general class of improved ratio type estimator t_2^p reduces to conventional product type estimator. Thus, $t_{2(0)}^p = \bar{y}_p$, is the particular member of general class of proposed product type estimator t_2^p.

Case II: For $q < 0$, consider the value of q as q= -1, the estimator t_2^p will become

$t_{2(-1)}^p = \bar{y}_p - \frac{1}{n}\bar{y}_r \frac{s_{xy}}{\bar{x}\bar{y}}$, the bias of the estimator is equal to zero. Thus, the proposed estimator is unbiased. (5.76)

Further, the relative variance of the estimator $t_{2(-1)}^p$ is as

$$RV\left(t_{2(-1)}^p\right) = RM\left(\bar{y}_p\right) + \frac{1}{n^2}[3C_{11}^2 + 4C_{20}C_{11} - 2C_{12} - 2C_{21}]. \tag{5.77}$$

From (5.77) and (5.3), it is observed that the relative mean squared error of the estimator $\bar{y}_p$ is identical to the estimator $t_{2(-1)}^p$upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Thus, $3C_{11}^2 + 4C_{20}C_{11} - 2C_{12} - 2C_{21} < 0$. (5.78)

For bivariate normal distribution, the above expression (5.78) reduces to

$$3C_{11}^2 + 4C_{20}C_{11} < 0. \tag{5.79}$$

Thus, it can be observed that $t_{2(-1)}^p$ is better than $\bar{y}_p$, if $0 < \theta < -\frac{4}{3}\rho$. (5.80)

From (5.77) and (5.7), it can be shown that estimator $t_{2(-1)}^p$ and t_R^p have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{2(-1)}^p$ is more efficient than t_R^p, if $4C_{11}^2 + 4C_{20}C_{11} < 0$. (5.81)

Thus, estimator $t^{p}_{2(-1)}$ will perform better than the estimator proposed by Robson (1957) upto order O (n^{-1}), if $0 < \theta < -\rho$.

From (5.77) and (5.11), it can be shown that estimator $t^{p}_{2(-1)}$ and t^{p}_{D} have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t^{p}_{2(-1)}$ is more efficient than estimator t^{p}_{D} upto order O (n^{-2}), if $2C_{20}C_{11} + 2C_{11}^{2} < 0$ (5.82)

Thus, the estimation $t^{p}_{2(-1)}$ will have smaller variance than the estimator t^{p}_{D}, if $0 < \theta < -\rho$.

The estimator $t^{p}_{2(-1)}$ has smaller relative mean squared error than $t^{p}_{s(-1)}$, if

$$3C_{11}^{2} - C_{20}^{2} + 2C_{20}C_{02} + 8C_{20}C_{11} - 2C_{12} + 2C_{30} < 0. \tag{5.83}$$

Under bivariate normal distribution, expression (5.83) reduces to

$$3C_{11}^{2} - C_{20}^{2} + 2C_{20}C_{02} + 8C_{20}C_{11} < 0. \tag{5.84}$$

Thus, the estimator $t^{p}_{2(-1)}$ is better than $t^{p}_{s(-1)}$ upto order O (n^{-2}), if

$\rho < \frac{1-5\theta^{2}}{8\theta}$and value of θ should lie between $0.449 < \theta < 1.710.$ (5.85)

Further, it can be observed that the estimator$t^{p}_{2(-1)}$has smaller relative mean squared error than $t^{p}_{1(-1)}$, if $3C_{11}^{2} - 3C_{20}^{2} + 6C_{20}C_{11} + 2C_{20}C_{02} - 2C_{12} + 2C_{30} < 0,$ (5.86)

under bivariate normal distribution, expression (5.86) reduces to

$$3C_{11}^{2} - 3C_{20}^{2} + 6C_{20}C_{11} + 2C_{20}C_{02} < 0. \tag{5.87}$$

Thus, the proposed estimator $t^{p}_{2(-1)}$will be more efficient than $t^{p}_{1(-1)}$ upto order O (n^{-2}), if $\rho < \frac{3-5\theta^{2}}{6\theta}$ and value of θ should lie between $1.100 < \theta < 1.849.$ (5.88)

Thus, the proposed estimator $t^p_{2(-1)}$ will be more efficient than $\bar{y}_p$, t^p_R, t^p_D, $t^p_{s(-1)}$ and $t^p_{1(-1)}$ upto order O (n^{-2}), if the value of θ should lie between $1.000 < \theta < 1.477$.

Case III: For $\boldsymbol{q} > 0$, consider the value of q as $q = 1$**,** the estimator t^p_2 will become

$$t^p_{2(1)} = \bar{y}_p + \frac{1}{n}\bar{y}_r \frac{s_{xy}}{\bar{x}\bar{y}}. \tag{5.89}$$

The relative bias and the relative mean squared error of the estimator $t^p_{2(1)}$ upto the order O (n^{-1}) and O (n^{-2}) respectively are as

$$RB\left(t^p_{2(1)}\right) = RB(\bar{y}_p) + \frac{1}{n}C_{11}, \tag{5.90}$$

$$RM\left(t^p_{2(1)}\right) = RM(\bar{y}_p) + \frac{1}{n^2}[2C_{12} + 2C_{21} - C_{11}^2 - 4C_{20}C_{11}]. \tag{5.91}$$

The proposed estimator $t^p_{2(1)}$ has smaller than estimator $\bar{y}_p$ upto order O (n^{-1}), if $\boldsymbol{C_{11}} < 0$**.**

The relative mean squared error of the estimator $t^p_{2(1)}$ is smaller than the estimator $\bar{y}_p$ upto order O (n^{-2}), if $2C_{12} + 2C_{21} - C_{11}^2 - 4C_{20}C_{11} < 0$, (5.92)

Under bivariate normal distribution, the expression (5.92) reduces to

$$C_{11}^2 + C_{20}C_{11} > 0, \tag{5.93}$$

Thus, the estimator $t^p_{2(1)}$ will be more efficient than $\bar{y}_p$ upto order O (n^{-2}), if $0 < \theta < -4\rho$.

Further, it is found that estimator $t^p_{2(1)}$ is more efficient than t^p_R upto order O (n^{-2}), if $\rho < 0$ and $\theta > 0$.

The estimator $t^p_{2(1)}$ performed better than estimator t^p_D upto order O (n^{-2}), if $0 < \theta < -3$

The estimator $t^p_{2(1)}$ has smaller bias than $t^p_{s(1)}$ and $t^p_{1(1)}$ upto order O (n^{-1}), if $C_{20} > C_{11}$.

The estimator $t^{p}_{2(1)}$ has smaller relative mean squared error than $t^{p}_{s(1)}$upto order O (n^{-2}), if

$$C_{11}^2 + C_{20}^2 + 8C_{20}C_{11} + 2C_{20}C_{02} + 2C_{12} - 2C_{30} > 0, \tag{5.94}$$

under bivariate normal distribution, the expression (5.94) reduces to

$$C_{11}^2 + C_{20}^2 + 8C_{20}C_{11} + 2C_{20}C_{02} > 0. \tag{5.95}$$

Thus, the estimator $t^{p}_{2(1)}$ will be more efficient than $t^{p}_{s(1)}$upto order O (n^{-2}), if

$$\rho < \frac{3\theta^2+1}{8\theta} \text{ and the value of } \theta \text{ should lie between } 0.578 < \theta < 2.535. \tag{5.96}$$

It can be observed that the estimator $t^{p}_{2(1)}$has smaller relative mean squared error than

upto order O (n^{-2}), if $C_{20}^2 - C_{11}^2 - 6C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30} < 0,$ (5.97)

under bivariate normal distribution, the expression (5.97) reduces to

$$C_{20}^2 - C_{11}^2 - 6C_{20}C_{11} - 2C_{20}C_{02} < 0. \tag{5.98}$$

Thus, the estimator $t^{p}_{2(1)}$ will be more efficient than $t^{p}_{1(1)}$upto order O (n^{-2}), if

$$\rho < \frac{1-3\theta^2}{6\theta} \text{ and the value of } \theta \text{ should lie between } 0.700 < \theta < 1.477. \tag{5.99}$$

Thus, it can be concluded that the estimator $t^{p}_{2(1)}$ performs better than $\bar{y}_p$, t^{p}_{R}, t^{*}, t^{p}_{D}, $t^{p}_{s(1)}$ and $t^{p}_{1(1)}$, if $1.003 < \theta < 1.477$.

5.3.3 The following improved class of product type estimators for $\bar{Y}$ have been proposed as

$$\boldsymbol{t_3^p = \bar{y}_p + \frac{q}{n}\bar{y}_r \frac{s_{xy}}{\bar{X}\bar{y}}}, \tag{5.100}$$

where, q is the scalars specifying the estimator.

Theorem 5.4: The relative bias and relative mean squared error of the estimator t^{p}_{3} upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB\left(t_3^p\right) = RB\left(\bar{y}_p\right) + \frac{q}{n}C_{11}, \tag{5.101}$$

$$RM\left(t_3^p\right) = RM\left(\bar{y}_p\right) + \frac{q}{n^2}\left[qC_{11}^2 - 2C_{20}C_{11} + 2C_{12} + 2C_{21}\right].1 \tag{5.102}$$

Proof: Expressing $\bar{y}$, $\bar{x}$ and s_{xy} in terms of U_y, U_x and W respectively in the expression (5.100), we have

$$\frac{t_3^p - \bar{Y}}{\bar{Y}} = \left(1 + U_y\right)(1 + U_x) + \frac{q}{n}(1 + U_x)^{-1}(W + C_{11}). \tag{5.103}$$

Expanding the terms on the right hand side of (5.103) and retaining the terms upto order O ($n^{-3/2}$), we have

$$\frac{t_3^p - \bar{Y}}{\bar{Y}} = d^{**}_{-\frac{1}{2}} + e^{**}_{-1} + f^{**}_{-\frac{3}{2}}, \tag{5.104}$$

where,

$$\left.\begin{aligned} d^{**}_{-\frac{1}{2}} &= U_y + U_x, \\ e^{**}_{-1} &= U_x U_y + \frac{q}{n}C_{11}, \\ f^{**}_{-\frac{3}{2}} &= \frac{q}{n}\left[W - U_x C_{11}\right]. \end{aligned}\right\} \tag{5.105}$$

Taking expectations of the (5.104) upto O (n^{-1}), we get

$$E\left[\frac{t_3^p - \bar{Y}}{\bar{Y}}\right] = E\left[d^{**}_{-\frac{1}{2}}\right] + E\left[e^{**}_{-1}\right], \tag{5.106}$$

where,

$$\left.\begin{aligned} E\left[d^{**}_{-\frac{1}{2}}\right] &= 0, \\ E\left[e^{**}_{-1}\right] &= \frac{C_{11}}{n} + \frac{q}{n}C_{11}. \end{aligned}\right\} \tag{5.107}$$

We get the expression (5.101) for relative bias, to the order of our approximation after substituting the expression (5.107) in (5.106) of t_3^p.

For relative mean squared error of t_3^p ,

$$E\left[\frac{t_3^p-\bar{Y}}{\bar{Y}}\right]^2 = E\left({d^{**}_{-\frac{1}{2}}}^2\right) + E\left({e^{**}_{-1}}^2\right) + 2E\left(d^{**}_{-\frac{1}{2}}e^{**}_{-1}\right) + 2E\left(d^{**}_{-\frac{1}{2}}f^{**}_{-\frac{3}{2}}\right). \tag{5.108}$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned}
E\left({d^{*}_{-\frac{1}{2}}}^2\right) &= \frac{1}{n}(C_{02} + C_{20} + 2C_{11}),\\
E\left({e^{**}_{-1}}^2\right) &= \frac{1}{n^2}[2C_{11}^2 + C_{20}C_{02}] + \frac{2}{n^2}qC_{11}^2 + \frac{q^2}{n^2}C_{11}^2,\\
E\left(d^{**}_{-\frac{1}{2}}e^{**}_{-1}\right) &= \frac{1}{n^2}[C_{21} + C_{12}],\\
E\left(d^{**}_{-\frac{1}{2}}f^{**}_{-\frac{3}{2}}\right) &= \frac{q}{n^2}[C_{21} + C_{12} - C_{11}^2 - C_{20}C_{11}].
\end{aligned}\right\} \tag{5.109}$$

The terms of order higher than order O (n^{-2}) have been dropped. Substituting the results of (5.109) in (5.108) and after algebraic simplification we obtained (5.102) of **Theorem** 5.4.

From (5.101), it is observed that the bias of the proposed estimator t_3^p will be zero, if $q = -1$.Thus the proposed estimator is unbiased.

From (5.102) and (5.3), it is observed that the relative mean squared error of both estimators i.e., t_3^p and $\bar{y}_p$, are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_3^p has smaller relative mean squared error than that of $\bar{y}_p$, if

$$q[qC_{11}^2 - 2C_{20}C_{11} + 2C_{12} + 2C_{21}] < 0. \tag{5.110}$$

For bivariate normal population, the expression (5.110) reduces to

$$q[qC_{11}^2 - 2C_{20}C_{11}] < 0. \tag{5.111}$$

Thus, the proposed estimator t_3^p will be more efficient than $\bar{y}_p$ upto order O (n^{-2}), if $q > 0$ and $qC_{11}^2 - 2C_{20}C_{11} < 0$. (5.112)

The comparison of relative mean squared error of t_3^p and t_R^p showed that t_3^p is more efficient than t_R^p, if

$$q[qC_{11}^2 - 2C_{20}C_{11} + 2C_{12} + 2C_{21}] + [C_{11}^2 + 2C_{12} + 2C_{21}] < 0. \tag{5.113}$$

which under bivariate normal distribution reduces to the following expression

$$q[qC_{11}^2 - 2C_{20}C_{11}] + [C_{11}^2] < 0. \tag{5.114}$$

From (5.102) and (5.11), it can be observed that the estimator t_3^p is more efficient than t_D^p, if

$$q[qC_{11}^2 - 2C_{20}C_{11} + 2C_{12} + 2C_{21}] - [2C_{20}C_{11} + C_{11}^2 - 2C_{12} - 2C_{21}] < 0. \tag{5.115}$$

Under bivariate normal population, the above expression (5.115) reduces to

$$q[qC_{11}^2 - 2C_{20}C_{11}] - [2C_{20}C_{11} + C_{11}^2] < 0.$$

From (5.101), (5.13) and (5.16), it can be observed that the estimator t_3^p has smaller bias than t_s^p and t_1^p upto order O (n^{-1}), if $q > 0$ and $C_{11} < C_{20}$.

From (5.102) and (5.14), it can be observed that the estimator t_3^p has smaller relative mean squared error than t_s^p upto order O(n^{-2}), if

$$q[qC_{11}^2 - qC_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30}] < 0 \tag{5.116}$$

which under bivariate normal distribution reduces to

$$q[qC_{11}^2 - qC_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02}] < 0. \tag{5.117}$$

Further, t_3^p will be more efficient than t_s^p, if

$q > 0$ and $qC_{11}^2 - qC_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02} < 0.$ (5.118)

From (5.102) and (5.17), it can be observed that the estimator t_3^p has smaller relative mean squared error than t_1^p upto order O (n^{-2}), if

$$q[qC_{11}^2 - qC_{20}^2 + 2C_{20}^2 - 4C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30}] < 0. \quad (5.119)$$

Under bivariate normal distribution, the expression (5.119) reduces to

$$q[qC_{11}^2 - qC_{20}^2 + 2C_{20}^2 - 4C_{20}C_{11} - 2C_{20}C_{02}] < 0. \quad (5.120)$$

Further, proposed estimator t_3^p will perform better than t_1^p, if

$q > 0$ and $qC_{11}^2 - qC_{20}^2 + 2C_{20}^2 - 4C_{20}C_{11} - 2C_{20}C_{02} < 0.$ (5.121)

From (5.101) and (5.54), it can be observed that the estimator t_3^p and t_2^p have identical bias upto order O (n^{-1}).

From (5.102) and (5.55), it can be observed that the estimator t_3^p has smaller relative mean squared error than t_2^p upto order O (n^{-2}), if $q[2C_{20}C_{11} + 2C_{11}^2] < 0.$ (5.122)

Thus, under bivariate normal distribution, the proposed estimator t_3^p will be more efficient than t_2^p upto order O (n^{-2}), if $q > 0$ and $0 < \theta < -\rho$.

5.3.3.1 Proposed product type estimators: For different values of q, different cases have been formed.

Case I: Consider q=0 in (5.100**),** the general class of improved ratio type estimator t_3^p reduces to conventional product type estimator. Hence, $t_{3(0)}^p = \bar{y}_p$, is the particular case of general class of proposed product type estimator t_3^p.

Case II: For $q < 0$, consider the value of q= -1**,** the estimator t_3^p will become $t_{3(-1)}^p = t_D^p$ *i.e.*, the estimator t_3^p reduces to the estimator proposed by Dubey (1993). Thus, it can be said that the estimator t_D^p is a particular case of the proposed general class of product type estimator.

Now, if we consider q=-2, the general class of proposed product type estimator reduces to t_3^p reduces to $t_{3(-2)}^p = \bar{y}_p - \frac{2}{n}\bar{y}_r \frac{s_{xy}}{\bar{X}\bar{y}}$, where, q is the scalars specifying the estimator.

(5.123)

The relative bias and relative mean squared error of the estimator $t_{3(-2)}^p$ upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB\left(t_{3(-2)}^p\right) = RB(\bar{y}_p) - \frac{2}{n}C_{11}, \tag{5.124}$$

$$RM\left(t_{3(-2)}^p\right) = RM(\bar{y}_p) + 4C_{11}^2 + 4C_{20}C_{11} - 4C_{12} - 4C_{21}. \tag{5.125}$$

From (5.125) and (5.3), it is observed that the relative mean squared error of the estimator $\bar{y}_p$ is identical to the estimator $t_{3(-2)}^p$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it has been observed that the estimator $t_{3(-2)}^p$ will be more efficient than $\bar{y}_p$ upto order O (n^{-2}), if $4C_{11}^2 + 4C_{20}C_{11} - 4C_{12} - 4C_{21} < 0.$ (5.126)

For bivariate normal distribution, estimator $t_{3(-2)}^p$ will performed better than $\bar{y}_p$, if $C_{20} > C_{11}$.

From (5.125) and (5.7), it can be shown that estimator $t_{3(-2)}^p$ and t_R^p have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

$$5C_{11}^2 + 4C_{20}C_{11} - 2C_{12} - 2C_{21} < 0. \tag{5.127}$$

For bivariate normal population, estimator $t_{3(-2)}^p$ will be more efficient than t_R^p, if $\rho < -1.25\theta$ which results to θ lies between $0.001 < \theta < 0.800$.

From (5.125) and (5.11), it can be observed that estimator $t_{3(-2)}^p$ and t_D^p have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t^p_{3(-2)}$ is more efficient than estimator t^p_D upto order O (n^{-2}), if $3C_{11}^2 + 2C_{20}C_{11} - 2C_{12} - 2C_{21} < 0.$ (5.128)

Under bivariate normal distribution, estimator $t^p_{3(-2)}$ will performed better than t^p_D, if

$\rho < -1.5\theta$ and the value of θ should lie between $0.001 < \theta < 0.667.$ (5.129)

The comparison of the estimator $t^p_{3(-2)}$ and $t^p_{s(-1)}$ showed that the estimator $t^p_{3(-2)}$ has smaller relative mean squared error than $t^p_{s(-1)}$, if

$4C_{11}^2 - C_{20}^2 + 8C_{20}C_{11} + 2C_{20}C_{02} - 4C_{12} - 2C_{21} + 2C_{30} < 0.$ (5.130)

Under bivariate normal population, the expression (5.130) reduces to

$4C_{11}^2 - C_{20}^2 + 8C_{20}C_{11} + 2C_{20}C_{02} < 0.$ (5.131)

Thus, estimator $t^p_{3(-2)}$ will be more efficient than $t^p_{s(-1)}$, if

$\rho < \frac{1-6\theta^2}{8\theta}$ and the value of θ should lie between $0.409 < \theta < 1.448.$ (5.132)

From (5.125) and (5.35), it can be observed that the estimator $t^p_{3(-2)}$ has smaller relative mean squared error than $t^p_{1(-1),}$if

$4C_{11}^2 - 3C_{20}^2 + 6C_{20}C_{11} + 2C_{20}C_{02} - 4C_{12} - 2C_{21} - 2C_{30} < 0.$ (5.133)

Under bivariate normal distribution, the above expression reduces to

$4C_{11}^2 - 3C_{20}^2 + 6C_{20}C_{11} + 2C_{20}C_{02} < 0.$ (5.134)

Thus, estimator $t^p_{3(-2)}$ will be more efficient than $t^p_{1(-1)}$, if

$\rho < \frac{1-2\theta^2}{2\theta}$and the value of θ should lie between $0.708 < \theta < 1.366.$ (5.135)

From (5.124) and (5.76), it can be observes that the estimator $t^p_{3(-2)}$ and $t^p_{2(-1)}$have identical bias.

From (5.125) and (5.77), it can be observed that the estimator $t_{3(-2)}^{p}$ has smaller relative mean squared error than $t_{2(-1),}^{p}$ for bivariate normal population, if $C_{11} < 0$.

(5.136)

Thus, it can be concluded that the proposed estimator $t_{3(-2)}^{p}$ will be more efficient than $\bar{y}_p,\ t_R^p, t^*,\ t_D^p,\ t_{s(-1)}^p$ and $t_{2(-1)}^p$, if the value of θ lies between $0.001 < \theta < 0.666$.

Case III: For $q > 0$, consider the value of q=1, the estimator t_3^p will become

$$\boldsymbol{t_{3(1)}^{p} = \bar{y}_p + \frac{1}{n}\bar{y}_r \frac{s_{xy}}{\bar{X}\bar{y}}}. \tag{5.137}$$

The relative bias and relative mean squared error of the estimator $t_{1(1)}^{p}$upto order O (n^{-1}) and O (n^{-2}) respectively are as

$$RB\left(t_{3(1)}^{p}\right) = RB(\bar{y}_p) + \frac{1}{n}C_{11}, \tag{5.138}$$

$$RM\left(t_{3(1)}^{p}\right) = RM(\bar{y}_p) + \frac{1}{n^2}[C_{11}^2 - 2C_{20}C_{11} + 2C_{12} + 2C_{21}]. \tag{5.139}$$

From (5.138) and (5.2), it is observed that the relative bias of $t_{3(1)}^{p}$is smaller than that of estimator $\bar{y}_p$ upto order O (n^{-1}), if $C_{11} < 0$.

From (5.139) and (5.3), it is observed that the relative mean squared error of the estimator $\bar{y}_p$ is identical to the estimator $t_{3(1)}^{p}$upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

From (5.139) and (5.3), it can be observed that the estimator $t_{3(1)}^{p}$ will be more efficient than $\bar{y}_p$ upto order O (n^{-2}), if $\boldsymbol{\theta} > 2$. (5.140)

From (5.139) and (5.7), it can be shown that estimator $t_{3(1)}^{p}$ and t_R^p have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{3(1)}^{p}$is more efficient than t_{R}^{p} upto order O (n^{-2}), if $C_{20} > C_{11}$.

From (5.139) and (5.11), it can be shown that estimator $t_{3(1)}^{p}$ and t_{D}^{p} have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{3(1)}^{p}$ is more efficient than estimator t_{D}^{p} upto order O (n^{-2}), if $2C_{11}^{2} + 2C_{02}C_{11} - 4C_{20}C_{11} < 0$ (5.141)

This gives $\rho < \frac{\theta}{\theta^{2}-2}$ and the value of θ should lie between $0.001 < \theta < 0.999$. (5.142)

Further, it can be observed that the estimator $t_{3(1)}^{p}$has smaller bias than $t_{s(1)}^{p}$ and $t_{1(1)}^{p}$upto order O (n^{-1}), if $C_{20} > C_{11}$.

The comparison of the estimator $t_{3(1)}^{p}$ and $t_{s(1)}^{p}$ showed that the estimator $t_{3(1)}^{p}$has smaller relative mean squared error than $t_{s(1)}^{p}$, if

$$C_{11}^{2} - C_{20}^{2} - 6C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30} < 0, \tag{5.143}$$

resulting in for bivariate normal population

$$\rho < \frac{-1-\theta^{2}}{6\theta} \text{ and the value of } \theta \text{ should lie between } 0.172 < \theta < 0.999. \tag{5.144}$$

From (5.139) and (5.46), it can be observed that the estimator $t_{3(1)}^{p}$has smaller relative mean squared error than $t_{1(1),}^{p}$if

$$C_{11}^{2} + C_{20}^{2} - 4C_{20}C_{11} - 2C_{20}C_{02} + 2C_{12} - 2C_{30} < 0, \tag{5.145}$$

under bivariate normal distribution, the above expression reduces to

$$C_{11}^{2} + C_{20}^{2} - 4C_{20}C_{11} - 2C_{20}C_{02} < 0, \tag{5.146}$$

which gives $\rho < \frac{-\theta^{2}+1}{4\theta}$and the value of θ should lie between $1.00 \leq \theta < 4.23$.

From (5.138) and (5.90), it can be observes that the estimator $t_{3(1)}^{p}$ and $t_{2(1)}^{p}$have identical bias.

From (5.139) and (5.91), it can be observed that the estimator $t_{3(1)}^{p}$has smaller relative mean squared error than $t_{2(1),}^{p}$ if $C_{20}C_{11}+C_{11}^{2}<0$ which results to θ lies between $0<\theta<-\rho$.

Thus, it can be concluded that the proposed estimator $t_{3(1)}^{p}$ will be more efficient than t_{D}^{p}, $t_{s(1)}^{p}$ and $t_{2(1)}^{p}$, if the value of θ lies between $0.172<\theta<0.999$.

5.3.4 The following improved class of ratio type estimators for $\bar{Y}$ have been proposed as

$$\boldsymbol{t_4^p=\bar{y}_p+\frac{1}{n}\bar{y}\left[p\frac{s_y^2}{\bar{y}^2}+q\frac{s_{xy}}{\bar{x}\bar{y}}\right]}, \tag{5.147}$$

where, p and q are scalars specifying the estimator.

Theorem 5.5: The relative bias and relative mean squared error of the estimator t_4^p upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB\left(t_4^p\right)=RB\left(\bar{y}_p\right)+\frac{1}{n}[pC_{02}+qC_{11}] \tag{5.148}$$

$$RM\left(t_4^p\right)=RM\left(\bar{y}_p\right)+\frac{p}{n^2}[4C_{02}C_{11}+2C_{02}^2+2C_{12}+2C_{03}]+\frac{q}{n^2}[2C_{12}+2C_{21}-2C_{20}C_{11}]+\frac{1}{n^2}[pC_{02}+qC_{11}].^2 \tag{5.149}$$

Proof: Expressing $\bar{y}$, $\bar{x}$, $s_y^2$and s_{xy}in terms of U_y, U_x , V_y and W respectively in the expression (5.147), we have

$$\frac{t_4^p-\bar{Y}}{\bar{Y}}=\left(1+U_y\right)(1+U_x)+\frac{p}{n}\left(1+U_y\right)^{-1}\left(V_y+C_{02}\right)+\frac{q}{n}(1+U_x)^{-1}(W+C_{11}). \tag{5.150}$$

Expanding the terms on the right hand side of (5.150) and retaining the terms upto order O ($n^{-3/2}$), we have $\frac{t_4^p-\bar{Y}}{\bar{Y}}=d_{-\frac{1}{2}}^{**}+d_{-1}^{**}+e_{-\frac{3}{2}}^{**}$, (5.151)

where,

$$\left.\begin{aligned} d^{**}_{-\frac{1}{2}} &= U_y + U_x; \\ f^{**}_{-1} &= U_x U_y + \frac{p}{n} C_{02} + \frac{q}{n} C_{11}; \\ f^{**}_{-\frac{3}{2}} &= \frac{p}{n}\left[V_y - U_y C_{02}\right] + \frac{q}{n}(W - U_x C_{11}). \end{aligned}\right\} \quad (5.152)$$

Taking expectations of the (5.151) upto O (n^{-1}), we get

$$E\left[\frac{t_4^p - \bar{Y}}{\bar{Y}}\right] = E\left[d^{**}_{-\frac{1}{2}}\right] + E[d^{**}_{-1}], \quad (5.153)$$

where,

$$\left.\begin{aligned} E\left[d^{**}_{-\frac{1}{2}}\right] &= 0; \\ E[d^{**}_{-1}] &= U_x U_y + \frac{p}{n} C_{20} + \frac{q}{n} C_{11}. \end{aligned}\right\} \quad (5.154)$$

We get the expression (5.148) for relative bias, to the order of our approximation after substituting the expression (5.154) in (5.153), of t_4^p

For relative mean squared error of t_4^p ,

$$E\left[\frac{t_2^p - \bar{Y}}{\bar{Y}}\right]^2 = E\left({d^{**}_{-\frac{1}{2}}}^2\right) + E\left({f^{**}_{-1}}^2\right) + 2E\left(d^{**}_{-\frac{1}{2}} f^{**}_{-1}\right) + 2E\left(d^{**}_{-\frac{1}{2}} f^{**}_{-\frac{3}{2}}\right). \quad (5.155)$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned} E\left({d^{*}_{-\frac{1}{2}}}^2\right) &= E\left(U_y^2 + U_x^2 - 2U_y U_x\right) \\ &= \frac{1}{n}(C_{02} + C_{20} + 2C_{11}), \\ E\left({f^{**}_{-1}}^2\right) &= \frac{1}{n^2}[2C_{11}^2 + C_{20}C_{02}] + \frac{2}{n^2}[pC_{02}C_{11} + qC_{11}^2] + \frac{1}{n^2}[pC_{02} + qC_{11}]^2, \end{aligned}\right\} \quad (5.156)$$

$$E\left(d^{**}_{-\frac{1}{2}} f^{**}_{-1}\right) = \frac{1}{n^2}[C_{21} + C_{12}],$$

$$E\left(d^{**}_{-\frac{1}{2}} f^{**}_{-\frac{3}{2}}\right) = \frac{p}{n^2}[C_{02}^2 + C_{02}C_{11} + C_{12} + C_{03}] + \frac{q}{n^2}[C_{12} + C_{21} - C_{11}^2 - C_{20}C_{11}].$$

The terms of order higher than O (n^{-2}) have been dropped. Substituting the results of (5.156) in (5.155) and after algebraic simplification we obtained (5.149) of Theorem 5.5.

From (5.148), it is observed that estimator t_4^p has smaller bias than conventional product type estimator $\bar{y}_p$ upto order O (n^{-1}), if p and q have different directions with $p < q$ and $C_{02} > C_{11}$.

From (5.149) and (5.3), it is observed that the relative mean squared error of both estimators i.e., t_4^p and $\bar{y}_p$, are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_4^p has smaller relative mean squared error than that of $\bar{y}_p$, if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] + q[2C_{12} + 2C_{21} - 2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 < 0. \tag{5.157}$$

For bivariate normal population, the expression (5.157) reduces to

$$p[4C_{02}C_{11} + 2C_{02}^2] - q[2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 < 0. \tag{5.158}$$

The comparison of relative mean squared error of t_4^p and t_R^p showed that t_4^p is more efficient than t_R^p upto order O (n^{-2}), if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] + q[2C_{12} + 2C_{21} - 2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 + [C_{11}^2 + 2C_{12} + 2C_{21}] < 0, \tag{5.159}$$

which under bivariate normal distribution reduces to the following expression

$$p[4C_{02}C_{11} + 2C_{02}^2] - q[2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 + [C_{11}^2] < 0. \qquad (5.160)$$

From (5.149) and (5.11), it can be observed that the estimator t_4^p is more efficient than t_D^p upto order O (n^{-2}), if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] + q[2C_{12} + 2C_{21} - 2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 - [2C_{20}C_{11} + C_{11}^2 - 2C_{12} - 2C_{21}] < 0. \qquad (5.161)$$

Under bivariate normal population, the above expression (5.161) reduces to

$$p[4C_{02}C_{11} + 2C_{02}^2] - q[2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 - [2C_{20}C_{11} + C_{11}^2] < 0. \qquad (5.162)$$

From (5.148) and (5.13), it can be observed that the estimator t_4^p has smaller bias than t_s^p, if $pC_{02} + q(C_{11} - C_{20}) < 0$.

From (5.149) and (5.14), it can be observed that the estimator t_4^p has smaller relative mean squared error than t_s^p, if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] + q[2C_{12} - 2C_{30} - qC_{20}^2 - 6C_{20}C_{11} - 2C_{20}C_{02}] + [pC_{02} + qC_{11}]^2 < 0, \qquad (5.163)$$

which under bivariate normal distribution gives

$$p[4C_{02}C_{11} + 2C_{02}^2] - q[qC_{20}^2 + 6C_{20}C_{11} + 2C_{20}C_{02}] + [pC_{02} + qC_{11}]^2 < 0. \qquad (5.164)$$

From (5.148) and (5.16), it can be observed that the estimator t_4^p has smaller bias than t_1^p upto order O (n^{-1}), if $pC_{02} + q(C_{11} - C_{20}) < 0$.

From (5.149) and (5.17), it can be observed that the estimator t_4^p has smaller relative mean squared error than t_1^p upto order O (n^{-2}), if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] + q[2C_{12} - 2C_{30} + 4C_{20}^2 - qC_{20}^2 - 4C_{20}C_{11} - 2C_{20}C_{02}] + [pC_{02} + qC_{11}]^2 < 0. \qquad (5.165)$$

Under bivariate normal distribution, the expression (5.165) reduces to

$$p[4C_{02}C_{11} + 2C_{02}^2] + q[4C_{20}^2 - qC_{20}^2 - 4C_{20}C_{11} - 2C_{20}C_{02}] + [pC_{02} + qC_{11}]^2 < 0. \quad (5.166)$$

From (5.148) and (5.54), it can be observed that the estimator t_4^p has smaller bias than t_2^p upto order O (n^{-2}), if $p < 0$ and $C_{02} > 0$.

From (5.149) and (5.55), it can be observed that the estimator t_4^p has smaller relative mean squared error than t_2^p upto order O (n^{-2}), if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] + q[2C_{11}^2 - qC_{11}^2 + 2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 < 0. \quad (5.167)$$

Under bivariate normal distribution, the expression (5.167) reduces to

$$p[4C_{02}C_{11} + 2C_{02}^2] + q[2C_{11}^2 - qC_{11}^2 + 2C_{20}C_{11}] + [pC_{02} + qC_{11}]^2 < 0. \quad (5.168)$$

From (5.148) and (5.101), it can be observed that the estimator t_4^p has smaller relative bias than t_3^p upto order O (n^{-2}), if $p < 0$ and $C_{02} > 0$.

From (5.149) and (5.102), it can be observed that the estimator t_4^p has smaller relative mean squared error than t_3^p upto order O (n^{-2}), if

$$p[4C_{02}C_{11} + 2C_{02}^2 + 2C_{12} + 2C_{03}] - qC_{11}^2 + [pC_{02} + qC_{11}]^2 < 0. \quad (5.169)$$

Under bivariate normal distribution, the expression (5.169) reduces to

$$p[4C_{02}C_{11} + 2C_{02}^2] - qC_{11}^2 + [pC_{02} + qC_{11}]^2 < 0. \quad (5.170)$$

5.3.4.1: Proposed product type estimator: For different values of p and q different cases have been formed.

Case I: Consider p=q=0 in (5.147), the general class of improved ratio type estimator t_4^p reduces to conventional product type estimator. Hence, $t_{4(0,0)}^p = \bar{y}_r$, is the particular member of proposed product type estimator t_4^p.

Case II: For $p > 0$ and $q < 0$, consider the values of p = 1 and q= -1, the estimator t_4^p will become $t_{4(1,-1)}^p = \bar{y}_p + \frac{1}{n}\bar{y}\left[\frac{s_y^2}{\bar{y}^2} - \frac{s_{xy}}{\bar{x}\bar{y}}\right]$. (5.171)

The relative bias and relative mean squared error of the estimator $t_{4(1,-1)}^p$ are as

$$RB\left(t_{4(1,-1)}^p\right) = RB(\bar{y}_p) + \frac{1}{n}[C_{02} - C_{11}], \quad (5.172)$$

$$RM\left(t_{4(1,-1)}^p\right) = RM(\bar{y}_p) + \frac{1}{n^2}[2C_{02}C_{11} + 3C_{02}^2 + 2C_{20}C_{11} + C_{11}^2 + 2C_{03} - 2C_{21}]. \quad (5.173)$$

From (5.172) and (5.2), it is observed that the relative bias of $t_{4(1,-1)}^p$ is more than that of estimator $\bar{y}_p$ upto order O (n^{-1}).

From (5.173) and (5.3), it is observed that the relative mean squared error of the estimator $\bar{y}_p$ is identical to the estimator $t_{4(1,-1)}^p$ upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Thus, the estimator $t_{4(1,-1)}^p$ will be more efficient than $\bar{y}_p$, if

$$2C_{02}C_{11} + 3C_{02}^2 + 2C_{20}C_{11} + C_{11}^2 + 2C_{03} - 2C_{21} < 0. \quad (5.174)$$

For bivariate normal distribution, the above expression (5.174) reduces to

$$2C_{02}C_{11} + 3C_{02}^2 + 2C_{20}C_{11} + C_{11}^2 < 0, \quad (5.175)$$

which gives $\rho > \frac{-3\theta^2-\theta}{2\theta^2+2}$ and the value of θ should lie between $0.001 < \theta < 1.000$.

From (5.173) and (5.7), it can be shown that estimator $t_{4(1,-1)}^p$ and t_R^p have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t_{4(1,-1)}^p$ is more efficient than t_R^p upto order O (n^{-2}), if

$$2C_{02}C_{11} + 2C_{20}C_{11} + 3C_{02}^2 + 2C_{11}^2 + 2C_{12} + 2C_{03} < 0. \tag{5.176}$$

Under bivariate normal distribution, expression (5.176) reduces to

$$2C_{02}C_{11} + 2C_{20}C_{11} + 3C_{02}^2 + 2C_{11}^2 < 0. \tag{5.177}$$

Thus, the estimator $t_{4(1,-1)}^p$ will be more efficient than t_R^p, if

$\rho > \frac{-3\theta^2-2\theta}{2\theta^2+2}$ and the value of θ should lie between $0.001 < \theta < 0.826$.

From (5.173) and (5.11), it can be shown that estimator $t_{4(1,-1)}^p$ and t_D^p have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t_{4(1,-1)}^p$ is more efficient than estimator t_D^p upto order O (n^{-2}), if $2C_{02}C_{11} + 3C_{02}^2 + 2C_{12} + 2C_{03} < 0.$ (5.178)

Under bivariate normal distribution, expression (5.178) reduces to

$$2C_{02}C_{11} + 3C_{02}^2 < 0 \tag{5.179}$$

Thus, the estimator $t_{4(1,-1)}^p$ will perform better than t_D^p, if the value of θ should lie between $0 < \theta < -\frac{2\rho}{3}$.

From (5.172), (5.13) and (5.34), it can be observed that the estimator $t_{4(1,-1)}^p$has more bias than $t_{s(-1)}^p$ and $t_{1(-1)}^p$ upto order O (n^{-1}).

Further, it can be observed that the estimator $t_{4(1,-1)}^p$has smaller relative mean squared error than $t_{s(-1)}^p$, if $2C_{02}C_{11} + 6C_{20}C_{11} + 2C_{20}C_{02} + 3C_{02}^2 - C_{20}^2 + C_{11}^2 + 2C_{03} + 2C_{30} < 0.$ (5.180)

Under bivariate normal distribution, expression (5.180) reduces to

$$2C_{02}C_{11} + 6C_{20}C_{11} + 2C_{20}C_{02} + 3C_{02}^2 - C_{20}^2 + C_{11}^2 < 0, \tag{5.181}$$

which gives $\rho < \frac{1-3\theta^2-3\theta^4}{2\theta^3+6\theta}$ and the value of θ should lie between $0.520 < \theta < 1.274$.

From (5.173) and (5.35), it can be observed that the estimator $t^p_{4(1,-1)}$ has smaller relative mean squared error than $t^p_{1(-1)}$ upto order O (n^{-2}), if

$$\rho < \frac{3-3\theta^2-3\theta^4}{2\theta^3+4\theta} \text{ and the value of } \theta \text{ should lie between } 0.800 < \theta < 1.240. \tag{5.182}$$

From (5.172) and (5.76), it can be observes that the estimator $t^p_{4(1,-1)}$ has more bias than $t^p_{2(-1)}$ upto order O (n^{-1}), if $C_{02} > 0$.

From (5.173) and (5.77), it can be observed that the estimator $t^p_{4(1,-1)}$ has smaller relative mean squared error than $t^p_{2(-1)}$ upto order O (n^{-2}), if

$\rho < \frac{2\theta-3\theta^3}{2\theta^2-2}$ and the value of θ should lie between $0.001 < \theta < 0.577$.

Further, the estimator $t^p_{4(1,-1)}$ was found to be more efficient than the estimator $t^p_{3(-2)}$ upto order O (n^{-2}), if $\rho < \frac{3\theta-3\theta^3}{2\theta^2-2}$ and the value of θ should lie between $0.001 < \theta < 0.666$.

Thus, the proposed estimator $t^p_{4(1,-1)}$ will be more efficient than $\bar{y}_p$, t^p_R, t^*, t^p_D, $t^p_{s(-1)}$, $t^p_{1(-1)}$, $t^p_{2(-1)}$ and $t^p_{3(-1)}$, if the value of θ should lie between $0.520 < \theta < 0.577$.

Case III: For $p < 0$ and $q > 0$, consider the values of p = -1 and q= 1, the estimator t^p_4 will be as

$$t^p_{4(-1,1)} = \bar{y}_p + \frac{1}{n}\bar{y}\left[\frac{s_{xy}}{\bar{x}\bar{y}} - \frac{s_y^2}{\bar{y}^2}\right]. \tag{5.183}$$

The relative bias and relative mean squared error of the estimator $t^p_{4(-1,1)}$ are as

$$RB\left(t^p_{4(-1,1)}\right) = RB\left(\bar{y}_p\right) + \frac{1}{n}[C_{11} - C_{02}], \tag{5.184}$$

$$RM\left(t^{p}_{4(-1,1)}\right) = RM(\bar{y}_p) + \frac{1}{n^2}[C_{11}^2 - 6C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 - 2C_{03} + 2C_{21}]. \quad (5.185)$$

From (5.184) and (5.2), it is observed that the relative bias of $t^{p}_{4(-1,1)}$is smaller than that of estimator $\bar{y}_p$ upto order O (n^{-1}), if $C_{02} > C_{11}$.

From (5.185) and (5.3), it is observed that the relative mean squared error of the estimator $\bar{y}_p$ is identical to the estimator $t^{p}_{4(-1,1)}$upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Thus, the estimator $t^{p}_{4(-1,1)}$ performed better than $\bar{y}_p$ upto order O (n^{-2}), if

$$C_{11}^2 - 6C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 - 2C_{03} + 2C_{21} < 0. \quad (5.186)$$

For bivariate normal distribution, the above expression (5.186) reduces to

$$C_{11}^2 - 6C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 < 0, \quad (5.187)$$

Thus the estimator $t^{p}_{4(-1,1)}$ will be more efficient than $\bar{y}_p$, if

$\rho > \frac{-\theta^3+\theta}{6\theta^2+2}$ and the value of θ should lie between $1.003 < \theta < 6.210$.

From (5.185) and (5.7), it can be shown that estimator $t^{p}_{4(-1,1)}$ and t^{p}_{R} have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is found that estimator $t^{p}_{4(-1,1)}$is more efficient than t^{p}_{R} upto order O (n^{-2}), if

$$2C_{11}^2 - 6C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 - 2C_{03} + 2C_{21} < 0. \quad (5.188)$$

Under bivariate normal distribution, expression (5.188) reduces to

$$2C_{11}^2 - 6C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 < 0, \quad (5.189)$$

Thus, the estimator $t^{p}_{4(-1,1)}$ will be more efficient than t^{p}_{R}, if

$\rho > \frac{-\theta^3+2\theta}{6\theta^2+2}$ and the value of θ should lie between $1.500 < \theta < 6.360$.

From (5.185) and (5.11), it can be shown that estimator $t^p_{4(-1,1)}$ and t^p_D have identical relative mean squared error upto order O (n^{-1}). They differ in terms of order O (n^{-2}).

Further, it is observed that estimator $t^p_{4(-1,1)}$ is more efficient than estimator t^p_D upto order O (n^{-2}), if $6C_{02}C_{11} + 4C_{20}C_{11} + C_{02}^2 - 2C_{03} - 2C_{21} > 0$. (5.190)

Under bivariate normal distribution, expression (5.190) reduces to

$$6C_{02}C_{11} + 4C_{20}C_{11} + C_{02}^2 > 0, \quad (5.191)$$

Thus, the estimator $t^p_{4(-1,1)}$ will be more efficient than t^p_D, if

$\rho > \frac{-\theta^3}{6\theta^2+4}$ and the value of θ should lie between $0.100 < \theta < 6.100$.

From (5.184), (5.13) and (5.45) it can be observes that the estimator $t^p_{4(-1,1)}$has smaller bias than $t^p_{s(1)}$and $t^p_{1(1)}$ upto order O (n^{-1}), if $C_{20} + C_{02} > C_{11}$.

Further, it can be observed that the estimator $t^p_{4(-1,1)}$has smaller relative mean squared error than $t^p_{s(1)}$, if $C_{11}^2 - 6C_{02}C_{11} - 6C_{20}C_{11} - 2C_{20}C_{02} - C_{02}^2 - C_{20}^2 - 2C_{03} - 2C_{30} < 0$. (5.192)

Under bivariate normal distribution, expression (5.192) reduces to

$$C_{11}^2 - 6C_{02}C_{11} - 6C_{20}C_{11} - 2C_{20}C_{02} - C_{02}^2 - C_{20}^2 < 0, \quad (5.193)$$

Thus, the estimator $t^p_{4(-1,1)}$ will be more efficient than $t^p_{s(1)}$, if

$\rho > \frac{-\theta^4-\theta^2-1}{6\theta^3-6\theta}$ and the value of θ should lie between $2.200 < \theta < 5.600$.

From (5.185) and (5.17), it can be observed that the estimator $t^p_{4(-1,1)}$ has smaller relative mean squared error than $t^p_{1(1)}$ upto order O (n^{-2}), if

$$C_{11}^2 - 6C_{02}C_{11} - 4C_{20}C_{11} - 2C_{20}C_{02} - C_{02}^2 + 3C_{20}^2 - 2C_{03} - 2C_{30} < 0, \tag{5.194}$$

Under bivariate normal distribution, expression (5.194) reduces to

$$C_{11}^2 - 6C_{02}C_{11} - 4C_{20}C_{11} - 2C_{20}C_{02} - C_{02}^2 + 3C_{20}^2 < 0. \tag{5.195}$$

Thus, the estimator $t_{4(-1,1)}^{p}$ will be more efficient than $t_{1(1)}^{p}$, if

$\rho > \frac{-\theta^4-\theta^2+3}{6\theta^3+4\theta}$ and the value of θ should lie between $1.000 < \theta < 5.954.$

From (5.184) and (5.90), it can be observes that the estimator $t_{4(-1,1)}^{p}$ has more bias than $t_{2(1)}^{p}$ upto order O (n^{-1}), if $C_{02} > 0$.

From (5.185) and (5.91), it can be observed that the estimator $t_{4(-1,1)}^{p}$has smaller relative mean squared error than $t_{2(1)}^{p}$ upto order O (n^{-2}), if

$$2C_{11}^2 - 6C_{02}C_{11} + 2C_{20}C_{11} - C_{02}^2 < 0, \tag{5.196}$$

Thus, the estimator $t_{4(-1,1)}^{p}$ will be more efficient than $t_{2(1)}^{p}$, if

$$\rho > \frac{\theta^4-2\theta^2}{-6\theta^3+2\theta} \text{ and the value of } \theta \text{ should lie between } 0.001 < \theta < 6.268. \tag{5.197}$$

From (5.184) and (5.138), it can be observes that the estimator $t_{4(-1,1)}^{p}$has more bias than $t_{3(1)}^{p}$ upto order O (n^{-1}), if $C_{02} > 0$.

From (5.185) and (5.139), it can be observed that the estimator $t_{4(-1,1)}^{p}$has smaller relative mean squared error than $t_{3(1)}^{p}$ upto order O (n^{-2}), if

$$6C_{02}C_{11} + C_{02}^2 - 2C_{12} - 2C_{03} < 0. \tag{5.198}$$

Thus, the proposed estimator $t_{4(-1,1)}^{p}$ will be more efficient than $t_{3(1)}^{p}$ upto order O (n^{-2}), if $\rho > -\frac{\theta}{6}$ and the value of θ should lie between $0.001 < \theta < 6.00.$

Thus, the proposed estimator $t^{p}_{4(-1,1)}$ will be more efficient than $\bar{y}_p$, t^{p}_{R}, t^{*}, t^{p}_{D}, $t^{p}_{s(1)}$, $t^{p}_{1(1)}$, $t^{p}_{2(1)}$ and $t^{p}_{3(1)}$, if the value of θ should lie between $1.000 < \theta < 5.600$.

5.3.5 Another proposed class of product type estimators have been developed by incorporating the linear combination of $\bar{y}_r$ with $\frac{s_y^2}{\bar{y}^2}$ and $\frac{s_{xy}}{\bar{x}\bar{y}}$ with the conventional product type estimator as follows:

$$\boldsymbol{t_5^p = \bar{y}_p + \frac{1}{n}\bar{y}_r\left[p\frac{s_y^2}{\bar{y}^2} + q\frac{s_{xy}}{\bar{x}\bar{y}}\right]}, \tag{5.199}$$

where, p and q are scalars specifying the estimator.

Theorem 5.6: The relative bias and relative mean squared error of the estimator t_5^p upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB(t_5^p) = RB(\bar{y}_p) + \frac{1}{n}(pC_{02} + qC_{11}), \tag{5.200}$$

$$RM(t_5^p) = RM(\bar{y}_p) + \frac{p}{n^2}[2C_{12} + 2C_{03} - 2C_{20}C_{11} - 2C_{20}C_{02} - 2C_{02}^2] + \frac{q}{n^2}[2C_{21} + 2C_{12} - 4C_{20}C_{11} - 2C_{11}^2] + \frac{1}{n^2}[pC_{02} + qC_{11}]^2. \tag{5.201}$$

Proof: Expressing $\bar{y}$, $\bar{x}$, s_y^2 and s_{xy} in terms of U_y, U_x , V_y and W respectively in the expression (5.199), we have

$$\frac{t_5^p - \bar{Y}}{\bar{Y}} = (1+U_y)(1+U_x) + \frac{p}{n}(1+U_x)^{-1}(1+U_y)^{-1}(V_y + C_{02}) + \frac{q}{n}(1+U_x)^{-2}(W + C_{11}). \tag{5.202}$$

Expanding the terms on the right hand side of (5.202) and retaining the terms to order O ($n^{-3/2}$), we have

$$\frac{t_5^p - \bar{Y}}{\bar{Y}} = d^{**}_{-\frac{1}{2}} + i^{**}_{-1} + i^{**}_{-\frac{3}{2}}, \tag{5.203}$$

where,

$$\left.\begin{aligned} d^{**}_{-\frac{1}{2}} &= U_y + U_x, \\ i^{**}_{-1} &= U_x U_y + \tfrac{p}{n} C_{02} + \tfrac{q}{n} C_{11}, \\ i^{**}_{-\frac{3}{2}} &= \tfrac{p}{n}\left[V_y - U_y C_{02} - U_x C_{02}\right] + \tfrac{q}{n}(W - 2U_x C_{11}). \end{aligned}\right\} \tag{5.204}$$

Taking expectations of the (5.203) upto O (n^{-1}), we get

$$E\left[\frac{t_5^p - \bar{Y}}{\bar{Y}}\right] = E\left[d^{**}_{-\frac{1}{2}}\right] + E[i^{**}_{-1}], \tag{5.205}$$

where,

$$\left.\begin{aligned} E\left[d^{**}_{-\frac{1}{2}}\right] &= 0; \\ E[i^{**}_{-1}] &= U_x U_y + \frac{p}{n} C_{02} + \frac{q}{n} C_{11} \end{aligned}\right\} \tag{5.206}$$

We get the expression (5.200) for relative bias, to the order of our approximation after substituting the expression (5.206) in (5.205), of t_5^p

For relative mean squared error of t_5^p,

$$E\left[\frac{t_5^p - \bar{Y}}{\bar{Y}}\right]^2 = E\left({d^{**}_{-\frac{1}{2}}}^2\right) + E\left({i^{**}_{-1}}^2\right) + 2E\left(d^{**}_{-\frac{1}{2}} i^{**}_{-1}\right) + 2E\left(d^{**}_{-\frac{1}{2}} i^{**}_{-\frac{3}{2}}\right). \tag{5.207}$$

By using the results as discussed in chapter 3, we have

$$\left.\begin{aligned} E\left({d^{*}_{-\frac{1}{2}}}^2\right) &= E\left(U_y^2 + U_x^2 - 2U_y U_x\right) \\ &= \tfrac{1}{n}(C_{02} + C_{20} + 2C_{11}), \\ E\left({i^{**}_{-1}}^2\right) &= \tfrac{1}{n^2}[2C_{11}^2 + C_{20}C_{02}] + \tfrac{1}{n^2}[pC_{02} + qC_{11}]^2 + \tfrac{1}{n^2}[2pC_{02}C_{11} + 2qC_{11}^2], \\ E\left(d^{**}_{-\frac{1}{2}} i^{**}_{-1}\right) &= \tfrac{1}{n^2}[C_{21} + C_{12}], \\ E\left(d^{**}_{-\frac{1}{2}} i^{**}_{-\frac{3}{2}}\right) &= \frac{p}{n^2}[C_{03} + C_{12} - C_{02}^2 - C_{20}C_{11} - C_{20}C_{02}] \\ &\quad + \frac{q}{n^2}[C_{21} + C_{12} - 2C_{20}C_{11} - 2C_{11}^2]. \end{aligned}\right\} \tag{5.208}$$

The terms of order higher than O (n^{-2}) have been dropped. Substituting the results of (5.208) in (5.207) and after algebraic simplification we obtained (5.201) of Theorem 5.6.

From (5.200), it is observed that estimator t_5^p has smaller bias than conventional product type estimator $\bar{y}_p$, if $qC_{11} < pC_{02}$.

From (5.201) and (5.3), it is observed that the relative mean squared error of both estimators i.e., t_5^p and $\bar{y}_p$ are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator t_5^p has smaller relative mean squared error than that of $\bar{y}_p$, if

$$p[2C_{12} + 2C_{03} - 2C_{20}C_{11} - 2C_{20}C_{02} - 2C_{02}^2] + q[2C_{21} + 2C_{12} - 4C_{20}C_{11} - 2C_{11}^2] + [pC_{02} + qC_{11}]^2 < 0. \tag{5.209}$$

For bivariate normal population, the expression (5.209) reduces to

$$p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2] + q[4C_{20}C_{11} + 2C_{11}^2] - [pC_{02} + qC_{11}]^2 > 0. \tag{5.210}$$

The comparison of relative mean squared error of t_5^p and t_R^p showed that t_5^p is more efficient than t_R^p, if

$$p[2C_{12} + 2C_{03} - 2C_{20}C_{11} - 2C_{20}C_{02} - 2C_{02}^2] + q[2C_{21} + 2C_{12} - 4C_{20}C_{11} - 2C_{11}^2] + [pC_{02} + qC_{11}]^2 + [C_{11}^2 + 2C_{12} + 2C_{21}] < 0. \tag{5.211}$$

This under bivariate normal distribution reduces to the following expression

$$p[-2C_{20}C_{11} - 2C_{20}C_{02} - 2C_{02}^2] + q[-4C_{20}C_{11} - 2C_{11}^2] + [pC_{02} + qC_{11}]^2 + [C_{11}^2] < 0. \tag{5.212}$$

From (5.201) and (5.11), it can be observed that the estimator t_5^p is more efficient than t_D^p, if

$$p[2C_{12} + 2C_{03} - 2C_{20}C_{11} - 2C_{20}C_{02} - 2C_{02}^2] + q[2C_{21} + 2C_{12} - 4C_{20}C_{11} - 2C_{11}^2] + [pC_{02} + qC_{11}]^2 - [2C_{20}C_{11} + C_{11}^2 - 2C_{12} - 2C_{21}] < 0, \quad (5.213)$$

Under bivariate normal population, the above expression (5.213) reduces to

$$p[-2C_{20}C_{11} - 2C_{20}C_{02} - 2C_{02}^2] + q[-4C_{20}C_{11} - 2C_{11}^2] + \frac{1}{n^2}[pC_{02} + qC_{11}]^2 - [2C_{20}C_{11} + C_{11}^2] < 0. \quad (5.214)$$

From (5.200), (5.13) and (5.16), it can be observed that the estimator t_5^p has smaller bias than t_s^p and t_1^p upto the order O (n^{-1}), if p and q have different directions with $p > q$ and the value of θ lies between $0.619 < \theta < 1.000$.

From (5.201) and (5.14), it can be observed that the estimator t_5^p has smaller relative mean squared error than t_s^p, if

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2 - 2C_{12} - 2C_{03}] + q[8C_{20}C_{11} + 2C_{11}^2 + qC_{20}^2 + 2C_{20}C_{02} - 2C_{21} + 2C_{30}], \quad (5.215)$$

under bivariate normal distribution, expression (5.215) reduces to

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2] + q[8C_{20}C_{11} + 2C_{11}^2 + qC_{20}^2 + 2C_{20}C_{02}]. \quad (5.216)$$

The comparison of t_5^p and t_1^p showed that the estimator t_5^p is more efficient than t_1^p upto the order O (n^{-2}), if

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2 - 2C_{12} - 2C_{03}] + q[6C_{20}C_{11} + 2C_{20}C_{02} + 2C_{11}^2 - 2C_{20}^2 + qC_{20}^2 - 2C_{12} + 2C_{30}], \quad (5.217)$$

for bivariate normal population, the expression (5.217) reduces to

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2] + q[6C_{20}C_{11} + 2C_{20}C_{02} + 2C_{11}^2 - 2C_{20}^2 + qC_{20}^2]. \quad (5.218)$$

Further, it has been found that the estimator t_5^p has smaller bias than t_2^p and t_3^p upto order O (n[-1]), we have $p < 0$ and $C_{02} > 0$.

From (5.201) and (5.55), it has been observed that the estimator t_5^p is more efficient than t_2^p upto order O (n[-2]), if

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2 - 2C_{12} - 2C_{03}] + q^2C_{11}^2, \qquad (5.219)$$

the expression (5.219) under bivariate normal distribution reduces to

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2] + q^2C_{11}^2. \qquad (5.220)$$

From (5.201) and (5.102), it can be observed that the estimator t_5^p has smaller relative mean squared error than t_3^p, if

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2 - 2C_{12} - 2C_{03}] + q[qC_{11}^2 + 2C_{11}^2 + 4C_{20}C_{11} - 2C_{20}C_{02}], \qquad (5.221)$$

under bivariate normal distribution, expression (5.221) reduces to

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 2C_{20}C_{02} + 2C_{02}^2] + q[qC_{11}^2 + 2C_{11}^2 + 4C_{20}C_{11} - 2C_{20}C_{02}]. \qquad (5.222)$$

From (5.200) and (5.148), it can be observed that the estimators t_5^p and t_4^p have identical bias upto order O (n[-2]).

From (5.201) and (5.149), it can be observed that the estimator t_5^p has smaller relative mean squared error than t_4^p, if

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 4C_{02}C_{11} + 2C_{20}C_{02} + 4C_{02}^2] + q[-2C_{03} + 2C_{21} + 2C_{20}C_{02} + 2C_{02}^2], \qquad (5.223)$$

under bivariate normal distribution, expression (5.223) reduces to

$$[pC_{02} + qC_{11}]^2 < p[2C_{20}C_{11} + 4C_{02}C_{11} + 2C_{20}C_{02} + 4C_{02}^2] + q[2C_{20}C_{02} + 2C_{02}^2]. \qquad (5.224)$$

5.3.5.1: Proposed product type estimator: For different values of p and q different cases have been formed**.**

Case I: Consider $p = 0$ and $q = 0$ in (5.199), the proposed product type estimator reduces to the conventional product type estimator i.e., conventional product estimator is a particular member of the general proposed class of product type estimator.

Case II: For $\boldsymbol{p} > 0$ and $\boldsymbol{q} < 0$, consider the values of scalars as p=1 and q=-1 in (5.199), the proposed estimator reduces to the following

$$\boldsymbol{t_5^p} = \overline{\boldsymbol{y}}_{\boldsymbol{p}} + \frac{\mathbf{1}}{\boldsymbol{n}} \overline{\boldsymbol{y}}_{\boldsymbol{r}} \left[\frac{\boldsymbol{s_y^2}}{\overline{\boldsymbol{y}}^{\mathbf{2}}} - \frac{\boldsymbol{s_{xy}}}{\overline{\boldsymbol{x}}\overline{\boldsymbol{y}}} \right]. \tag{5.225}$$

The relative bias and relative mean squared error of the estimator $t_{5(1,-1)}^p$ upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB\left(t_{5(1,-1)}^p\right) = RB\left(\bar{y}_p\right) + \frac{1}{n}(C_{02} - C_{11}), \tag{5.226}$$

$$RM\left(t_{5(1,-1)}^p\right) = RM\left(\bar{y}_p\right) + \frac{1}{n^2}[2C_{20}C_{11} + 2C_{02}C_{11} - 2C_{20}C_{02} + 3C_{11}^2 - C_{02}^2 + 2C_{03} - 2C_{21}]. \tag{5.227}$$

From (5.226), it is observed that estimator $t_{5(1,-1)}^p$ has smaller bias than conventional product type estimator $\bar{y}_p$, if $C_{02} < C_{11}$.

From (5.227) and (5.3), it is observed that the relative mean squared error of both estimators i.e., $t_{5(1,-1)}^p$ and $\bar{y}_p$, are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator $t_{5(1,-1)}^p$ has smaller relative mean squared error than that of $\bar{y}_p$, if

$$[2C_{20}C_{11} + 2C_{02}C_{11} - 2C_{20}C_{02} + 3C_{11}^2 - C_{02}^2 + 2C_{03} - 2C_{21}] < 0. \tag{5.228}$$

For bivariate normal population, the expression (5.228) reduces to

$$[2C_{20}C_{11} + 2C_{02}C_{11} - 2C_{20}C_{02} + 3C_{11}^2 - C_{02}^2] < 0, \tag{5.229}$$

Thus, the proposed estimator $t^{p}_{5(1,-1)}$ will be more efficient than $\bar{y}_p$, if

$\rho < \frac{\theta^3-\theta}{2\theta^2+2}$ and the value of θ should lie between $0.4745 < \theta \leq 1.000.$

The comparison of relative mean squared error of $t^{p}_{5(1,-1)}$ and t^{p}_{R} showed that $t^{p}_{5(1,-1)}$ is more efficient thant^{p}_{R}, if

$$[2C_{20}C_{11} + 2C_{02}C_{11} - 2C_{20}C_{02} + 4C_{11}^2 - C_{02}^2 + 2C_{03} - 2C_{21}] < 0. \tag{5.230}$$

This under bivariate normal distribution reduces to the following expression

$$[2C_{20}C_{11} + 2C_{02}C_{11} - 2C_{20}C_{02} + 4C_{11}^2 - C_{02}^2] < 0, \tag{5.231}$$

Thus, the proposed estimator $t^{p}_{5(1,-1)}$ will be more efficient than t^{p}_{R}, if

$\rho < \frac{\theta^3-2\theta}{2\theta^2+2}$ and the value of θ should lie between $0.61 < \theta \leq 1.40.$

From (5.227) and (5.11), it can be observed that the estimator $t^{p}_{5(1,-1)}$is more efficient than t^{p}_{D}, if

$$[2C_{02}C_{11} - 2C_{20}C_{02} + 2C_{11}^2 - C_{02}^2 + 2C_{03} + 2C_{12}] < 0. \tag{5.232}$$

Under bivariate normal population, the above expression (5.232) reduces to

$$[2C_{02}C_{11} - 2C_{20}C_{02} + 2C_{11}^2 - C_{02}^2] < 0, \tag{5.233}$$

Thus, the proposed estimator $t^{p}_{5(1,-1)}$ will be more efficient than t^{p}_{D}, if $\theta > 2\rho$.

From (5.226), (5.13) and (5.34) it can be observed that the estimator $t^{p}_{5(1,-1)}$ has smaller bias than $t^{p}_{s(-1)}$and $t^{p}_{1(-1)}$have identical bias upto the order O (n^{-1}), if

$C_{02} < C_{20} + C_{11}$which results to the value of θ lies between $0.619 < \theta < 1.000.$

From (5.227) and (5.14), it can be observed that the estimator $t^{p}_{5(1,-1)}$ has smaller relative mean squared error than $t^{p}_{s(-1)}$, if

$$6C_{20}C_{11} + 2C_{02}C_{11} + 3C_{11}^2 - C_{02}^2 - C_{20}^2 + 2C_{30} + 2C_{03} < 0, \tag{5.234}$$

Under bivariate normal distribution, expression (5.234) reduces to

$$6C_{20}C_{11} + 2C_{02}C_{11} + 3C_{11}^2 - C_{02}^2 - C_{20}^2 < 0, \tag{5.235}$$

Thus, the proposed estimator $t_{5(1,-1)}^{p}$ will be more efficient than $t_{s(-1)}^{p}$, if the value of θ lies between $0.619 < \theta < 1.051$.

The comparison of $t_{5(1,-1)}^{p}$and $t_{1(-1)}^{p}$ showed that the estimator $t_{5(1,-1)}^{p}$ is more efficient than $t_{1(-1)}^{p}$ upto the order O (n^{-2}), if

$$4C_{20}C_{11} + 2C_{02}C_{11} + 3C_{11}^2 - C_{02}^2 - 3C_{20}^2 + 2C_{30} + 2C_{03} < 0 \tag{5.236}$$

For bivariate normal population, the expression (5.236) reduces to

$$4C_{20}C_{11} + 2C_{02}C_{11} + 3C_{11}^2 - C_{02}^2 - 3C_{20}^2 < 0 \tag{5.237}$$

Thus, the proposed estimator $t_{5(1,-1)}^{p}$ will be more efficient than $t_{1(-1)}^{p}$, if

$\rho > \frac{-\theta^4+3\theta^2-3}{2\theta^3+4\theta}$ and the value of θ should lie between $0.560 < \theta < 2.890$.

The relative bias of the estimator $t_{5(1,-1)}^{p}$ has been observed to be more than $t_{2(-1)}^{p}$ upto order O (n^{-1}). From (5.227) and (5.77), it has been observed that the estimator $t_{5(1,-1)}^{p}$ is more efficient than $t_{2(-1)}^{p}$ upto order O (n^{-2}), if

$$2C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 + 2C_{12} + 2C_{03} < 0. \tag{5.238}$$

The expression (5.238) under bivariate normal distribution reduces to

$$2C_{02}C_{11} - 2C_{20}C_{11} - C_{02}^2 < 0, \tag{5.239}$$

Thus, the proposed estimator $t_{5(1,-1)}^{p}$ will be more efficient than $t_{2(-1)}^{p}$, if

$\rho < \frac{\theta^4}{2\theta^3-2\theta}$ and the value of θ should lie between $0.158 < \theta < 0.839$.

The proposed estimator $t^{p}_{5(1,-1)}$ performed better than $t^{p}_{3(-2)}$ upto order O (n^{-2}), if

$\rho < \frac{\theta^3+3\theta}{2\theta^2-2}$ and the value of θ should lie between $0.001 < \theta < 0.476$.

The estimator $t^{p}_{5(1,-1)}$ and $t^{p}_{4(1,-1)}$ have identical bias upto order O (n^{-1}).

From (5.227) and (5.173), it can be observed that the estimator $t^{p}_{5(1,-1)}$has smaller relative mean squared error than $t^{p}_{4(1,-1)}$, if

$$2C_{11}^2 - 4C_{02}^2 - 2C_{20}C_{02} < 0 \text{ and } \theta > 0. \tag{5.240}$$

Thus, it can be concluded that if we sacrifice the property of unbiasedness, the proposed estimator $t^{p}_{5(1,-1)}$ will be more efficient than $\bar{y}_p$, t^{p}_{R}, t^{*}, t^{p}_{D}, $t^{p}_{s(-1)}$, $t^{p}_{1(-1)}$, $t^{p}_{2(-1)}$, $t^{p}_{3(-1)}$ and $t^{p}_{4(1,-1)}$, if the value of θ should lie between $0.619 < \theta < 0.839$.

Case III: For $p < 0$ and $q > 0$, consider the value of scalars as p=-1 and q=1, the estimator t_5 becomes

$$t^{p}_{5(-1,1)} = \bar{y}_p + \frac{1}{n}\bar{y}_r\left[\frac{s_{xy}}{\bar{x}\bar{y}} - \frac{s_y^2}{\bar{y}^2}\right], \tag{5.241}$$

where, p and q are scalars specifying the estimator.

The relative bias and relative mean squared error of the estimator $t^{p}_{5(-1,1)}$ upto order O (n^{-1}) and O (n^{-2}) respectively, as

$$RB\left(t^{p}_{5(-1,1)}\right) = RB(\bar{y}_p) + \frac{1}{n}(C_{11} - C_{02}), \tag{5.242}$$

$$RM\left(t^{p}_{5(-1,1)}\right) = RM(\bar{y}_p) + \frac{1}{n^2}[2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{02}C_{11} - C_{11}^2 + 3C_{02}^2 + 2C_{12} - 2C_{03}]. \tag{5.243}$$

From (5.242), it is observed that estimator $t^{p}_{5(-1,1)}$ has smaller bias than conventional product type estimator $\bar{y}_p$, if $C_{11} < C_{02}$.

From (5.243) and (5.3), it is observed that the relative mean squared error of both estimators i.e., $t^{p}_{5(-1,1)}$ and $\bar{y}_p$, are identical upto order O (n^{-1}).

Now, comparing the relative mean squared error of both estimators upto O (n^{-2}), we find that the estimator $t^{p}_{5(-1,1)}$ has smaller relative mean squared error than that of $\bar{y}_p$, if

$$2C_{20}C_{02} - 2C_{02}C_{11} - 2C_{02}C_{02} - C_{11}^2 - 3C_{02}^2 - 2C_{03} + 2C_{21} < 0 \qquad (5.244)$$

For bivariate normal population, the expression (5.244) reduces to

$$2C_{20}C_{02} - 2C_{02}C_{11} - 2C_{02}C_{02} - C_{11}^2 - 3C_{02}^2 < 0, \qquad (5.245)$$

Thus, the proposed estimator $t^{p}_{5(-1,1)}$ will be more efficient than $\bar{y}_p$, if

$\rho < \frac{-(3\theta^3+\theta)}{2\theta^2+2}$ and the value of θ should lie between $0.00 < \theta \leq 1.000$.

The comparison of relative mean squared error of $t^{p}_{5(-1,1)}$ and t^{p}_{R} showed that $t^{p}_{5(-1,1)}$ is more efficient than t^{p}_{R}, if

$$[2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{02}C_{11} + 3C_{02}^2 - 4C_{21} + 2C_{12} - 2C_{03}] < 0. \qquad (5.246)$$

This under bivariate normal distribution reduces to the following expression

$$[2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{02}C_{11} + 3C_{02}^2] < 0, \qquad (5.247)$$

Thus, the proposed estimator $t^{p}_{5(-1,1)}$ will be more efficient than t^{p}_{R}, if

$\rho < -\left(\frac{3\theta^3+2\theta}{2\theta^2+2}\right)$ and the value of θ should lie between $0.00 < \theta \leq 0.732$.

From (5.243) and (5.11), it can be observed that the estimator $t^{p}_{5(-1,1)}$is more efficient than t^{p}_{D}, if

$$[2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{02}C_{11} - 2C_{11}^2 + 3C_{02}^2 - 2C_{03} + 2C_{12} + 4C_{21}] < 0. \qquad (5.248)$$

Under bivariate normal population, the above expression (5.248) reduces to

$$[2C_{20}C_{02} - 2C_{20}C_{11} - 2C_{02}C_{11} - 2C_{11}^2 + 3C_{02}^2] < 0, \tag{5.249}$$

Thus, the proposed estimator $t_{5(-1,1)}^p$ will be more efficient than t_D^p, if

$\rho < -\left(\frac{3\theta^3}{2\theta^2+4}\right)$ and the value of θ should lie between $0.000 < \theta < 1.373$.

Further, it can be observed that the estimator $t_{5(-1,1)}^p$ has smaller bias than $t_{s(1)}^p$ and $t_{(1)}^p$ have identical bias upto the order O (n^{-1}), if $C_{11} < C_{20}+C_{02}$.

From (5.243) and (5.14), it can be observed that the estimator $t_{5(-1,1)}^p$has smaller relative mean squared error than $t_{s(1)}^p$, if

$$3C_{02}^2 - C_{20}^2 - C_{11}^2 - 2C_{02}C_{11} - 6C_{20}C_{11} - 2C_{30} - 2C_{03} < 0. \tag{5.250}$$

Under bivariate normal distribution, expression (5.250) reduces to

$$3C_{02}^2 - C_{20}^2 - C_{11}^2 - 2C_{02}C_{11} - 6C_{20}C_{11} < 0, \tag{5.251}$$

Thus, the proposed estimator $t_{5(-1,1)}^p$ will be more efficient than $t_{s(1)}^p$, if

$\rho > \frac{3\theta^4-\theta^2-1}{2\theta^3+6\theta}$ and the value of θ should lie between $0.17 < \theta < 0.873$.

The comparison of $t_{5(-1,1)}^p$and $t_{1(1)}^p$ showed that the estimator $t_{5(-1,1)}^p$ is more efficient than $t_{1(1)}^p$ upto the order O (n^{-2}), if

$$C_{20}^2 + 3C_{02}^2 - 4C_{20}C_{11} - 2C_{02}C_{11} - C_{11}^2 - 2C_{30} - 2C_{03} < 0. \tag{5.252}$$

For bivariate normal population, the expression (5.252) reduces to

$$C_{20}^2 + 3C_{02}^2 - 4C_{20}C_{11} - 2C_{02}C_{11} - C_{11}^2 < 0, \tag{5.253}$$

Thus, the proposed estimator $t_{5(-1,1)}^p$ will be more efficient than $t_{1(1)}^p$, if

$\rho > \frac{3\theta^4-\theta^2+1}{2\theta^3+4\theta}$ and the value of θ should lie between $0.659 < \theta < 1.433$.

The relative bias of the estimator $t^{p}_{5(-1,1)}$ has been observed to be smaller than $t^{p}_{2(1)}$ and $t^{p}_{3(1)}$upto order O (n^{-1}), if $C_{02} > 0$.

From (5.243) and (5.91), it has been observed that the estimator $t^{p}_{5(-1,1)}$ more efficient than $t^{p}_{2(1)}$ upto order O (n^{-2}), if

$$2C_{20}C_{02} + 2C_{20}C_{11} - 2C_{02}C_{11} + 3C_{02}^{2} - 2C_{12} - 2C_{03} < 0. \qquad (5.254)$$

The expression (5.254) under bivariate normal distribution reduces to

$$2C_{20}C_{02} + 2C_{20}C_{11} - 2C_{02}C_{11} + 3C_{02}^{2} < 0. \qquad (5.255)$$

Thus, the proposed estimator $t^{p}_{5(-1,1)}$ will be more efficient than $t^{p}_{2(1)}$, if

$\rho < \frac{3\theta^{3}+2\theta}{2\theta^{2}-2}$ and the value of θ should lie between $0.001 < \theta < 0.519$.

From (5.243) and (5.139), it can be observed that the estimator $t^{p}_{5(-1,1)}$ has smaller relative mean squared error than $t^{p}_{3(1)}$, if

$$2C_{20}C_{02} - 2C_{02}C_{11} - 2C_{11}^{2} + 3C_{02}^{2} - 2C_{03} - 2C_{21} < 0. \qquad (5.256)$$

Under bivariate normal distribution, expression (5.256) reduces to

$$2C_{20}C_{02} - 2C_{02}C_{11} - 2C_{11}^{2} + 3C_{02}^{2} < 0, \qquad (5.257)$$

Thus, the proposed estimator $t^{p}_{5(-1,1)}$ will be more efficient than $t^{p}_{3(1)}$, if

$\rho < -1.5\theta$ and the value of θ should lie between $0.001 < \theta < 0.659$.

The relative bias of the estimators $t^{p}_{5(-1,1)}$ and $t^{p}_{4(-1,1)}$ have been found to be identical upto the order O (n^{-1}).

From (5.243) and (5.185), it can be observed that the estimator $t^{p}_{5(-1,1)}$ has smaller relative mean squared error than $t^{p}_{4(-1,1)}$, if

$$2C_{20}C_{02} + 4C_{02}C_{11} - 2C_{11}^2 + 4C_{02}^2 < 0 \text{ and } \theta < 1. \tag{5.258}$$

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Empirical Study through Simulation

CHAPTER 6

EMPERICAL STUDY THROUGH SIMULTION

6.1 INTRODUCTION

The process of designing a model of a real system and conducting experiments with this model for the purpose either of understanding the behavior of the system or of evaluating various strategies (within the limits imposed by a criterion or set of criteria) for the operation of the system (Shannon, 1975). Simulation facilitates the researchers to create a model or to test the proposed model in order to evaluate the complex models. It usually takes longer time to run and evaluate the more complex systems. Simulation modelling helps in developing a mathematical model of the system for better understanding of the real life problems. Simulation helps in testing the hypotheses related to the feasibility of the system. By altering the model of the system, the effect of certain informational, organizational, environmental and policy changes on the operation of a system can be studied without disturbing the real system. It can significantly reduce the risk of experimenting with it. Simulation can reduce system development time by developing well designed and robust systems. Simulation helps in identification of those variables whose performance measures are more sensitive.

In this chapter, the theoretical results obtained in the present study have been illustrated numerically. For numerical evaluation of the properties related to the proposed ratio and product type estimators for population mean under study, different bivariate population sets of sizes 200 have been generated through simulation by using R and SAS softwares with defined correlation. Samples of sizes 30, 60 and 120 have been drawn from the population of size 200 using simple random sampling without replacement.

6.2 IMPROVED RATIO ESTIMATORS OF POPULATION MEAN

Improved ratio type estimators of population mean have been theoretically developed and their large sample properties are worked out in the chapter IV and their efficiencies have been tested using two populations/datasets P_1 and P_2. Different types of estimators have been developed by using the values of p and q with respect to different cases. The estimators holding properties of unbiasedness, most efficient and consistency have been considered.

6.2.1: Population I (P_1)

Table 6.1: Descriptive statistics of the variable under study and auxiliary variable for P_1

Variable	Population mean	Population variance	ρ_{XY}
Y	11.923	133.327	0.918
X	0.503	0.243	

From table 6.1, it has been observed that the correlation between the two variables (Y and X) was found to be 0.918 whereas the mean and variance of the study variable Y were 11.923 and 133.327 whereas that of auxiliary variable X were 0.503 and 0.243 respectively.

Table 6.2: Relative bias of the proposed ratio estimators $t_{1(-3,1)}$ and $t_{1\,(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.05566	0.01257	0.01016
$t_{s\,(-1,-1)}$	-0.05430	-0.00503	0.00319
$t_{s\,(-3,1)}$	-0.36366	-0.03532	-0.01659
$t_{1\,(-1,-1)}$	0.00000	0.00000	0.00000
$t_{1\,(-3,1)}$	-0.02281	-0.03250	-0.01158

From the table 6.2, it has been observed that the proposed estimator $t_{1(-3,1)}$ was negatively biased and the biases were -0.02281, -0.03250 and -0.01158 at samples of sizes 30, 60 and 120 respectively. For p=-1 and q= -1, the proposed estimator t_1 was found to be unbiased. Further, for the estimator $t_{s\,(-3,1)}$, biases at sample of sizes 30, 60 and 120 were -0.36366, -0.03532 and -0.01659 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were -0.05430, -0.00503 and 0.00319 at samples of sizes 30, 60 and 120 respectively.

Table 6.3: Relative bias of the proposed ratio estimators $t_{2(-3,1)}$ and $t_{2\,(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.05566	0.01257	0.01016
$t_{s\,(-3,1)}$	-0.36366	-0.03532	-0.01659
$t_{s\,(-1,-1)}$	-0.05430	-0.00503	0.00319
$t_{2\,(-3,1)}$	-0.02281	-0.03250	-0.01158
$t_{2(-1,-1)}$	0.00000	0.00000	0.00000

From the table 6.3, the bias of the proposed estimator $t_{2\,(-3,1)}$ were -0.02281, -0.03250 and -0.01158 at samples of sizes 30, 60 and 120 respectively. For p=-1 and q= -1, the proposed estimator t_2 found to be unbiased. Further, for the estimator $t_{s\,(-3,1)}$, biases were -0.36366, -0.03532 and -0.01659 at sample of sizes 30, 60 and 120 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were -0.05430, -0.00503 and 0.00319 at samples of sizes 30, 60 and 120 respectively. Thus, the proposed estimator $t_{2(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator.

Table 6.4: Relative bias of the proposed ratio estimators $t_{3(-3,1)}$ and $t_{3\,(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.05566	0.01257	0.01016
$t_{s\,(-3,1)}$	-0.36366	-0.03532	-0.01659
$t_{s\,(-1,-1)}$	-0.05430	-0.00503	0.00319
$t_{3\,(-3,1)}$	-0.02281	-0.03250	-0.01158
$t_{3(-1,-1)}$	0.00000	0.00000	0.00000

From the table 6.4, the bias of the proposed estimator $t_{3\,(-3,1)}$ were -0.02281, -0.03250 and -0.01158 at samples of sizes 30, 60 and 120 respectively. An unbiased estimator t_3 have been obtained at p=-1 and q=-1. Further, for the estimator $t_{s\,(-3,1)}$, biases were -0.36366, -0.03532 and -0.01659 at sample of sizes 30, 60 and 120 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were -

0.05430, -0.00503 and 0.00319 at samples of sizes 30, 60 and 120 respectively. Thus, the proposed estimator $t_{3(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator.

Table 6.5: Relative bias of the proposed ratio estimators $t_{4(-3,1)}$ and $t_{4\,(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.05566	0.01257	0.01016
$t_{s\,(-3,1)}$	-0.36366	-0.03532	-0.01659
$t_{s\,(-1,-1)}$	-0.05430	-0.00503	0.00319
$t_{4\,(-3,1)}$	-0.02281	-0.03250	-0.01158
$t_{4(-1,-1)}$	0.00000	0.00000	0.00000

From the table 6.5, it has been observed that the proposed estimator $t_{1(-3,1)}$ was negatively biased and the biases have been found to be -0.02281, -0.03250 and -0.01158 at samples of sizes 30, 60 and 120 respectively. For p=-1 and q= -1, the proposed estimator t_4 was found to be unbiased. Further, for the estimator $t_{s\,(-3,1)}$, biases at sample of sizes 30, 60 and 120 were -0.36366, -0.03532 and -0.01659 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were -0.05430, -0.00503 and 0.00319 at samples of sizes 30, 60 and 120 respectively. Thus, the proposed estimator $t_{4(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator.

Table 6.6: Relative bias of the proposed ratio estimator $t_{5(-3,1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.05566	0.01257	0.01016
$t_{s(-3,1)}$	-0.36366	-0.03532	-0.01659
$t_{5(-3,1)}$	-0.02281	-0.03250	-0.01158

From the table 6.6, it has been observed that the proposed estimator $t_{5(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator. The values of the relative bias of the proposed estimator $t_{5(-3,1)}$ were -0.02281, -0.03250 and -

0.01158 at samples of sizes 30, 60 and 120 respectively whereas that for $t_{s(-3,1)}$ the values were -0.36366, -0.03532 and -0.01659 at the specified sample sizes.

Thus, it has been observed that all the proposed estimators i.e., $t_{1(-3,1)}$, $t_{2(-3,1)}$, $t_{3(-3,1)}$, $t_{4(-3,1)}$ and $t_{5(-3,1)}$ were biased with same magnitude whereas the proposed estimators $t_{1(-1,-1)}$, $t_{2(-1,-1)}$, $t_{3(-1,-1)}$ and $t_{4(-1,-1)}$ were found to be unbiased.

Table 6.7: Relative mean squared error / relative variance of the proposed ratio estimators $t_{1(-3,1)}$ and $t_{1(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.24678	0.02983	0.02134
$t_{s(-3,1)}$	0.51606	0.03406	0.02269
$t_{s(-1,-1)}$	0.28649	0.03096	0.02149
$t_{1(-3,1)}$	0.17906	0.02788	0.02024
$t_{1(-1,-1)}$	0.22736	0.02892	0.02079
Percent relative efficiencies of proposed estimators			
$t_{1(-3,1)}$ w.r.t $\bar{y}_r$	137.82	107.00	105.44
$t_{1(-3,1)}$ w.r.t $t_{s(-3,1)}$	288.20	122.18	112.11
$t_{1(-1,-1)}$ w.r.t $\bar{y}_r$	108.54	103.14	102.64
$t_{1(-1,-1)}$ w.r.t $t_{s(-1,-1)}$	126.01	107.06	103.37

From table 6.7, it has been observed that the proposed estimator $t_{1(-3,1)}$ was more efficient than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). Further, the percent relative efficiency of the proposed estimator $t_{1(-3,1)}$ was found to lie between 105.44 to 137.82 with respect to conventional ratio estimator. Further, the value of the relative mean squared error of the proposed estimator $t_{1(-3,1)}$ at samples of sizes 30, 60 and 120 were 0.17906, 0.02788 and 0.02024 respectively. Also, the percent relative efficiency of the proposed estimator $t_{1(-3,1)}$ was found to lie between 112.11 and 288.20 with respect to $t_{s(-3,1)}$ and was maximum among different estimators of the class. An unbiased ratio estimator $t_{1(-1,-1)}$ has also been proposed. The relative variance of the proposed estimator $t_{1(-1,-1)}$ were 0.22736, 0.02892 and 0.02079 at sample of sizes 30, 60 and 120 respectively. The percent relative efficiency of the proposed unbiased estimator $t_{1(-1,-1)}$ was found to lies between 102.64 to 108.54 with respect to conventional ratio estimator. Further, the percent relative efficiency of the proposed unbiased estimator $t_{1(-1,-1)}$ with respect to $t_{s(-1,-1)}$ was lying between 103.37 to 126.01.

Table 6.8: Relative mean squared error / relative variance of the proposed ratio estimators $t_{2(-3,1)}$ and $t_{2(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.24678	0.02983	0.02134
$t_{s(-3,1)}$	0.51606	0.03406	0.02269
$t_{s(-1,-1)}$	0.28649	0.03096	0.02149
$t_{2(-3,1)}$	0.22523	0.02977	0.02125
$t_{2(-1,-1)}$	0.24026	0.02958	0.02110
Percent relative efficiencies of proposed estimators			
$t_{2(-3,1)}$ w.r.t. $\bar{y}_r$	109.57	100.18	100.43
$t_{2(-3,1)}$ w.r.t. $t_{s(-3,1)}$	229.13	114.40	106.78
$t_{2(-1,-1)}$ w.r.t. $\bar{y}_r$	102.71	100.83	101.15
$t_{2(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	119.24	104.66	101.87

Table 6.8 revealed that the proposed estimator $t_{2(-3,1)}$ have smaller relative mean squared error than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). Further, the value of the relative mean squared error of the proposed estimator $t_{2(-3,1)}$ at samples of sizes 30, 60 and 120 were 0.22523, 0.02977 and 0.02125 respectively. The percent relative efficiency of the proposed estimator $t_{2(-3,1)}$ was found to lie between 100.43 to 109.57 with respect to conventional ratio estimator and was lying between 106.78 to 229.13 with respect to $t_{s(-3,1)}$. For the unbiased proposed estimator $t_{2(-1,-1)}$, the values of relative variance were 0.24026, 0.02958 and 0.02110 at samples of sizes 30, 60 and 120 respectively. The percent relative efficiency of the proposed estimator $t_{2(-1,-1)}$ was found to lie between 100.83 to 102.71 with respect to conventional ratio estimator and lying between 101.87 to 119.24 with respect to $t_{s(-1,-1)}$.

Table 6.9: Relative mean squared error/relative variance of the proposed ratio estimators $\boldsymbol{t_{3(-3,1)}}$ and $\boldsymbol{t_{3(-1,-1)}}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for $\mathbf{P_1}$

Estimator	n		
	30	60	120
$\bar{y}_r$	0.24678	0.02983	0.02134
$t_{s(-3,1)}$	0.51606	0.03406	0.02269
$t_{s(-1,-1)}$	0.28649	0.03096	0.02149
$t_{3(-3,1)}$	0.19932	0.02656	0.01974
$t_{3(-1,-1)}$	0.22665	0.02857	0.02054
Percent relative efficiencies of proposed estimators			
$t_{3(-3,1)}$ w.r.t. $\bar{y}_r$	123.81	112.30	108.08
$t_{3(-3,1)}$ w.r.t. $t_{s(-3,1)}$	258.91	128.24	114.92
$t_{3(-1,-1)}$ w.r.t. $\bar{y}_r$	102.71	100.83	101.15
$t_{3(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	119.24	104.66	101.87

Perusal of the table 6.9 revealed that the proposed estimator $\boldsymbol{t_{3(-3,1)}}$ performs better than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). The values of the relative mean squared error of the proposed estimator $t_{3(-3,1)}$ were 0.19932, 0.02656 and 0.01974 at samples of sizes 30, 60 and 120 respectively. Further, the percent relative efficiency of the proposed estimator $t_{3(-3,1)}$ lies between 108.08 to 123.81 with respect to conventional ratio estimator. Further, the percent relative efficiency of the proposed estimator was lying between 114.92 to 258.91 with respect to $t_{s(-3,1).}$ An unbiased estimator has also been obtained at p=-1 and q=-1 that was more efficient than conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010) whose percent relative efficiency was found to lie between 100.83 to 119.24 and the variances of the estimator at samples of sizes 30, 60 and 120 were 0.22665, 0.02857 and 0.02054. Further, the percent relative efficiency of the proposed unbiased estimator $t_{3(-1,-1)}$ was lying between 101.87 to 119.24 with respect to $t_{s(-1,-1)}$.

Table 6.10: Relative mean squared error/relative variance of the proposed ratio estimators $\boldsymbol{t_{4(-3,1)}}$ and $\boldsymbol{t_{4(-1,-1)}}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for $\mathbf{P_1}$

Estimator	n		
	30	60	120
$\bar{y}_r$	0.24678	0.02983	0.02134
$t_{s(-3,1)}$	0.51606	0.03406	0.02269
$t_{s(-1,-1)}$	0.28649	0.03096	0.02149
$t_{4(-3,1)}$	0.19497	0.02621	0.01989
$t_{4(-1,-1)}$	0.24003	0.02982	0.02129
Percent relative efficiencies of proposed estimators			
$t_{4(-3,1)}$ w.r.t. $\bar{y}_r$	126.58	113.78	107.28
$t_{4(-3,1)}$ w.r.t. $t_{s(-3,1)}$	264.69	129.93	114.07
$t_{4(-1,-1)}$ w.r.t. $\bar{y}_r$	102.81	100.03	100.21
$t_{4(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	119.36	103.74	100.93

From table 6.10, it has been observed that the values of relative mean squared error of the proposed estimator $\boldsymbol{t_{4(-3,1)}}$ were 0.19497, 0.02621 and 0.01989 at samples of sizes 30, 60 and 120 respectively. Further the proposed estimator $t_{4(-3,1)}$ was more efficient than conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). The percent relative efficiency of the proposed estimator $t_{4(-3,1)}$ lies between 107.28 to 126.58 with respect to conventional ratio type estimator and with respect to $t_{s(-3,1)}$, it was found to lie between 114.07 to 264.69. For the unbiased proposed estimator $t_{4(-1,-1)}$, the values of relative mean squared error were 0.24003, 0.02982 and 0.02129 at samples of sizes 30, 60 and 120 respectively. The percent relative efficiency of the proposed unbiased estimator $t_{4(-1,-1)}$ was found to lie between 100.21to 102.81 with respect to conventional ratio estimator and was lying between 100.93 to 119.36 with respect to $t_{s(-1,-1)}$.

Table 6.11: Relative mean squared error of the proposed ratio estimator $t_{5(-3,1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_1

Estimator	n		
	30	60	120
$\bar{y}_r$	0.24678	0.02983	0.02134
$t_{s(-3,1)}$	0.51606	0.03406	0.02269
$t_{5(-3,1)}$	0.21791	0.02751	0.02036
Percent relative efficiencies of proposed estimators			
$t_{5(-3,1)}$ w.r.t. $\bar{y}_r$	113.25	108.43	104.79
$t_{5(-3,1)}$ w.r.t. $t_{s(-3,1)}$	236.82	123.82	111.42

Table 6.11 revealed that the proposed estimator $t_{5(-3,1)}$ have smaller relative mean squared error than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). Further, the value of the relative mean squared error of the proposed estimator $t_{5(-3,1)}$ at samples of sizes 30, 60 and 120 were 0.21791, 0.02751 and 0.02036 respectively. The percent relative efficiency of the proposed estimator $t_{5(-3,1)}$ was found to lie between 104.79 to 113.25 with respect to conventional ratio estimator. Further, the percent relative efficiency of the proposed estimator $t_{5(-3,1)}$ was found to lie between 111.42 to 236.82 with respect to the estimator $t_{s(-3,1)}$.

6.2.2: Population II (P_2)

Table 6.12: Descriptive statistics of the variable under study and auxiliary variable for P_2

Variable	Population mean	Population variance	ρ_{XY}
Y	21.943	229.667	0.853
X	10.432	39.358	

From the table 6.12, it has been observed that the correlation between the two variables (Y and X) was found to be 0.853 whereas the mean and variance of the study variable Y were 21.943 and 229.667 whereas that of auxiliary variable X were 10.432 and 39.358 respectively.

Table 6.13: Relative bias of the proposed ratio estimators $\boldsymbol{t_{1(-3,1)}}$ and $\boldsymbol{t_{1(-1,-1)}}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for $\mathbf{P_2}$

Estimator	n		
	30	60	120
$\bar{y}_r$	0.01016	0.00681	0.00252
$t_{s(-3,1)}$	-0.04515	-0.02356	-0.00884
$t_{s(-1,-1)}$	0.02033	0.01362	0.00504
$t_{1(-3,1)}$	-0.02131	-0.01499	-0.00648
$t_{1(-1,-1)}$	0.00000	0.00000	0.00000

From the table 6.13, it has been observed that the proposed estimator $t_{1(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator. Further, the biases of the proposed estimator $t_{1(-3,1)}$ have been found to be -0.02131, -0.01499 and -0.00648 at sample of sizes 30, 60 and 120 respectively. For p=-1 and q= -1, the proposed estimator t_1 found to be unbiased. Further, for the estimator $t_{s\,(-3,1)}$, biases at sample sizes 30, 60 and 120 were -0.04515, -0.02356 and -0.00884 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were 0.02033, 0.01362 and 0.00504 at sample of sizes 30, 60 and 120 respectively.

Table 6.14: Relative bias of the proposed ratio estimators $\boldsymbol{t_{2(-3,1)}}$ and $\boldsymbol{t_{2(-1,-1)}}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for $\mathbf{P_2}$

Estimator	n		
	30	60	120
$\bar{y}_r$	0.01016	0.00681	0.00252
$t_{s(-3,1)}$	-0.04515	-0.02356	-0.00884
$t_{s(-1,-1)}$	0.02033	0.01362	0.00504
$t_{2(-3,1)}$	-0.02131	-0.01499	-0.00648
$t_{2(-1,-1)}$	0.00000	0.00000	0.00000

Table 6.14 revealed that the proposed estimator $t_{2(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator. The bias of the proposed estimator $t_{2\,(-3,1)}$ were -0.02131, -0.01499 and -0.00648 at samples of sizes 30, 60 and 120 respectively. For p=-1 and q= -1, the proposed estimator t_2 found to be unbiased.

Further, for the estimator $t_{s\,(-3,1)}$, biases were -0.04515, -0.02356 and -0.00884 at sample size 30, 60 and 120 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were 0.02033, 0.01362 and 0.00504 at samples of sizes 30, 60 and 120 respectively.

Table 6.15: Relative bias of the proposed ratio estimators $t_{3(-3,1)}$ and $t_{3(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_2

Estimator	n		
	30	60	120
$\bar{y}_r$	0.01016	0.00681	0.00252
$t_{s(-3,1)}$	-0.04515	-0.02356	-0.00884
$t_{s(-1,-1)}$	0.02033	0.01362	0.00504
$t_{3(-3,1)}$	-0.02131	-0.01499	-0.00648
$t_{3(-1,-1)}$	0.00000	0.00000	0.00000

From the table 6.15, it has been observed that the proposed estimator t_3 is unbiased at p=-1 and q=-1. The bias of the proposed estimator $t_{3\,(-3,1)}$ were -0.02131, -0.01499 and -0.00648 at samples of sizes 30, 60 and 120 respectively. Further, for the estimator $t_{s\,(-3,1)}$, biases were -0.04515, -0.02356 and -0.00884 at sample of sizes 30, 60 and 120 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were 0.02033, 0.01362 and 0.00504 at samples of sizes 30, 60 and 120 respectively.

Table 6.16: Relative bias of the proposed ratio estimators $t_{4(-3,1)}$ and $t_{4(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_2

Estimator	n		
	30	60	120
$\bar{y}_r$	0.01016	0.00681	0.00252
$t_{s(-3,1)}$	-0.04515	-0.02356	-0.00884
$t_{s(-1,-1)}$	0.02033	0.01362	0.00504
$t_{4(-3,1)}$	-0.02131	-0.01499	-0.00648
$t_{4(-1,-1)}$	0.00000	0.00000	0.00000

Table 6.16 revealed that the proposed estimator $t_{4(-3,1)}$ was negatively biased and the biases have been found to be -0.02131, -0.01499 and -0.00648 at sample of sizes 30, 60 and 120 respectively. For p=-1 and q= -1, the proposed estimator t_4 was found to be

unbiased. Further, for the estimator $t_{s\,(-3,1)}$, biases at sample of sizes 30, 60 and 120 were -0.04515, -0.02356 and -0.00884 respectively. Further, the relative bias of t_s at p=-1 and q=-1 were 0.02033, 0.01362 and 0.00504 at sample of sizes 30, 60 and 120 respectively.

Table 6.17: Relative bias of the proposed ratio estimator $t_{5(-3,1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_2

Estimator	n		
	30	60	120
$\bar{y}_r$	0.01016	0.00681	0.00252
$t_{s(-3,1)}$	-0.04515	-0.02356	-0.00884
$t_{5(-3,1)}$	-0.02131	-0.01499	-0.00648

From the table 6.17, it has been observed that the proposed estimator $t_{5(-3,1)}$ was negatively biased and have smaller bias than the conventional ratio estimator. The values of the relative bias of the proposed estimator $t_{5(-3,1)}$ were -0.02131, -0.01499 and -0.00648 at samples of sizes 30, 60 and 120 respectively whereas that for $t_{s(-3,1)}$ the values were -0.04515, -0.02356 and -0.00884 at the specified sample sizes.

Thus, it has been observed that all the estimators i.e., $t_{1(-3,1)}$, $t_{2(-3,1)}$, $t_{3(-3,1)}$, $t_{4(-3,1)}$ and $t_{5(-3,1)}$ were biased with same magnitude whereas estimators $t_{1(-1,-1)}$, $t_{2(-1,-1)}$, $t_{3(-1,-1)}$ and $t_{4(-1,-1)}$ were found to be unbiased.

Table 6.18: Relative mean squared error /relative variance of the proposed ratio estimators $\boldsymbol{t_{1(-3,1)}}$ and $\boldsymbol{t_{1(-1,-1)}}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for $\mathbf{P_2}$

Estimator	n		
	30	60	120
$\bar{y}_r$	0.02970	0.01738	0.00623
$t_{s(-3,1)}$	0.03578	0.01907	0.00645
$t_{s(-1,-1)}$	0.03130	0.01783	0.00629
$t_{1(-3,1)}$	0.02726	0.01645	0.00613
$t_{1(-1,-1)}$	0.02913	0.01712	0.00619
Percent relative efficiencies of proposed estimators			
$t_{1(-3,1)}$ w.r.t. $\bar{y}_r$	108.95	105.63	101.78
$t_{1(-3,1)}$ w.r.t. $t_{s(-3,1)}$	131.27	115.95	105.26
$t_{1(-1,-1)}$ w.r.t. $\bar{y}_r$	101.93	101.51	100.77
$t_{1(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	107.42	104.15	101.60

From table 6.18, it has been observed that the proposed estimator $\boldsymbol{t_{1(-3,1)}}$ was more efficient than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). Further, the percent relative efficiency of the proposed estimator $t_{1(-3,1)}$ was found to lie between 101.78 to 108.95 with respect to conventional ratio estimator and was lying between 105.26 to 131.27 with respect to $t_{s(-3,1)}$. The value of the relative mean squared error of the proposed estimator $t_{1(-3,1)}$ at samples of sizes 30, 60 and 120 were 0.02726, 0.01645 and 0.00613 respectively. The value of the relative variances of the proposed estimator $t_{1(-1,-1)}$ were 0.02913, 0.01712 and 0.00619 at samples of sizes 30, 60 and 120 respectively. The percent relative efficiency of the proposed estimator $t_{1(-1,-1)}$ was found to lies between 100.77 to 101.93. Further, with respect to the estimator $t_{s(-1,-1)}$ the percent relative efficiency of the proposed estimator $t_{1(-1,-1)}$ was found to lie between 101.60 to 107.42.

Table 6.19: Relative mean squared/relative variance of the proposed ratio estimators $t_{2(-3,1)}$ and $t_{2(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_2

Estimator	n		
	30	60	120
$\bar{y}_r$	0.02970	0.01738	0.00623
$t_{s(-3,1)}$	0.03578	0.01907	0.00645
$t_{s(-1,-1)}$	0.03130	0.01783	0.00629
$t_{2(-3,1)}$	0.02902	0.01706	0.00622
$t_{2(-1,-1)}$	0.02969	0.01732	0.00622
Percent relative efficiencies of proposed estimators			
$t_{2(-3,1)}$ w.r.t. $\bar{y}_r$	102.34	101.85	100.22
$t_{2(-3,1)}$ w.r.t. $t_{s(-3,1)}$	123.31	111.80	103.64
$t_{2(-1,-1)}$ w.r.t. $\bar{y}_r$	100.04	100.30	100.23
$t_{2(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	105.29	102.91	101.06

Table 6.19 revealed that the proposed estimator $t_{2(-3,1)}$ have smaller relative mean squared error than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). Further, the value of the relative mean squared error of the proposed estimator $t_{2(-3,1)}$ at samples of sizes 30, 60 and 120 were 0.02902, 0.01706 and 0.00622 respectively. The percent relative efficiency of the proposed estimator $t_{2(-3,1)}$ was found to lie between 100.22 to 102.34 with respect to conventional ratio estimator and with respect to $t_{s(-3,1)}$, it was found to lie between 103.64 to 123.31. For the unbiased proposed estimator $t_{2(-1,-1)}$, the values of relative variance were 0.02969, 0.01732 and 0.00622 at sample of sizes 30, 60 and 120 respectively. The percent relative efficiency of the proposed estimator $t_{2(-1,-1)}$ was found to lie between 100.04 to 101.30 with respect to conventional ratio estimator and with respect to $t_{s(-1,-1)}$, it was lying between 101.06 to 105.29.

Table 6.20: Relative mean squared error/relative variance of the proposed ratio estimators $t_{3(-3,1)}$ and $t_{3(-1,-1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_2

Estimator	n		
	30	60	120
$\bar{y}_r$	0.02970	0.01738	0.00623
$t_{s(-3,1)}$	0.03578	0.01907	0.00645
$t_{s(-1,-1)}$	0.03130	0.01783	0.00629
$t_{3(-3,1)}$	0.02712	0.01619	0.00609
$t_{3(-1,-1)}$	0.02910	0.01704	0.00618
Percent relative efficiencies of proposed estimators			
$t_{3(-3,1)}$ w.r.t. $\bar{y}_r$	109.51	107.34	102.34
$t_{3(-3,1)}$ w.r.t. $t_{s(-3,1)}$	131.94	117.82	105.83
$t_{3(-1,-1)}$ w.r.t. $\bar{y}_r$	102.05	101.97	100.89
$t_{3(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	107.55	104.63	101.72

Perusal of the table 6.20 revealed that the proposed estimator $\boldsymbol{t_{3(-3,1)}}$ performs better than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). The values of the relative mean squared error of the proposed estimator $t_{3(-3,1)}$ were 0.02712, 0.01619 and 0.00609 at sample of sizes 30, 60 and 120 respectively. Further, the percent relative efficiency of the proposed estimator $t_{3(-3,1)}$ lies between 102.34 to 109.51 with respect to conventional ratio estimator whereas it was lying between 105.83 to 131.94 with respect to $t_{s(-3,1).}$ An unbiased estimator has also been obtained at p=-1 and q=-1 that was more efficient than conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010) whose percent relative efficiency was found to lie between 100.89 to 102.05 and the variances of $t_{3(-1,-1)}$ at samples of sizes 30, 60 and 120 were 0.02910, 0.01704 and 0.00618. The percent relative efficiency of the proposed unbiased estimator was found to lie between 101.72 to 107.55 with respect to $t_{s(-1,-1)}$.

Table 6.21: Relative mean squared error/ relative variance of the proposed ratio estimators $\boldsymbol{t_{4(-3,1)}}$ and $\boldsymbol{t_{4(-1,-1)}}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for $\mathbf{P_2}$

Estimator	n		
	30	60	120
$\bar{y}_r$	0.02970	0.01738	0.00623
$t_{s(-3,1)}$	0.03578	0.01907	0.00645
$t_{s(-1,-1)}$	0.03130	0.01783	0.00629
$t_{4(-3,1)}$	0.02640	0.01595	0.00608
$t_{4(-1,-1)}$	0.02968	0.01734	0.00623
Percent relative efficiencies of proposed estimators			
$t_{4(-3,1)}$ w.r.t. $\bar{y}_r$	112.46	108.95	102.55
$t_{4(-3,1)}$ w.r.t. $t_{s(-3,1)}$	135.50	119.60	106.05
$t_{4(-1,-1)}$ w.r.t. $\bar{y}_r$	100.06	100.19	100.06
$t_{4(-1,-1)}$ w.r.t. $t_{s(-1,-1)}$	105.37	102.80	100.88

From table 6.21, it has been observed that the values of relative mean squared error of the proposed estimator $\boldsymbol{t_{4(-3,1)}}$ were 0.02640, 0.01595 and 0.00608 at samples of sizes 30, 60 and 120 respectively. Further the proposed estimator $t_{4(-3,1)}$ was more efficient than conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). The percent relative efficiency of the proposed estimator $t_{4(-3,1)}$ lies between 102.55 to 112.46 with respect to conventional ratio type estimator whereas with respect to $t_{s(-3,1)}$, it was lying between 106.05 to 135.50. If the property of unbiased is sacrificed a most efficient estimator $t_{4(-3,1)}$ can be employed. For the proposed unbiased estimator $t_{4(-1,-1)}$, the values of relative mean squared error were 0.02968, 0.01734 and 0.00623 at sample of sizes 30, 60 and 120 respectively. The percent relative efficiency of the proposed estimator $t_{4(-1,-1)}$ was found to lie between 100.06 to 100.19 with respect to conventional ratio estimator and with respect to $t_{s(-1,-1)}$, it was lying between 100.88 to 105.37.

Table 6.22: Relative mean squared error of the proposed ratio estimator $t_{5(-3,1)}$ with respect to conventional ratio estimator and estimator proposed by Sharma *et al.* (2010) for P_2

Estimator	n		
	30	60	120
$\bar{y}_r$	0.02970	0.01738	0.00623
$t_{s(-3,1)}$	0.03578	0.01907	0.00645
$t_{5(-3,1)}$	0.02774	0.01647	0.00613
Percent relative efficiencies of proposed estimators			
$t_{5(-3,1)}$ w.r.t. $\bar{y}_r$	107.06	105.53	101.70
$t_{5(-3,1)}$ w.r.t. $t_{s(-3,1)}$	128.99	115.83	105.17

Table 6.22 revealed that the proposed estimator $t_{5(-3,1)}$ have smaller relative mean squared error than the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010). Further, the value of the relative mean squared error of the proposed estimator $t_{5(-3,1)}$ at samples of sizes 30, 60 and 120 were 0.02774, 0.01647and 0.00613 respectively. The percent relative efficiency of the proposed estimator $t_{5(-3,1)}$ was found to lie between 101.70 to 107.06 with respect to conventional ratio estimator. Further, the percent relative efficiency of the proposed estimator $t_{5(-3,1)}$ was lying between 105.17 to 128.99 with respect to $t_{s(-3,1)}$.

6.3 IMPROVED PRODUCT TYPE ESTIMATORS

Improved product type of estimators of population mean have been theoretically developed and their large sample properties are worked out in the chapter V. The efficiencies of the developed estimators have been tested through two bivariate population datasets (P_3 and P_4). The correlations among the variables Y and X have been considered high and negative. Further, different values of p and q have been used, the estimators with the properties of unbiasedness, efficiency and consistency are proposed.

6.3.1 POPULATION III (P_3):

Table 6.23: Descriptive statistics of the variable under study and auxiliary variable for P_3

Variable	Population Mean	Population Variance	ρ_{XY}
Y	0.758	17.597	-0.909
X	4.571	342.054	

From table 6.23, it has been observed that the correlation between the two variables (Y and X) was found to be -0.909 which was high and negative whereas the mean of the study variable Y and auxiliary variable X were 0.758 and 4.571 whereas the variances were 17.597 and 342.054 respectively.

Table 6.24: Relative bias of the proposed product estimator $t_{1(1)}^{p}$ with respect to existing product estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	-0.13498	-0.01894	-0.00699
t_R^p	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t_D^p	0.00000	0.00000	0.00000
$t_{s(1)}^p$	1.20381	0.12268	0.16895
$t_{1(1)}^p$	1.20381	0.12268	0.16895

From the table 6.24, it can be observed that the proposed estimator $t_{1(1)}^p$ is positively biased and has identical bias to that of the estimator as proposed by Sharma *et al.* (2007). The bias of the proposed estimator $t_{1(1)}^p$ were 1.20381, 0.12268 and 0.16895 at sample sizes 30, 60 and 120 respectively. Estimators proposed by Robson (1957), Singh (1989) and Dubey (1993) were unbiased.

Table 6.25: Relative bias of the proposed product estimator $t_{2(3)}^{p}$ and $t_{2(-1)}^{p}$ with respect to existing product estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	-0.13498	-0.01894	-0.00699
t_R^p	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t_D^p	0.00000	0.00000	0.00000
$t_{s(3)}^p$	3.88137	0.40591	0.52082
$t_{s(-1)}^p$	-1.47376	-0.16055	-0.18292
$t_{2(3)}^p$	-0.53991	-0.07575	-0.02795
$t_{2(-1)}^p$	0.00000	0.00000	0.00000

Table 6.25 revealed that the proposed product type estimator $\boldsymbol{t_{2(3)}^{p}}$ was negatively biased and has more bias than the conventional estimator and less than the estimator proposed by Sharma *et al.* (2007). The biases of the proposed estimator $t_{2(3)}^{p}$ at samples of sizes 30, 60 and 120 were -0.53991, -0.07575 and -0.02795 respectively. Further, estimator at q=-1 of the proposed class t_2 has been proposed which was unbiased like Robson (1957), Singh (1989) and Dubey (1993). For samples of sizes 30, 60 and 120, the biases of the estimator t_s at q=3 were 3.88137, 0.40591 and 0.52082 whereas at q=-1 were -1.47376, -0.16055 and -0.18292 respectively.

Table 6.26: Relative bias of the proposed product estimator $\boldsymbol{t_{3(2)}^{p}}$ and $\boldsymbol{t_{3(-1)}^{p}}$ with respect to existing product estimators for $\mathbf{P_3}$

Estimator	n		
	30	60	120
$\bar{y}_p$	-0.13498	-0.01894	-0.00699
t_R^p	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t_D^p	0.00000	0.00000	0.00000
$t_{s(-1)}^p$	-1.47376	-0.16055	-0.18292
$t_{s(2)}^p$	2.54259	0.26430	0.34488
$t_{3(2)}^p$	-0.40494	-0.05681	-0.02096
$t_{3(-1)}^p$	0.00000	0.00000	0.00000

From the table 6.26, it can be seen that the proposed estimator $t_{3(2)}^{p}$ was negatively biased and have more bias than the conventional product estimator and less than the estimator proposed by Sharma *et al.* (2007) at q=2. The bias of the proposed estimator $t_{3(2)}^{p}$ were -0.40494, -0.05681 and -0.02096 at samples of sizes 30, 60 and 120 respectively. An unbiased estimator $t_{3(-1)}^{p}$ has also been proposed. Further, it has been observed that t_3 estimator at q=-1 was unbiased like Robson (1957), Singh (1989), and Dubey (1993) upto order O (n^{-1}).

Table 6.27: Relative bias of the proposed product estimator $t^{p}_{4(-2,1)}$ with respect to existing product estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	-0.13498	-0.01894	-0.00699
t^p_R	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t^p_D	0.00000	0.00000	0.00000
$t^p_{s(1)}$	1.20381	0.12268	0.16895
$t^p_{4(-2,1)}$	-0.74319	-0.19438	-0.20044

Table 6.27 showed that the proposed product estimator $t^p_{4(-2,1)}$ was negatively biased and have more bias than the conventional product estimator and less than estimator proposed by Sharma *et al.* (2007**)** which is positive. The bias of the proposed estimator $t^p_{4(-2,1)}$ at samples of size 30, 60 and 120 were -0.74319, -0.19438 and -0.20044 respectively. The biases of the estimator $t^p_{s(1)}$ at samples of size 30, 60 and 120 were 1.20381, 0.12268 and 0.16895 respectively

Table 6.28: Relative bias of the proposed product estimator $t^{p}_{5(3,1)}$ with respect to existing product estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	-0.13498	-0.01894	-0.00699
t^p_R	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t^p_D	0.00000	0.00000	0.00000
$t^p_{s(1)}$	1.20381	0.12268	0.16895
$t^p_{5(3,1)}$	0.43989	0.19688	0.26573

From the table 6.28, it has been observed that the proposed product estimator $t^p_{5(3,1)}$ was positively biased and have smaller bias than the estimator proposed by Sharma *et al.* (2007**).** The bias of the estimator $\boldsymbol{t^p_{s(1)}}$ at samples of size 30, 60 and 120 were 1.20381, 0.12268 and 0.16895 respectively. The biases of the estimator $t^p_{5(3,1)}$ at samples of size 30, 60 and 120 were 0.43989, 0.19688 and 0.26573 respectively.

Table 6.29: Relative mean squared error and percent relative efficiency of the proposed product estimator $t_{1(1)}^{p}$ with respect to different product type estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	1.57413	1.73250	0.01072
t_R^p	1.61000	1.73190	0.01077
t^*	1.61000	1.73190	0.01077
t_D^p	1.46049	1.51157	0.01077
$t_{s(1)}^p$	2.86801	3.34149	0.01079
$t_{1(1)}^p$	1.27593	1.11355	0.01061
Percent relative efficiencies of proposed estimators			
$t_{1(1)}^p$ w.r.t. $\bar{y}_p$	123.37	155.58	101.06
$t_{1(1)}^p$ w.r.t. t_R^p and t^*	126.18	155.53	101.51
$t_{1(1)}^p$ w.r.t. t_D^p	114.46	135.74	101.54
$t_{1(1)}^p$ w.r.t. $t_{s(1)}^p$	224.78	300.08	101.69

Table 6.29 reveals that the proposed estimator $t_{1(1)}^p$ performed better than the conventional product estimator and the estimator proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The relative man squared error of the proposed estimator was found to be 1.27593, 1.11355 and 0.01061 at samples of size 30, 60 and 120 respectively and was minimum among the other estimators. Further, the percent relative efficiency of the proposed estimator with respect to conventional estimator was found to lie in the range between 101.06 to 155.58. The highest relative efficiency of the proposed estimator was found to be 155.58 with respect to conventional estimator for sample of size 60. Further, the percent relative efficiency of the proposed estimator $t_{1(1)}^p$ was found to lie between 101.51 to 155.53 with respect to t_R^p and t^*. With respect to t_D^p and $t_{s(1)}^p$, the percent relative efficiency of the proposed estimator $t_{1(1)}^p$ was lying between 101.54 to 135.74 and 101.69 to 224.78 respectively.

Table 6.30: Relative mean squared error and percent relative efficiency of the proposed estimator $t_{2(3)}^{p}$ with respect to different product type estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	1.15778	1.28830	0.36744
t_R^p	1.18587	1.13633	0.36914
t^*	1.18587	1.13633	0.36914
t_D^p	1.18970	1.59871	0.37198
$t_{s(3)}^p$	2.70947	2.64334	0.50991
$t_{2(3)}^p$	1.05132	0.79463	0.34913
Percent relative efficiencies of proposed estimators			
$t_{2(3)}^p$ w.r.t. $\bar{y}_p$	110.13	162.13	105.25
$t_{2(3)}^p$ w.r.t. t_R^p and t^*	112.80	143.00	105.73
$t_{2(3)}^p$ w.r.t. t_D^p	113.16	201.19	106.54
$t_{2(3)}^p$ w.r.t. $t_{s(3)}^p$	257.72	332.65	146.05

It has been observed from the table 6.30, that the proposed estimator $t_{2(3)}^p$ have smallest relative mean squared error than the conventional estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). Further, the values of the relative mean squared error of the proposed estimator $t_{2(3)}^p$ at samples of sizes 30, 60 and 120 were found to be 1.05132, 0.79463 and 0.34913 respectively. The percent relative efficiency of the proposed estimator lies between 105.25 to 162.13 with respect to conventional product type estimator. Further, the relative efficiency of the proposed estimator $t_{2(3)}^p$ was found to lie between 105.73 to 143.00 with respect to t_R^p and t^*. With respect to t_D^p and $t_{s(3)}^p$, the percent relative efficiency of the proposed estimator $t_{2(3)}^p$ was lying between 106.54 to 201.19 and 146.05 to 332.65 respectively.

Table 6.31: Relative mean squared error and percent relative efficiency of the proposed estimator $t_{3(2)}^{p}$ with respect to different product type estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	1.15301	2.03022	0.31280
t_R^p	1.16923	2.00930	0.30753
t^*	1.16923	2.00930	0.30753
t_D^p	1.34137	2.28904	0.34201
$t_{s(2)}^p$	4.54621	6.60589	0.46706
$t_{3(2)}^p$	0.83269	1.66506	0.29436
Percent relative efficiencies of proposed estimators			
$t_{3(2)}^p$ w.r.t. $\bar{y}_p$	138.47	121.93	106.26
$t_{3(2)}^p$ w.r.t. t_R^p and t^*	140.42	120.67	104.47
$t_{3(2)}^p$ w.r.t. t_D^p	161.09	137.48	116.19
$t_{3(2)}^p$ w.r.t. $t_{s(2)}^p$	545.97	396.74	158.67

Table 6.31 showed that the proposed estimator $t_{3(2)}^p$ though inconsistent but have smallest relative mean squared error than the conventional product estimator and estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The values of the relative mean squared error of the proposed estimator $t_{3(2)}^p$ at samples of sizes 30, 60 and 120 were 0.83269, 1.66506 and 0.29436 respectively. The percent relative efficiency of the proposed estimator $t_{3(2)}^p$ with respect to conventional product type estimators, t_D^p and $t_{s(2)}^p$ was found to lie between 106.26 to 138.47, 116.19 to 161.09 and 158.67 to 545.97 respectively. Further, the percent relative efficiency of the proposed estimator was lying between 104.47 to 140.42 with respect to t_R^p and t^*.

Table 6.32: Relative mean squared error and percent relative efficiency of the proposed estimator $t^{p}_{4(-2,1)}$ with respect to different product type estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	1.15301	0.47102	0.26303
t^p_R	1.16923	0.46472	0.26331
t^*	1.16923	0.46472	0.26331
t^p_D	1.34137	0.46856	0.26458
$t^p_{s(1)}$	2.22427	0.55277	0.31200
$t^p_{4(-2,1)}$	0.89040	0.39890	0.25602
Percent relative efficiencies of proposed estimators			
$t^p_{4(-2,1)}$ w.r.t. $\bar{y}_p$	124.33	135.57	107.45
$t^p_{4(-2,1)}$ w.r.t. t^p_R and t^*	126.08	133.76	107.56
$t^p_{4(-2,1)}$ w.r.t. t^p_D	144.64	134.86	108.08
$t^p_{4(-2,1)}$ w.r.t. $t^p_{s(1)}$	239.85	159.10	127.45

From the table 6.32, it has been observed that the proposed estimator $t^p_{4(-2,1)}$ was consistent and performed better than other product type estimators *viz.*,$\bar{y}_p$, t^p_R,t^*, t^p_D and t^p_s. Further, the values of the relative mean squared error of the proposed estimator $t^p_{4(-2,1)}$ at samples of sizes 30, 60 and 120 were found to be 0.89040, 0.39890 and 0.25602 respectively. The percent relative efficiency of the proposed estimator lies between 107.45 to 135.57 with respect to conventional product type estimator whereas with respect to t^p_R and t^*, it was lying between 107.56 to 133.76. Further, the relative efficiency of the proposed estimator $t^p_{4(-2,1)}$ was found to lie between 108.08 to 144.64 and 127.45 to 239.85 with respect to t^p_D and $t^p_{s(1)}$ respectively.

Table 6.33: Relative mean squared error and percent relative efficiency of the proposed estimator $t^p_{5(3,1)}$ with respect to different product type estimators for P_3

Estimator	n		
	30	60	120
$\bar{y}_p$	1.29847	0.33081	0.19728
t_R^p	1.29974	0.33154	0.19848
t^*	1.29974	0.33154	0.19848
t_D^p	1.46729	0.32801	0.19785
$t_{s(1)}^p$	2.06052	0.40326	0.22569
$t^p_{5(3,1)}$	0.57929	0.26024	0.16521
Percent relative efficiencies of proposed estimators			
$t^p_{5(3,1)}$ w.r.t. $\bar{y}_p$	224.15	127.12	119.41
$t^p_{5(3,1)}$ w.r.t. t_R^p and t^*	224.37	127.40	120.14
$t^p_{5(3,1)}$ w.r.t. t_D^p	253.29	126.04	119.76
$t^p_{5(3,1)}$ w.r.t. $t_{s(1)}^p$	355.70	154.96	136.61

Table 6.33 showed that the proposed product estimator $t^p_{5(3,1)}$ was more efficient than the conventional product estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007). The values of the relative mean squared error of the proposed estimator $t^p_{5(3,1)}$ were 0.57929, 0.26024 and 0.16521 respectively. Further, the proposed estimator was found to be consistent. The percent relative efficiency of the proposed estimator with respect to conventional product type estimators was found to lie between 119.41 to 224.15, 119.76 to 253.29 and 136.61 to 355.70 with respect to t_D^p and $t_{s(1)}^p$ respectively. Further, with respect to t_R^p and t^*, the percent relative efficiency of the proposed estimator was lying between 120.14 to 224.37.

6.3.2: POPULATION IV (P_4)

Table 6.34: Descriptive statistics of the variable under study and auxiliary variable for P_4

Variable	Population Mean	Population Variance	ρ_{XY}
Y	4.279	48.022	-0.981
X	8.664	511.084	

From table 6.34, it has been observed that the correlation between the two variables (Y and X) was found to be -0.981 which was high and negative whereas the mean of the study variable Y and auxiliary variable X were 4.279 and 8.664 whereas the variances were 48.022 and 511.084 respectively.

Table 6.35: Relative bias of the proposed product estimator $t^p_{1(1)}$ with respect to existing product estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.07878	-0.01138	-0.00229
t^p_R	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t^p_D	0.00000	0.00000	0.00000
$t^p_{s(1)}$	0.97993	0.31672	0.12868
$t^p_{1(1)}$	0.97993	0.31672	0.12868

From the table 6.35, it can be observed that the proposed estimator $t^p_{1(1)}$ is positively biased and has identical bias to that of the estimator as proposed by Sharma *et al.* (2007). The bias of the proposed estimator $t^p_{1(1)}$ were 0.97993, 0.31672 and 0.12868 at sample sizes 30, 60 and 120 respectively. Estimators proposed by Robson (1957), Singh (1989) and Dubey (1993) were unbiased.

Table 6.36: Relative bias of the proposed product estimator $t^p_{2(3)}$ and $t^p_{2(-1)}$ with respect to conventional and other product estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.07878	0.00397	0.01107
t^p_R	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t^p_D	0.00000	0.00000	0.00000
$t^p_{s(3)}$	2.78224	1.72514	1.44100
$t^p_{s(-1)}$	-0.82237	-0.56975	-0.46557
$t^p_{2(3)}$	0.31513	0.01589	0.04430
$t^p_{2(-1)}$	0.00000	0.00000	0.00000

Table 6.36 revealed that the proposed product type estimator $t^p_{2(3)}$ was positively biased and has more bias than the conventional estimator but smaller bias than the

estimator proposed by Sharma *et al.* (2007) at q=3. The biases of the proposed estimator $t_{2(3)}^{p}$ at samples of sizes 30, 60 and 120 were 0.31513, 0.01589 and 0.04430 respectively. Another estimator at q=-1 of the proposed class t_2 has been proposed which was unbiased. For samples of sizes 30, 60 and 120, the biases of the estimator t_s at q=3 were 2.78224, 1.72514 and 1.44100 whereas at q=-1 were -0.82237, -0.56975 and -0.46557 respectively.

Table 6.37: Relative bias of the proposed product estimator $t_{3(2)}^{p}$ and $t_{3(-1)}^{p}$ with respect to conventional and other product estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.02516	0.00397	0.01107
t_R^p	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t_D^p	0.00000	0.00000	0.00000
$t_{s(-1)}^p$	-0.19395	-0.56975	-0.46557
$t_{s(2)}^p$	0.46338	1.15142	0.96436
$t_{3(2)}^p$	0.07548	0.01192	0.03322
$t_{3(-1)}^p$	0.00000	0.00000	0.00000

From the table 6.37, it can be seen that the proposed estimator $t_{3(2)}^{p}$ was positively biased and have smaller bias than estimator proposed by Sharma *et al.* (2007). The bias of the proposed estimator $t_{3(2)}^{p}$ were 0.07548, 0.01192 and 0.03322 at samples of sizes 30, 60 and 120 respectively. At q=-1, the bias of the proposed estimator is zero like estimators proposed by Robson (1957), Singh (1989) and Dubey (1993) and hence it is unbiased.

Table 6.38: Relative bias of the proposed product estimator $t^{p}_{4(-2,1)}$ with respect to existing product estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.07878	0.00301	0.00635
t^p_R	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t^p_D	0.00000	0.00000	0.00000
$t^p_{s(1)}$	0.97993	0.07881	0.10933
$t^p_{4(-2,1)}$	0.00191	-0.05907	-0.03682

Table 6.38 showed that the proposed product estimator $t^p_{4(-2,1)}$ have smaller bias than the conventional product estimator and estimator proposed by Sharma *et al.* (2007**).** The bias of the proposed estimator $t^p_{4(-2,1)}$ at samples of size 30, 60 and 120 were 0.00191, -0.05907 and 0.03682 respectively. The biases of the estimator $t^p_{s(1)}$ at samples of size 30, 60 and 120 were 0.97993, 0.07881 and 0.10933 respectively.

Table 6.39: Relative bias of the proposed product estimator $t^{p}_{5(3,1)}$ with respect to existing product estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.07878	0.00301	0.00635
t^p_R	0.00000	0.00000	0.00000
t^*	0.00000	0.00000	0.00000
t^p_D	0.00000	0.00000	0.00000
$t^p_{s(1)}$	0.97993	0.07881	0.10933
$t^p_{5(3,1)}$	0.00191	-0.05907	-0.03682

From the table 6.39, it has been observed that the proposed product estimator $t^p_{5(3,1)}$ have smaller bias than the estimator proposed by Sharma *et al.* (2007**).** The bias of the proposed estimator $t^p_{5(3,1)}$ at samples of size 30, 60 and 120 were 0.00191, -0.05907 and 0.03682 respectively. The biases of the estimator $t^p_{5(3,1)}$ at samples of size 30, 60 and 120 were 0.00191, -0.05907 and -0.03682 respectively.

Table 6.40: Relative mean squared error and percent relative efficiency of the proposed product estimator $t^{p}_{1(1)}$ with respect to different product type estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	1.22013	0.37540	0.14232
t^p_R	1.21288	0.37108	0.14204
t^*	1.21288	0.37108	0.14204
t^p_D	1.36729	0.36387	0.14145
$t^p_{s(1)}$	2.47281	0.49460	0.16049
$t^p_{1(1)}$	0.70667	0.28677	0.12678
Percent relative efficiencies of proposed estimators			
$t^p_{1(1)}$ w.r.t. $\bar{y}_p$	172.66	130.91	112.26
$t^p_{1(1)}$ w.r.t. t^p_R and t^*	171.63	129.40	112.04
$t^p_{1(1)}$ w.r.t. t^p_D	193.48	126.89	111.57
$t^p_{1(1)}$ w.r.t. $t^p_{s(1)}$	349.92	172.47	126.59

Table 6.40 reveals that the proposed estimator $t^p_{1(1)}$ performed better than the conventional product estimator and the estimator proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The relative mean squared error of the proposed estimator was found to be 0.70667, 0.28677 and 0.12678 at samples of size 30, 60 and 120 respectively and was minimum among the other estimators. Further, the percent relative efficiency of the proposed estimator with respect to $\bar{y}_p$, t^p_D and $t^p_{s(1)}$ was found to lie between 112.26 to 172.66, 111.57 to 193.48 and 126.59 to 349.92 respectively whereas with respect to t^p_R and t^*, the percent relative efficiency of the proposed estimator $t^p_{1(1)}$ was found to lie between 112.04 to 171.63.

Table 6.41: Relative mean squared error and percent relative efficiency of the proposed estimator $t_{2(3)}^{p}$ with respect to different product type estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.82175	0.64007	0.54379
t_R^p	0.83452	0.64994	0.54378
t^*	0.83452	0.64994	0.54378
t_D^p	0.85945	0.65454	0.55458
$t_{s(3)}^p$	5.86103	3.69052	2.69521
$t_{2(3)}^p$	0.63704	0.58311	0.48048
Percent relative efficiencies of proposed estimators			
$t_{2(3)}^p$ w.r.t. $\bar{y}_p$	129.00)	109.77	113.18
$t_{2(3)}^p$ w.r.t. t_R^p and t^*	131.00	111.46	113.18
$t_{2(3)}^p$ w.r.t. t_D^p	131.00	111.46	113.18
$t_{2(3)}^p$ w.r.t. $t_{s(3)}^p$	134.91	112.25	115.42

It has been observed from the table 6.41, that the proposed estimator $t_{2(3)}^p$ have smallest relative mean squared error than the conventional estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). Further, the values of the relative mean squared error of the proposed estimator $t_{2(3)}^p$ at samples of sizes 30, 60 and 120 were found to be 0.63704, 0.58311 and 0.48048 respectively. The percent relative efficiency of the proposed estimator with respect to $\bar{y}_p$, t_D^p and $t_{s(3)}^p$ lies between 109.77 to 129.00, 111.46 to 131.00 and 112.25 to 134.91 respectively. The percent relative efficiency of the proposed estimator $t_{2(3)}^p$ was lying between 111.46 to 131.00 with respect to t_R^p and t^*.

Table 6.42: Relative mean squared error and percent relative efficiency of the proposed estimator $t_{3(2)}^{p}$ with respect to different product type estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.34588	0.64007	0.54379
t_R^p	0.35301	0.64994	0.54377
t^*	0.35301	0.64994	0.54378
t_D^p	0.36531	0.65453	0.55458
$t_{s(2)}^p$	0.65631	2.01539	1.52369
$t_{3(2)}^p$	0.31084	0.61124	0.52294
Percent relative efficiencies of proposed estimators			
$t_{3(2)}^p$ w.r.t. $\bar{y}_p$	111.28	104.72	103.99
$t_{3(2)}^p$ w.r.t. t_R^p and t^*	113.57	106.33	103.99
$t_{3(2)}^p$ w.r.t. t_D^p	117.52	107.08	106.05
$t_{3(2)}^p$ w.r.t. $t_{s(2)}^p$	211.14	329.72	291.37

Table 6.42 showed that the proposed estimator $t_{3(2)}^p$ have smallest relative mean squared error than the conventional product estimator and estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The values of the relative mean squared error of the proposed estimator $t_{3(2)}^p$ were 0.31084, 0.61124 and 0.52294. The percent relative efficiency of the proposed estimator lies between 103.99 to 111.28, 106.05 to 117.52 and 211.14 to 329.72 with respect to $\bar{y}_p$, t_D^p and $t_{s(2)}^p$ respectively. Further, with respect to t_R^p and t^*, the percent relative efficiency of the proposed estimator was lying between 103.99 to 113.57.

Table 6.43: Relative mean squared error and percent relative efficiency of the proposed estimator $t^{p}_{4(-2,1)}$ with respect to different product type estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	1.22013	0.11643	0.14306
t^p_R	1.21288	0.11685	0.14302
t^*	1.21288	0.11685	0.14302
t^p_D	1.36729	0.11732	0.14441
$t^p_{s(1)}$	2.47281	0.12723	0.16161
$t^p_{4(-2,1)}$	1.00019	0.11271	0.13994
Percent relative efficiencies of proposed estimators			
$t^p_{4(-2,1)}$ w.r.t. $\bar{y}_p$	121.99	103.30	102.23
$t^p_{4(-2,1)}$ w.r.t. t^p_R and t^*	121.26	103.67	102.20
$t^p_{4(-2,1)}$ w.r.t. t^p_D	136.70	104.09	103.19
$t^p_{4(-2,1)}$ w.r.t. $t^p_{s(1)}$	247.23	112.88	115.48

From the table 6.43, it has been observed that the proposed estimator $t^p_{4(-2,1)}$ performed better than other product type estimators *viz.*,$\bar{y}_p$, t^p_R, t^*, t^p_D and $t^p_{s(1)}$. Further, the values of the relative mean squared error of the proposed estimator $t^p_{4(-2,1)}$ at samples of sizes 30, 60 and 120 were found to be 1.00019, 0.11271 and 0.13994 respectively. The percent relative efficiency of the proposed estimator lies between 102.23 to 121.99, 103.19 to 136.70 and 112.88 to 247.23 with respect to $\bar{y}_p$, t^p_D and $t^p_{s(1)}$ respectively. Further, with respect to t^p_R and t^*, the percent relative efficiency of the proposed estimator was lying between 102.20 to 121.26.

Table 6.44: Relative mean squared error and percent relative efficiency of the proposed estimator $t^{p}_{5(3,1)}$ with respect to different product type estimators for P_4

Estimator	n		
	30	60	120
$\bar{y}_p$	0.45693	0.64007	0.54379
t^p_R	0.45035	0.64994	0.54378
t^*	0.45035	0.64994	0.54378
t^p_D	0.47323	0.65454	0.55458
$t^p_{s(1)}$	0.63068	0.99857	0.80655
$t^p_{5(3,1)}$	0.25501	0.46510	0.41297
Percent relative efficiencies of proposed estimators			
$t^p_{5(3,1)}$ w.r.t. $\bar{y}_p$	179.18	137.62	131.68
$t^p_{5(3,1)}$ w.r.t. t^p_R and t^*	76.60	139.74	131.67
$t^p_{5(3,1)}$ w.r.t. t^p_D	185.57	140.73	134.29
$t^p_{5(3,1)}$ w.r.t. $t^p_{s(1)}$	247.31	214.70	195.30

Table 6.44 showed that the proposed product estimator $t^p_{5(3,1)}$ was more efficient than the conventional product estimator and the estimators proposed by Robson (1957), Dubey (1993), Singh (1989) and Sharma *et al.* (2007). Further, the proposed estimator was found to be consistent. The percent relative efficiency of the proposed estimator with respect to conventional product estimators was found to lie between 131.68 to 179.18. The values of the relative mean squared error of the proposed estimator were 0.25501, 0.46510 and 0.41297. Further, the percent relative efficiency of the proposed estimator $t^p_{5(3,1)}$ was found to lie between 131.67 to 176.60 with respect to t^p_R and t^*. Further, the percent relative efficiency of the proposed estimator was lying between 134.29 to 185.57 and 195.30 to 247.31 with respect to t^p_D and $t^p_{s(1)}$ respectively.

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Discussion

CHAPTER 7

DISCUSSION

In the present study, improved classes of ratio and product type estimators have been proposed by considering the linear combination of conventional ratio and product estimator with $\frac{s_y^2}{\bar{y}^2}$, $\frac{s_x^2}{\bar{x}^2}$ or $\frac{s_{xy}}{\bar{x}\bar{y}}$.

In the chapter IV and V, five general classes of ratio type estimators t_1, t_2, t_3, t_4 and t_5 and five product type estimators *viz.*, t_1^p, t_2^p, t_3^p, t_4^p and t_5^p have been proposed and their relative bias and relative mean squared error have been worked out upto order O (n^{-1}) and O (n^{-2}) respectively.

Different values of scalars *viz.*, p and q have been used for developing ratio and product type estimators. The efficiencies of these proposed ratio and product have been worked under different conditions and empirically analyzed through simulation data with respect to population datasets P_1 and P_2 for ratio estimators and P_3 and P_4 for product estimators.

The discussions of the results have been done under four headings as:

7.1 Relative bias and relative mean squared error of the ratio type estimators for population I (P_1).

7.2 Relative bias and relative mean squared error of the ratio type estimators for population II (P_2).

7.3 Relative bias and relative mean squared error of the product type estimators for population III (P_3).

7.4 Relative bias and relative mean squared error of the product type estimators for population IV (P_4).

7.1. Relative bias and relative mean squared error of the ratio type estimator for population I.

In case of population I, the proposed member of general class of estimators t_1 was found to be unbiased at p=-1 and q=-1. Further, the smallest value of relative mean squared errors of the proposed estimator t_1 under different values of p and q was observed to be at p=-3 and q=1. The proposed estimator $t_{1(-3,1)}$ was negatively biased. The relative mean squared errors of the proposed estimator $t_{1(-3,1)}$ were 0.17906, 0.02788 and 0.02024 at sample sizes 30, 60 and 120 whereas the relative variance of proposed unbiased ratio estimator $t_{1(-1,-1)}$ were 0.22736, 0.02892 and 0.02079 at sample sizes 30, 60 and 120 respectively. The range of percent relative efficiency of the proposed biased estimator $t_{1(-3,1)}$ with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010) was 105.44 to 288.20 whereas for proposed unbiased ratio estimator, it was 102.64 to 126.01. Similar findings have been reported by Sharma *et al.* (2010).

Second proposed class of ratio type estimators t_2 in case of population I, it was observed to be unbiased at p=-1 and q=-1. The relative variances of the proposed unbiased ratio estimator were 0.24026, 0.02958 and 0.02110 at sample sizes 30, 60 and 120 respectively. The class of estimator was found to be more efficient at p=-3 and q=1 than at other values of p and q. Further, the smallest relative mean squared error of the proposed estimator $t_{2\,(-3,\,1)}$ was observed to be 0.22523 at sample size 30 and the estimator was consistent in nature as well. The percent relative efficiency of the proposed biased ratio estimator $t_{2(-3,1)}$ was lying in the range 100.18 to 229.13 whereas for unbiased ratio estimator $t_{2(-1,-1)}$ it was lying in the range 100.83 to 119.24 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010). Similar results have been proposed by Sharma and Singh (2014).

Under different values of p and q of ratio type estimator, the proposed member of class t_3 was more efficient at p=-3 and q=1. The values of relative mean squared error of the proposed estimator $t_{3(-3,1)}$ at different sample sizes 30, 60 and 120 were 0.19932, 0.02656 and 0.01974 respectively. The proposed estimator was unbiased at p=-1 and q=-

1 and the relative variances were 0.22665, 0.02857 and 0.02054 respectively. Further, the proposed estimator $t_{3(-1,-1)}$ was unbiased and more efficient than $\bar{y}_r$ and t_s. The percent relative efficiency of the proposed biased ratio estimator $t_{3(-3,1)}$ was lying in the range 108.08 to 258.91 whereas for unbiased ratio estimator $t_{3(-1,-1)}$ it was lying in the range 100.83 to 119.24 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

Regarding the proposed class of ratio type estimators t_4, it was observed to be unbiased at p=-1 and q=-1. The class of estimators was found to be more efficient at p=-3 and q=1 than at other values of p and q and was negatively biased. Further, the smallest relative mean squared error of the proposed estimator $t_{4\,(-3,1)}$ at sample sizes 30, 60, and 120 were 0.19497, 0.02621 and 0.01989 respectively and thus the estimator was consistent in nature as well. Further, the proposed estimator was more efficient than conventional estimator and estimator proposed by Sharma *et al.* (2010) at p=-3 and q=1. The relative variances of the proposed unbiased ratio estimator $t_{4\,(-1-1)}$ at sample sizes 30, 60 and 120 were 0.24003, 0.02982 and 0.02129 respectively. Further, the percent relative efficiency of the proposed biased ratio estimator $t_{4(-3,1)}$ was lying in the range 107.28 to 264.69 whereas for unbiased ratio estimator $t_{4(-1,-1)}$ it was lying in the range 100.03 to 119.36 with respect to conventional estimator and estimator proposed by Solanki *et al.* (2012).

Next proposed general class of estimators t_5 was biased. Further, the minimum value of relative mean squared error of the proposed estimator t_5 under different values of p and q was observed to be at p=-3 and q=1. The relative mean squared errors of the proposed estimator $t_{5\,(-3,1)}$ were 0.21791, 0.02751 and 0.02036 at sample sizes 30, 60 and 120 respectively. The proposed estimator was more efficient than the conventional estimator and the estimator proposed by Sharma *et al.* (2010). The percent relative efficiency of the proposed estimator $t_{5\,(-3,1)}$ was found to lie in the range 104.79 to 236.82 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

7.2. Relative bias and relative mean squared error of the ratio type estimators in case of population II.

Proposed class of estimators t_1, t_2, t_3 and t_4 were unbiased at p=-1 and q=-1 whereas t_5 has been found to be biased.

For the proposed class of ratio type estimators t_1, the smallest value of relative mean squared error of the proposed estimator t_1 under different values of p and q was observed to be at p=-3 and q=1. The relative mean squared errors of the proposed estimator $t_{1(-3.1)}$ were 0.02726, 0.01645 and 0.00613 at sample sizes 30, 60 and 120 respectively. The proposed estimator was more efficient than the conventional estimator and the estimator proposed by Sharma *et al*., (2010). The relative variances of the proposed unbiased estimator $t_{1(-1,-1)}$ were 0.02913, 0.01712 and 0.00619 at sample sizes 30, 60 and 120 respectively. Further, the percent relative efficiency of the proposed estimator $t_{1\,(-3,1)}$ was found to lie in the range 101.78 to 131.27 whereas for unbiased ratio estimator $t_{1(-1,-1),}$ it was lying in the range 100.77 to 107.42 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

Secondly, the proposed class of ratio type estimators t_2 in case of population II, it was found that the proposed member of the class to be more efficient at p=-3 and q=1. Further, the smallest relative mean squared errors of the proposed estimator $t_{2\,(-3,1)}$ at sample sizes 30, 60 and 120 were 0.02902, 0.01706 and 0.00622 respectively and thus the estimator was consistent in nature as well. Further, the relative variance of the proposed unbiased ratio estimator $t_{2\,(-1,-1)}$ at sample sizes 30, 60 and 120 were 0.02969, 0.01732 and 0.00622 respectively. Further, the percent relative efficiency of the proposed estimator $t_{2\,(-3,1)}$ was found to lie in the range 100.22 to 123.31 whereas for unbiased ratio estimator $t_{2(-1,-1),}$ it was lying in the range 100.04 to 105.29 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

Thirdly, the proposed class of estimator t_3 were more efficient at p=-3 and q=1, The values of relative mean squared errors at different sample sizes 30, 60 and 120 were 0.02712, 0.01619 and 0.00609 respectively. Further, the proposed estimator $t_{3\,(-1,-1)}$ was more efficient than the other estimators $\bar{y}_r$ and t_s. Further, the relative variance of the proposed unbiased ratio estimator $t_{3\,(-1,-1)}$ at sample sizes 30, 60 and 120 were 0.02910, 0.01704 and 0.00618 respectively. Further, the percent relative efficiency of the proposed estimator $t_{3\,(-3,1)}$ was found to lie in the range 102.34 to 131.94 whereas for unbiased ratio estimator $t_{3(-1,-1),}$ it was lying in the range 100.89 to 107.55 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

Regarding the proposed class of ratio type estimators t_4, it was found to be more efficient at p=-3 and q=1. Further, the smallest relative mean squared errors of the proposed estimator $t_{4\,(-3,1)}$ at sample sizes 30, 60 and 120 were 0.02640, 0.01595 and 0.00608 respectively and thus the estimator was consistent in nature as well. Further, the proposed estimator was more efficient than the conventional estimator, $\bar{y}_r$ and t_s. At p=-1 and q=-1, the proposed member of the class $t_{4,}$ was unbiased. The relative variance of the proposed unbiased ratio estimator $t_{4\,(-1,-1)}$ at sample sizes 30, 60 and 120 were 0.02968, 0.01734 and 0.00623 respectively. Further, the percent relative efficiency of the proposed estimator $t_{4\,(-3,1)}$ was found to lie in the range 102.55 to 135.50 whereas for unbiased ratio estimator $t_{4(-1,-1),}$ it was lying in the range 100.06 to 105.37 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

The proposed general class of estimators t_5 have the minimum value of relative mean squared error under different values of p and q was observed to be at p= -3 and q=1. The relative mean squared errors of the proposed estimator $t_{5\,(-3,1)}$ were 0.02774, 0.01647 and 0.00613 at sample sizes 30, 60 and 120 respectively. Further, the proposed estimator was more efficient than the conventional estimator and the estimator proposed by Sharma *et al.* (2010). The percent relative efficiency of the proposed estimator $t_{5\,(-3,1)}$ was found to lie in the range 101.70 to 128.99 with respect to conventional estimator and estimator proposed by Sharma *et al.* (2010).

7.3 Relative bias and relative mean squared error of the product type estimators in case of population III.

The relative bias of the proposed estimator $t_{1(1)}^{p}$ was identical to that of the estimator as proposed by Sharma *et al.* (2007) and was positively biased. Further, the proposed estimator $t_{1(1)}^{p}$ was more efficient than the conventional product estimator and the estimator proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The relative mean squared error of the proposed estimator was found to be 1.27593, 1.11355 and 0.01061 at sample sizes 30, 60 and 120 respectively and having relative mean squared error minimum among the other estimators. The highest relative efficiency of the proposed estimator was found to be 300.08 with respect to Sharma *et al.* (2007) estimator for sample at size 60. The range of percent relative efficiency of the proposed estimator $t_{1(1)}^{p}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was 101.06 to 300.08.

The general class of product type estimators t_{2}^{p} was unbiased at q= -1. Regarding the proposed product type estimator $t_{2(3)}^{p}$, it was negatively biased and has smaller bias than the conventional estimator and the estimator proposed by Sharma *et al.* (2007). It has been observed that the proposed estimator $t_{2(3)}^{p}$ have smallest relative mean squared errors than the conventional estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). Further, the values of the relative mean squared error of the proposed estimator $t_{2(3)}^{p}$ at sample sizes 30, 60 and 120 were found to be 1.05132, 1.74519 and 0.34913 respectively. Further, the range of percent relative efficiency of the proposed estimator $t_{2(3)}^{p}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was from 105.25 to 332.65. This is in confirmatory with the findings of Singh (1987).

The proposed class of product type estimators t_{3}^{p} was unbiased at q=-1. Further, the proposed estimator $t_{3(2)}^{p}$ was negatively biased and have smaller bias than the conventional product estimator as well as estimator proposed by Sharma *et al.* (2007).

The proposed estimator $t^{p}_{3(2)}$ though inconsistent have smallest relative mean squared error than the conventional product estimator and estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The values of relative mean squared errors of the proposed estimator $t^{p}_{3(2)}$ were 0.83269, 1.66506 and 0.29436 at sample sizes 30, 60 and 120. Further, the percent relative efficiency of the proposed estimator $t^{p}_{3(2)}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was found to lie in the range of 104.47 to 545.97. Similar findings have been proposed by Kataria and Singh (1989).

The proposed product estimator $t^{p}_{4(-2,1)}$ was negatively biased and have smaller bias than the conventional product estimator and estimator proposed by Sharma *et al.* (2007).Further, it has been observed that the proposed estimator $t^{p}_{4(-2,1)}$ was consistent and performed better than other product type estimators *viz.*,$\bar{y}_p$, t^{p}_{R} ,t^{*}, t^{p}_{D} and t^{p}_{s} . The values of the relative mean squared error of the proposed estimator $t^{p}_{4(-2,1)}$was found to be 0.89040, 0.39890 and 0.25602 at sample sizes 30, 60 and 120 respectively. Further, the percent relative efficiency of the proposed estimator $t^{p}_{4(-2,1)}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was found to lie between 107.45 to 239.85. This supports the findings of Vishwakarma *et al.* (2016).

Further, the proposed product estimator $t^{p}_{5(3,1)}$, it was positively biased and have smaller bias than the estimator proposed by Sharma *et al.* (2007). Further, the proposed product estimator $t^{p}_{5(3,1)}$ was more efficient than the conventional product estimator and the estimators proposed by Robson (1957), Dubey (1993), Singh (1989), Sharma *et al.* (2007). Further, the proposed estimator was found to be consistent. The values of the relative mean squared error of the proposed estimator $t^{p}_{5(3,1)}$was found to be 0.57929, 0.26024 and 0.16521 at sample sizes 30, 60 and 120 respectively. Further, the percent relative efficiency of the proposed estimator $t^{p}_{5(3,1)}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was found to lie in the range 119.41 to 355.70.

7.4. Relative bias and relative mean squared error of the product type estimators in case of population IV.

The proposed estimator $t_{1(1)}^{p}$ was more efficient than the conventional product estimator and the estimator proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The relative man squared error of the proposed estimator was found to be 0.70667, 0.28677 and 0.12678 at sample sizes 30, 60 and 120 respectively and was minimum among the other estimators. Further, the range of percent relative efficiency of the proposed estimator $t_{1(1)}^{p}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was 111.57 to 349.92. Similar findings have been reported by Sharma *et al.* (2007). Similar results have been reported by Vishwakarma *et al.* (2016).

The proposed estimator $t_{2(3)}^{p}$, it has smallest relative mean squared error than the conventional estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). Further, the value of the relative mean squared errors of the proposed estimator $t_{2(3)}^{p}$ at sample sizes 30, 60 and 120 were found to be 0.63704, 0.58311 and 0.48048 respectively. The percent relative efficiency of the proposed estimator $t_{2(3)}^{p}$ was maximum with respect to the estimator proposed by Sharma *et al.* (2007) at sample of size 30. Further, the range of percent relative efficiency of the proposed estimator $t_{2(3)}^{p}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was from 109.77 to 134.91.

The proposed estimator $t_{3(2)}^{p}$ have smallest relative mean squared error than the conventional product estimator and estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007). The relative mean squared error of the proposed estimator $t_{3(2)}^{p}$ at samples sizes 30, 60 and 120 were 0.31084, 0.61124 and 0.52294 respectively. Further, the range of percent relative efficiency of the proposed estimator $t_{3(2)}^{p}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007) was from 103.99 to 329.72. Similar results have been reported by Bhatnagar (1996).

The proposed estimator $t^{p}_{4(-2,1)}$ performed better than other product type estimators *viz.*,$\bar{y}_p$, t^{p}_{R}, $t^{*}t^{p}_{D}$ and t^{p}_{s}. Further, the values of the relative mean squared error of the proposed estimator $t^{p}_{4(-2,1)}$ at sample sizes 30, 60 and 120 were found to be 1.00019, 0.11271 and 0.13994 respectively. The percent relative efficiency of the proposed estimator $t^{p}_{4(-2,1)}$ has been found to maximum with respect to the estimator proposed by Sharma *et al.* (2007) at sample of size 30. The percent relative efficiency of the proposed estimator $t^{p}_{4(-2,1)}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was found to lie in the range 102.20 to 247.23.

The proposed product estimator $t^{p}_{5(3,1)}$ was more efficient than the conventional product estimator and the estimators proposed by Robson (1957), Dubey (1993), Singh (1989), Sharma *et al.* (2007). The values of the relative mean squared error of the proposed estimator $t^{p}_{5(3,1)}$were 0.25501, 0.46510 and 0.41297 at sample sizes 30, 60 and 120 respectively. Further, the range of percent relative efficiency of the proposed estimator $t^{p}_{5(3,1)}$ with respect to conventional estimator, estimators proposed by Robson (1957), Singh (1989), Dubey (1993), Sharma *et al.* (2007) was from 131.67 to 247.31.

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Summary and Conclusions

CHAPTER 8

SUMMARY AND CONCLUSIONS

In this thesis, some improved general class of ratio and product type estimators (may be biased or unbiased) for estimation of population mean have been developed by making use of the auxiliary information using simple random sampling scheme. Further, discussions have been done for both the theoretical as well as empirical results.

The panorama of the work done in chapter IV, V and VI is as:

Some improved general class of ratio type estimators for estimation of population mean *viz.*, t_1, t_2, t_3, t_4 and t_5 have been proposed by utilizing the auxiliary information. The expressions for relative bias and relative mean squared error have been worked out for the proposed ratio estimators have been derived upto order O (n^{-1}) and O (n^{-2}) respectively. The efficiencies of the proposed estimators have been compared with that of the conventional ratio estimator and the estimator proposed by Sharma *et al.* (2010) under different values of p and q. Some of the proposed estimators *i.e.*, t_1, t_2, t_3 and t_4 were unbiased. Further, proposed estimators have been found to be more efficient than the conventional estimator and the estimators proposed by Sharma *et al.* (2010) and were consistent in nature.

In the chapter V, some improved general class of product type estimators *viz.*, t_1^p, t_2^p, t_3^p, t_4^p and t_5^p for estimation of population mean have been proposed by utilizing the auxiliary information. The expressions for relative bias and relative mean squared error for the proposed product estimators have been derived upto order O (n^{-1}) and O (n^{-2}) respectively. The efficiencies of the proposed estimators have been compared with that of the conventional product estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007) under different values of the scalars used in the estimators. Two proposed product type estimator have been found to be unbiased at q=-1. The proposed product estimators were found to more efficient than conventional product estimator and the estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007).

An empirical study has been carried out by generating the different datasets/populations through simulation using SAS and R softwares in order to support the efficiency of the proposed estimators. Using these datasets, the efficiencies of the proposed ratio and product type estimators have been analyzed empirically.

CONCLUSIONS: From the present study it can be concluded that the if the auxiliary is utilized at the estimation stage, an improved ratio and product type estimators can be developed which can help in estimation of population mean with greater precision.

Since, the general classes of improved ratio type estimators have been proposed with p and q as scalars. In case of population I, different values of p and q with optimum value of p i.e., -3, -2 and -1 and q as -1 and 1 respectively have been used to find out the relative bias and relative mean squared error of different estimators. The unbiased ratio type estimators and estimator with minimum relative mean squared error have been proposed i.e., if one is prepared to sacrifice the unbiasedness, the proposed ratio estimator $t_{1(-3,1)} = \bar{y}_r - \frac{3}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}\bar{X}} + \frac{s_{xy}}{\bar{y}\bar{X}}\right]$ can be used more efficiently than $\bar{y}_r$ and t_s . Another unbiased ratio type estimator has also been proposed as $t_{3(-1,-1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{X}^2} - \frac{s_{xy}}{\bar{x}\bar{y}}\right]$.

Further, in order to find out the relative bias and relative mean squared error of different estimators for population II, different values of p and q with optimum value of p i.e., -3, -2 and -1 and q as -1 and 1 respectively have been used. The unbiased ratio type estimators and estimator with minimum relative mean squared error have been proposed i.e., if one is prepared to sacrifice the unbiasedness, the proposed estimator $t_{4(-3,1)} = \bar{y}_r - \frac{3}{n}\bar{y}\left[\frac{s_x^2}{\bar{X}^2} + \frac{s_{xy}}{\bar{x}\bar{y}}\right]$ can be used more efficiently. Another unbiased ratio type estimator has also been proposed as $t_{3(-1,-1)} = \bar{y}_r - \frac{1}{n}\bar{y}\left[\frac{s_x^2}{\bar{x}^2} - \frac{s_{xy}}{\bar{x}\bar{y}}\right]$.

Thus, the proposed ratio biased estimators *viz.*, $t_{1\,(-3,1)}, t_{2\,(-3,1)}, t_{3\,(-3,1)}, t_{4\,(-3,1)}$ and $t_{5\,(-3,1)}$ were found to be more efficient with respect to $\bar{y}_r$, and t_s in case of two different populations P_1 and P_2.

Likewise, the general classes of improved product type estimators have been proposed by taking with q as scalar for estimators t_1^p, t_2^p and t_3^p whereas for estimators t_4^p and t_5^p, two scalars p and q have been considered.

In case of population III and IV, different values of q have been used to find out the relative bias and relative mean squared error of first three estimators i.e., t_1^p, t_2^p and t_3^p. The unbiased product type estimators and estimator with minimum relative mean squared error have been proposed *i.e.*, if one is prepared to sacrifice the unbiasedness, the proposed estimator $t_{2(3)}^p = \boldsymbol{\bar{y}_p + \frac{3}{n}\bar{y}_r \frac{s_{xy}}{\bar{x}\bar{y}}}$ can be used more efficiently. Another unbiased product type estimator has also been proposed as $t_{2(-1)}^p = \bar{y}_p - \frac{1}{n}\bar{y}_r \frac{s_{xy}}{\bar{x}\bar{y}}$**.** Among all the estimators t_4^p and t_5^p have been proposed under different values of p and q. The estimator $t_{5(3,1)}^p$ was found to be more efficient than the conventional product type estimator $t_{4(3,1)}^p$ estimators proposed by Robson (1957), Singh (1989), Dubey (1993) and Sharma *et al.* (2007).

Thus, the proposed product estimators *viz.*, $t_{1(1)}^p$, $t_{2(3)}^p$, $t_{3(2)}^p$, $t_{4(-2,1)}^p$ and $t_{5(3,1)}^p$ were found to be more efficient with respect to $\bar{y}_p$, t_R^p , t^*, t_D^p and t_s^p in case of two different populations P_3 and P_4.

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CERTIFICATE-IV

Certified that all the necessary corrections as suggested by the external examiner and the Advisory Committee have been duly incorporated in the thesis entitled **"Estimation of Population Mean through Improved Ratio and Product Type Estimators using Auxiliary Information"** submitted by **Mr. Banti Kumar,** Regd. No. **J-13-D-203-A.**

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