BAYESIAN ESTIMATION IN WEIGHTED XGAMMA DISTRIBUTION

Thesis

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By

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CERTIFICATE-I

This is to certify that the thesis entitled **"BAYESIAN ESTIMATION IN WEIGHTED XGAMMA DISTRIBUTION"**, submitted in partial fulfilment of the requirements for the degree of **Master of Science** with major in **Statistics** and minor in **Mathematics** of the College of Post-Graduate Studies, G. B. Pant University of Agriculture and Technology, Pantnagar, India, is a record of bonafide research carried out by **Ms. Priya Agrawal, Id No. 49392** under my supervision and no part of the thesis has been submitted for any other degree or diploma.

The assistance and help received during the course of this investigation and source of literature have been duly acknowledged.

Pantnagar
December, 2017

(Vinod Kumar)
Chairman
Advisory Committee

<u>CERTIFICATE-II</u>

We, the undersigned, members of the Advisory Committee of **Ms. Priya Agrawal**, **Id. No. 49392,** a candidate for the degree of **Master of Science** with major in **Statistics** and minor in **Mathematics,** agree that the thesis entitled **"BAYESIAN ESTIMATION IN WEIGHTED XGAMMA DISTRIBUTION"** may be submitted in partial fulfilment of the requirements for the degree.

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Introduction

Bayesian Inference is a tool for statistical analysis that exploits the simple idea that the only satisfactory description of uncertainty can be done by means of probability. It involves the use of the Bayes theorem to update the probability of an event when another event has occurred. It utilizes the information already available to us about the unknown parameter and thus helpful to get better results.

Earlier the notion of Bayesian analysis was being greatly criticized and overlooked as the fundamental purpose of estimation is to know about the unknown population but if we have prior information than there is no need of estimating the said population.

Later Bayesian approach developed extensively and is being used nowadays in most of the analysis to obtain the better results of the estimation with the available prior information. It can be used in any distribution.

In recent times Bayesian analysis is being extensively used in lifetime distributions but its implementation is so tough because it may result in analytically inflexible posterior models which are very challenging from the conventional numerical perspective. In recent times the statistical analysis of life time and failure time data has a great importance. The main purpose of analyzing life time models is to gather information concerning failure. This information is used in order to quantify reliability, improve product reliability and find out whether safety and reliability goals are being achieved. Some very important probability distributions which are used as life testing models are exponential, Weibull, normal, gamma, generalized gamma etc. It was Bhattacharya who first introduced the concept of Bayesian analysis in reliability and life testing and in estimating the parameter and reliability of one-parameter exponential distribution under type II censoring.

The exponential and gamma distributions are familiar probability distributions used for modeling lifetime data. The exponential distribution being a special case of the gamma distribution has been used in modeling time-to-event data or modeling waiting times and their various extensions can be obtained in the literature for describing the uncertainty behind real life phenomena arising in the area of survival modeling and reliability engineering. In many real situations it has been observed that their extensions fit better than their corresponding standard ones. In recent times the utility of finite mixture distributions is increasing widely in the statistical literature as they arise in a large variety of fields like from atomic physics to life testing, reliability and microbiology and also because of a little interest in their mathematical properties. The finite mixture distributions generated from the standard distributions model the real-life phenomena in a better way than the standard ones.

The xgamma distribution is a continuous distribution obtained as a special finite mixture of exponential and gamma distributions and hence the name proposed. It provides an adequate fit for the data set. The weighted xgamma distribution is a weighted version of xgamma distribution. It can be shown as a generalization of xgamma distribution by considering a non negative weight function. When the weight function depends on the lengths of units of interest (i.e., w(x)=x), the resulting distribution is called length-biased thus a special case of weighted xgamma distribution is called as length biased xgamma distribution.

In many observational studies it was observed that the sampling frames were not well defined and the recorded observations were biased. They did not have an equal chance of being recorded. Such observations don't follow the original distribution unless an equal chance is given to each observation for recording and thus their modeling gave rise to the theory of weighted distributions. Thus the concept of weighted distributions was laid by Fisher in 1934 to study the effect of methods of ascertainment upon estimation of frequencies. Further C.R. Rao (1965) provided a unified approach and identified the situations that can be modeled using weighted distributions. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Zelen (1974) introduced the weighted distribution in context of cell kinetics and early detection of disease and perceived it as length-biased sampling. Suppose that the original observation X has $f_o(x)$ as the pdf and that the probability of recording the observation x is $0 < w(x) < 1$, then the pdf of X^w the recorded observation is

$$f(x) = \frac{w(x) f_o(x)}{E(w(x))} \tag{1.1}$$

Where w is the normalizing factor obtained to make the total probability equal to unity. Thus w may be referred to as the visibility factor. Rao (1985) introduced such kind of distributions where w(x) is a non-negative function which may exceed unity and gave practical examples where w(X) = X or X^r are appropriate and called such distributions with arbitrary w(x) as weighted distributions. The length biased distribution has enormous applications in biomedical field such as early detection of a disease viz. breast cancer. Rao (1985) made use of length biased distribution in the study of human families and wild-life population. Further it has been used in a cardiology study involving two phases. Thus due to its wider applicability it became necessary to study various structural properties of weighted random variables with respect to original random variables.

The xgamma density for a parameter θ is given by

$$f_o(x) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)\exp(-\theta x) \qquad .....(1.2)$$

The weighted xgamma density for a parameter θ is given by

$$f(x) = \frac{8\theta^2}{(4\theta+1)}\left(1+\frac{\theta}{2}x^2\right)\exp(-2\theta x); \theta > 0, x > 0 \qquad ....(1.3)$$

The weighted xgamma distribution is a special finite mixture distribution of Gamma(3,2θ) and Exp(2θ) in mixing proportion 1/(4θ+1) and 4θ/(4θ+1), respectively.

Reliability Function

It is the probability of survival beyond a particular time period t. It is the complement of cumulative distribution function and obtained by subtracting CDF from 1.

The Bayesian-Weighted xgamma model uses the available prior information on parameter θ to fit a given set of data.

Hazard Rate Function

It is also called instantaneous failure rate and is defined by

$h(t) = \dfrac{f(t)}{R(t)}$ where R(t) is the reliability function at time t.

Objectives

The present study was undertaken with the following objectives:

1. To obtain Bayes estimators of the parameters of the weighted xgamma distribution under different priors.
2. To obtain Bayes estimators of the reliability and hazard rate functions of the weighted xgamma distribution under different priors.
3. To compare the above said estimators by using a suitable loss function.
4. To illustrate the above methodology by means of numerical examples.

Review of Literature

Patil and Rao (1978) examined some general models that lead to weighted distributions, for examples probability sampling in sample surveys, visibility bias dependent on the nature of data collection, additive damage models and two-stage sampling. Several important distributions and their size-biased versions were also obtained and gave a few theorems on the inequalities between the mean values of two weighted distributions. They applied their results to the study of data relating to human populations and wildlife management.

Tierney and Kadane (1986) proposed a more convenient method of approximation than Lindley's method as Lindley's method involve computation of third order derivative of log-likelihood function which is cumbersome to obtain in case of vector valued parameters. Their method provided an approximation of posterior means and variances of a positive function of a real or vector valued parameter and to the marginal posterior densities of the arbitrary parameters which can be used to evaluate approximate predictive densities and they are more accurate than the usual normal approximations.

Pandey et al. (1993) proposed Bayesian estimation for estimating the parameters of the linear hazard- rate model. They worked out the Bayes estimates of the 2-parameters from a type-2 censored sample. The Bayes risk of the regression estimator is compared with the minimum Bayes risk using Monte Carlo simulation study. They validated their results for continuous distribution as well.

Nanda and Jain (1999) derived some partial ordering results concerning the original and the weighted distributions of random variables and random vectors. They also studied length biased and equilibrium distributions and derived their results. They discussed bivariate weighted distributions and obtained some of the results regarding them. They further derived some characterization results of different ageing properties in terms of residual life distributions.

Gove (2003) surveyed some of the possible uses of size- biased distribution theory in forestry and its related fields, since the traditional equal probability method was supposed to be inappropriate in such situations and size-biased theory provided better estimate. He reviewed some more results on size-biased distributions related to parameter estimation in forestry.

Marin et al. (2005) introduced the construction, prior modeling estimation and computation of mixture distribution in Bayesian paradigm and they showed that mixture distributions provide a more flexible expression for statistical analysis. They also depicted some of the properties of mixture distributions along with two different motivations.

Abu-Taleb et al. (2007) studied the Bayes estimates of the parameters of lifetime distributions, assuming that both the survival and censoring time are independent exponentially distributed. They made use of conjugate inverted gamma priors to derive the Bayes estimates, marginal posterior and credible sets.

Shukla and Kumar (2008) obtained the Bayes estimators of the scale parameter of generalized gamma type model under different priors using Lindley's approximation and they further obtained the Bayes estimators of reliability function and hazard rate function.

Kundu and Pradhan (2009) worked on the Bayesian estimation of the unknown parameters of the progressively censored generalized exponential distribution under an assumption that its scale and shape parameter have independent gamma priors. To obtain the closed form of the Bayes estimates of the parameters, they suggested the use of Lindley's approximation method. Further they calculated the approximate Bayes estimates and constructed the highest posterior density credible intervals using the Markov Chain Monte Carlo method. To compare two different sampling schemes and to find the optimal one they proposed optimum censoring scheme. Thus they observed that if we have proper prior information, then the Bayesian inference has a clear advantage over the classical inference.

Shukla and Kumar (2009) worked out the Bayes estimators of the shape parameter of the generalized gamma type model with the help of Lindley's approximation under different priors. They also obtained the Bayes estimators of the hazard rate and reliability functions of the model.

Riabi et al. (2010) derived the β-entropy for Pareto-type and related distributions. They further obtained the β-entropy for some weighted versions of these distributions, such as order statistics, proportional hazards, proportional reversed hazards, probability weighted moments, upper record and lower record.

Pradhan and Kundu (2011) considered the Bayes estimates of two parameter gamma distribution and assumed that scale parameter has a gamma prior and shape parameter has any log-concave prior which are independently distributed. They further used Gibbs sampling technique to generate random samples from posterior density functions and on the basis of those samples they calculated the Bayes estimates of unknown parameters. They further observed that the Gibbs sampling technique can be used quite effectively, for estimating the posterior predictive density and also for constructing predictive interval of the order statistics from the future sample.

Singh et al. (2011) suggested the Bayes estimators of the parameter of the exponentiated gamma distribution and associated reliability function under General Entropy loss function for a censored sample. Through the simulated risk of the estimators they compared the proposed estimator with the corresponding Bayes estimators obtained under squared error loss function and maximum likelihood estimators.

Azimi et al. (2012) obtained the Bayesian estimators of the parameters and reliability function of a Rayleigh distribution given a progressively type II censored sample from a Rayleigh distribution, under asymmetric loss functions such as LINEX loss function, Precautionary loss function, entropy loss function for the parameter and reliability function. Further simulation study was carried out for the comparison purpose. They found that the Bayesian estimators for a given progressively type II censored sample from a Rayleigh distribution are superior to MLEs.

Feroze and Aslam (2012) studied the posterior analysis of the exponentiated gamma distribution for type II censored samples and derived the Bayes estimators and associated risk for the exponentiated gamma distribution under different priors. They obtained the posterior predictive distributions and constructed corresponding intervals. The objective of their study was to find a suitable estimator of the parameter and they found that the under gamma prior using entropy loss function the performance of estimators is the best.

Al-kadeem and Hantoosh (2013) defined and discussed the Even-power Weighted Distribution and its statistical properties. They discussed the Even-power

Weighted Normal Distribution, its shape and derived its mode, reliability function, hazard function, moment generating function thus also found its mean and variance.

Guure and Bosomprah (2013) determined the Bayesian and Non-Bayesian estimation of the parameters and survival function of the generalized exponential distribution. They employed different data sets to estimate the parameters. They computed standard errors of the estimators and made a comparison about the estimators on the basis of the standard error. They also computed the mean squared error and absolute bias and carried out a simulation study for comparison.

Al-Kadim and Hussein (2014) introduced a new class of length-biased of weighted exponential and Rayleigh distributions. They studied some statistical properties and application of these new distributions. They derived the moment generating function, reliability function, hazard function, reverse hazard function and MLE's of the unknown parameters of these distributions and considered some of their sub-models. Finally they found that length biased weighted Rayleigh distribution was quite flexible for modeling the quality of the data protection devices for the Electronic Industries.

Dutta and Borah (2014) investigated size-biased Poisson-Lindley distribution (SBPL) and derived some of the distributional properties of size-biased Poisson-Lindley distribution including moments, cumulants, harmonic mean, coefficient of variation, reliability function etc. Estimation of parameters was carried out by employing method of moments and ratio of the first two relative frequencies. Finally, they fitted the distribution to two reported data sets for empirical comparison.

Jain et al. (2014) introduced the weighted gamma distribution and observed that the hazard function is increasing or upside-down bathtub depending upon the values of the parameters. They further obtained the expressions for its moment generating functions and the moments. They computed MLEs through simulations and also for a real data set. They observed that weighted gamma fit better than its submodels Weighted Exponential (WE), Generalized Exponential (GE), Weibull and Exponential distributions.

Kishan (2014) in his studies made a comparison between maximum likelihood estimators and Bayes estimator of the shape parameter of the generalized gamma type model under squared error loss function, provided shape parameter was known. He also computed the relative efficiency of the Bayes estimator w.r.t. the maximum likelihood estimator.

Khan and Hakkak (2014) on the basis of complete sample analyzed and estimated the parameters of the generalized exponential distribution. The Bayes estimates of the parameters of the models were computed using the Markov Chain Monte Carlo (MCMC). Further a comparative study of Bayes estimators with the classical estimator was done with the computation of the maximum likelihood estimate and associated confidence interval.

Kumar et al. (2014) considered a problem where progressive type-II censoring scheme was applied to a life-testing experiment in which each unit under test was a series system and the investigator seek to obtain the estimates of the reliability of individual components. Their estimation was based on an assumption that the lifetimes of the components followed Rayleigh distribution, and thus provided the maximum likelihood and Bayes estimator of the lifetime parameters, mean lives and reliability measures of individual components using masked system lifetime data. Bayes estimates were computed using Lindley's approximation and Gibbs Sampler and finally they carried out simulation study.

Singh et al. (2014) used binomial removal scheme to obtain the Maximum likelihood and Bayes estimators of the unknown parameters of the exponentiated Pareto model based on progressive type II censored data. The Bayes estimates of the parameters were computed using the Markov Chain Monte Carlo method and by considering the generalized entropy loss function and squared error loss function. Further the Bayes estimators were compared with MLEs using Monte Carlo Simulation.

Alqallaf et al. (2015) used Monte Carlo simulations to compare the finite sample properties of the estimates of the parameters of the weighted exponential distribution obtained by five estimation methods. They made use of bias and mean-squared error as the criterion for comparison. The conclusions drawn by simulation study were also supported with the analysis of two real data sets.

Singh et al. (2015) derived the Bayes estimates of the parameter and reliability function of exponentiated gamma distribution under the assumption of independent gamma prior using three different approximations methods namely Lindley's approximation, Tierney-Kadane and Markov Chain Monte Carlo methods. They further made a comparison of the Bayes estimates with corresponding maximum likelihood estimates through simulation. They illustrated their study in realistic phenomenon using a real data set.

Shukla and Kumar (2015) introduced a finite range failure model and derived the Bayes estimators of the parameter of this finite range failure model under different priors using Tierney Kadane approximation method. They also obtained their expected losses.

Das and Kundu (2016) discussed various reliability properties of the weighted exponential distribution proposed by Gupta and Kundu (2009). They further discussed different properties and inferential issues of its length biased version. They obtained the maximum likelihood estimators of the unknown parameters of the proposed length biased weighted exponential distribution. The applicability of the model was illustrated with the help of a data set and they observed that it provided a better fit in comparison of some of the existing three-parameter models based on K-S statistics.

Bashir and Naqvi (2016) derived a new weighted exponential distribution and observed that the distribution so obtained was positively skewed. They presented various graphs showing the shape of the weighted exponential distribution with different values of parameters and weights. Further they derived moments and their various measures including cdf, moments, median, skewness etc. The estimation of parameter of the weighted exponential distribution was carried out by using maximum likelihood and moment estimation methods. The applicability of model has been illustrated on waiting time data.

Sen et al. in 2016 studied and proposed a new distribution called the xgamma distribution generated as a finite mixture of gamma and exponential distribution. He derived various distributional, structural, mathematical and survival properties of xgamma distribution and it was observed that xgamma distribution is more flexible than the exponential distribution. He proposed the method of moments and method of maximum likelihood to estimate the parameters and indicated a simulation algorithm to generate random samples from the xgamma distribution.

Sen and Chandra in 2017 introduced a two-parameter probability distribution for the purpose of modeling lifetime data as an extension of xgamma distribution and called it as quasi xgamma distribution. Important distributions of its order statistics have been obtained and they studied various structural and distributional properties of quasi xgamma distribution. To estimate the parameters of the proposed distribution they depicted the method of moments and maximum likelihood estimation. The suitability and applicability of the model has been illustrated with the help of a bladder cancer survival data.

Sen et al. 2017 introduced the weighted version of xgamma distribution and studied its properties by considering weight function as rth power of x. And finally studied its distributional and survival properties for the special case when r=1. They obtained MLE's and moment estimators of the parameter of the proposed distribution. They illustrated the theory with the help of random samples of various sizes generated through simulation technique. Further, a real life data set was used to illustrate the results and for comparing with other lifetime and length biased models.

Materials and Methods

This chapter depicts various methods used and the mathematical expressions solved in order to get the required results. Following headings have been used to precisely describe the procedure and complete the objectives:

3.1 Methodology and Synthesis

3.2 Classical Estimation

3.3 Bayes estimation of θ under prior 1

3.4 Bayes estimation of θ under prior 2

3.5 Simulation study

3.6 Real life application

3.1 Methodology and Synthesis

The classical estimates as well as the Bayes estimates of the parameter θ of the weighted xgamma distribution have been evaluated for the purpose of comparison. The methods and techniques used have been well described in this section as under:

3.1.1 Likelihood Function

The likelihood function of a set of parametric values, say θ, given outcomes $x_1, x_2, x_3, \ldots, x_n$ is the joint probability of $x_1, x_2, x_3, \ldots, x_n$ given the parametric values, i.e.

$L(\theta|x_1, x_2, x_3, \ldots, x_n) = P(x_1, x_2, x_3, \ldots, x_n|\theta)$

3.1.2 Newton Raphson Method

Newton Raphson method named after Sir Isaac Newton and Joseph Raphson is used to find successive better approximations to the roots of a real-valued function f(θ). Mathematically the method can be expressed as:

$$\theta_k = \theta_{k-1} - \frac{f(\theta_{k-1})}{f'(\theta_{k-1})}$$

where,

$f(\theta_{k-1})$ is the value of function of θ at $(k-1)^{th}$ iteration.

$f'(\theta_{k-1})$ is the first derivative of the function of θ at $(k-1)^{th}$ iteration.

3.1.3 Maximum Likelihood Estimate

The values of the parameters that maximize the sample likelihood function are known as the Maximum Likelihood Estimates.

3.1.4 Bayes Estimation

It is an approach to statistical inference which utilizes the available prior information about the unknown population parameter. The Bayesian theory is based on incorporating the information from the data as well as the prior information of the parameters to obtain the posterior distribution of the parameters.

3.1.5 Prior Distribution

The prior distribution is an appropriately chosen probability density function that summarizes the available information about the parameters in the form of a model. It expresses an investigator's beliefs about the unknown parameters even before some evidence is taken into account.

3.1.6 Posterior Distribution

The posterior distribution is the distribution of unobserved data conditional on the observed data. It incorporates the information from the data as well as the prior information.

3.1.7 Tierney and Kadane Method

Tierney and Kadane method can be used to find the Bayes estimate of any parametric function when the integral formed is not in closed form or cannot be solved analytically. Bayes estimates of a function g(θ) can be evaluated by:

$$\tilde{g} = \sqrt{\frac{|\Sigma_g^*|}{|\Sigma|}} \exp\left[n\{\delta_g^*(\hat{\theta}) - \delta(\hat{\theta})\}\right]$$

where $|\Sigma|$ and $\left|\Sigma_g^*\right|$ are the negative of inverse of Hessians of $\delta(\theta)$ and $\delta_g^*(\theta)$ evaluated at $\hat{\theta}$. They are calculated as:

$$|\Sigma| = \left[\frac{\partial^2}{\partial\theta^2}\delta(\theta)\right]^{-1}$$

$$|\Sigma_g^*| = \left[\frac{\partial^2}{\partial\theta^2}\delta_g^*(\theta)\right]^{-1}$$

Here $\hat{\theta}$ maximize $\delta(\theta)$ and $\delta_g^*(\theta)$ respectively.

3.1.8 The weighted xgamma distribution and its characteristics

The PDF of the weighted xgamma distribution can be obtained by using equation (1.1)

We have from equation (1.2)

$$f_o(x) = \frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}x^2\right)\exp(-\theta x) \qquad ....(3.1)$$

Let $w(x) = \exp(-\theta x), where\ 0 < w(x) < 1$

Then

$$f(x,\theta) = \frac{w(x) f_o(x)}{E[w(x)]}$$

$$f(x,\theta) = \frac{8\theta^2}{(4\theta+1)}\left(1+\frac{\theta}{2}x^2\right)\exp(-2\theta x) \qquad ....(3.2)$$

Its CDF is

$$F(x) = 1-\exp(-2\theta x)\left[1+2\theta x\frac{(\theta x+1)}{(4\theta+1)}\right] \qquad ....(3.3)$$

Let F(x)=u

$$1-\exp(-2\theta x)\left[1+2\theta x\frac{(\theta x+1)}{(4\theta+1)}\right] = u$$

Reliability Function R(t) is

R(t)=1-F(t)

$$R(t) = \exp(-2\theta t)\left[1+2\theta t\frac{(\theta t+1)}{(4\theta+1)}\right] . \qquad ....(3.4)$$

Hazard rate function H(t) is

$$h(t) = \frac{f(t)}{S(t)}$$

$$h(t)=\frac{8\theta^2\left(1+\frac{\theta}{2}t^2\right)}{\left(2\theta^2t^2+2\theta t+4\theta+1\right)} \quad ....(3.5)$$

Moments and associated measures:

The moment generating function of X is given by $M_X(t)$

$$M_X(t)=\frac{8\theta^2}{(4\theta+1)}\frac{1}{(2\theta-t)}\left[1+\frac{\theta}{(2\theta-t)^2}\right]; \quad |t|<2\theta \quad ....(3.6)$$

The characteristic function (CF) of X is given by

$$\phi_X(t)=E\left(e^{itX}\right)=\frac{8\theta^2}{(4\theta+1)}\frac{1}{(2\theta-it)}\left[1+\frac{\theta}{(2\theta-it)^2}\right];\theta>0,t\in R \quad ....(3.7)$$

Cumulant Generating Function (CGF) of X is given by

$$K_X(t)=\log M_X(t)=\log 8\theta^2-\log(4\theta+1)-3\log(2\theta-t)+\log\left(4\theta^2+t^2-4\theta t+\theta\right);\theta>0,t\in R ..(3.8)$$

The r[th] order raw moment of weighted xgamma distribution in (3.2) is given by:

$$E(X^r)=\int_0^\infty x^r f(x,\theta)dx$$

$$E(X^r)=\int_0^\infty \frac{8\theta^2}{(4\theta+1)}\left(1+\frac{\theta}{2}x^2\right)\exp(-2\theta x)$$

$$E(X^r)=\frac{r!\theta}{(4\theta+1)}\frac{\left[8\theta+(r+1)(r+2)\right]}{(2\theta)^{r+1}} \quad ....(3.9)$$

$$E(X)=\frac{(4\theta+3)}{2\theta(4\theta+1)}$$

$$Mean=\frac{(4\theta+3)}{2\theta(4\theta+1)} \quad ....(3.10)$$

Variance= $E\left(X^2\right)-\left[E(X)\right]^2$

$$E(X^2)=\frac{(2\theta+3)}{\theta^2(4\theta+1)}$$

Therefore,

$$V(X) = \frac{(16\theta^2 + 32\theta + 3)}{4\theta^2 (4\theta + 1)^2} \qquad \text{....(3.11)}$$

Coefficient of skewness (β_1)

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_3 = \frac{64\theta^3 + 24\theta^2 + 36\theta + 3}{4\theta^3 (4\theta + 1)^3} \qquad \text{....(3.12)}$$

$$\beta_1 = \frac{4(64\theta^3 + 24\theta^2 + 36\theta + 3)^2}{(16\theta^2 + 32\theta + 3)^3} \qquad \text{....(3.13)}$$

Coefficient of kurtosis (β_2)

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$\mu_4 = \frac{3((768\theta^4 + 4096\theta^3 + 1632\theta^2 + 288\theta + 15)}{16\theta^4 (4\theta + 1)^4} \qquad \text{....(3.14)}$$

$$\beta_2 = \frac{3(768\theta^4 + 4096\theta^3 + 1632\theta^2 + 288\theta + 15)}{(16\theta^2 + 32\theta + 3)^2} \qquad \text{....(3.15)}$$

3.2 Classical Estimation

3.2.1 Method of moment estimator

Let X_1, X_2, X_3, ...,X_n be a random sample of size n drawn from the weighted xgamma distribution

If $\overline{X}$ denotes the sample mean, then by applying the method of moments, we have

$$\overline{X} = \frac{(4\theta + 3)}{2\theta(4\theta + 1)}$$

Let $\hat{\theta}_M$ be the method of moment estimator of θ then

$$\hat{\theta}_M = \frac{-(\overline{X}-2)+\sqrt{\overline{X}^2+20\overline{X}+4}}{8\overline{X}} \quad \text{....(3.16)}$$

3.2.2 Maximum Likelihood Estimator

Let $\tilde{x}' = (x_1, x_2, ..., x_n)$ be sample observations on X_1, X_2, ..., X_n. The likelihood function of θ given $\tilde{x}$ is written as:

$$l(\theta \mid \tilde{x}) = \prod_{i=1}^{n} \frac{8\theta^2}{(4\theta+1)}\left(1+\frac{\theta}{2}x_i^2\right)\exp(-2\theta x_i)$$

$$= \frac{(8\theta^2)^n}{(4\theta+1)^n}\exp(-2\theta\sum_{i=1}^{n} x_i)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right) \quad \text{....(3.17)}$$

The log-likelihood function is given by

$$L(\theta \mid \tilde{x}) = n\log 8\theta^2 - n\log(4\theta+1) - 2\theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log\left(1+\frac{\theta}{2}x_i^2\right) \quad \text{....(3.18)}$$

Differentiating it with respect to θ and equating to 0, we have the log-likelihood equation as

$$\frac{\partial}{\partial\theta}L(\theta \mid \tilde{x}) = \frac{2n}{\theta} - \frac{4n}{(4\theta+1)} - 2\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} = 0 \quad \text{....(3.19)}$$

Differentiating it twice with respect to θ, we get

$$\frac{\partial^2}{\partial\theta^2}L(\theta \mid \tilde{x}) = \frac{-2n}{\theta^2} + \frac{16n}{(4\theta+1)^2} - \sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} \quad \text{....(3.20)}$$

The equation cannot be solved analytically thus to find the maximum likelihood estimate $(\hat{\theta})$ of θ we have applied Newton-Raphson method.

$$M.S.E.(\hat{\theta}) = E\left(\hat{\theta}-\theta\right)^2 \quad \text{....(3.21)}$$

Then MLE's $(\hat{R}(t), \hat{h}(t))$ of reliability function R(t) and hazard rate function h(t) are obtained by using invariance property.

$$\hat{R}(t) = \exp\left(-2\hat{\theta}t\right)\left[1 + 2\hat{\theta}t\frac{\left(\hat{\theta}t + 1\right)}{\left(4\hat{\theta} + 1\right)}\right] \qquad \text{....(3.22)}$$

$$M.S.E.\left(\hat{R}(t)\right) = E\left(\hat{R}(t) - R(t)\right)^2 \qquad \text{....(3.23)}$$

$$\hat{h}(t) = \frac{8\hat{\theta}^2\left(1 + \frac{\hat{\theta}}{2}t^2\right)}{\left(2\hat{\theta}^2 t^2 + 2\hat{\theta}t + 4\hat{\theta} + 1\right)} \qquad \text{....(3.24)}$$

$$M.S.E.\left(\hat{h}(t)\right) = E\left(\hat{h}(t) - h(t)\right)^2 \qquad \text{....(3.25)}$$

3.3 Bayes Estimation of θ under Prior 1

Let the prior distribution of θ be

$$g_1(\theta) = \frac{1}{\theta}, \theta > 0$$

Posterior distribution of θ under Prior 1 is given by

$$g_1(\theta \mid \tilde{x}) = \frac{l(\theta \mid \tilde{x}).g_1(\theta)}{\int_0^\infty l(\theta \mid \tilde{x}).g_1(\theta)d\theta}$$

$$g_1(\theta \mid \tilde{x}) = \frac{\frac{8^n\theta^{2n}}{(4\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n\left(1 + \frac{\theta}{2}x_i^2\right).\frac{1}{\theta}}{\int_0^\infty \frac{8^n\theta^{2n}}{(4\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n\left(1 + \frac{\theta}{2}x_i^2\right).\frac{1}{\theta}d\theta}$$

$$g_1(\theta \mid \tilde{x}) = \frac{\frac{\theta^{2n-1}}{(4\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n\left(1 + \frac{\theta}{2}x_i^2\right)}{\int_0^\infty \frac{\theta^{2n-1}}{(4\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n\left(1 + \frac{\theta}{2}x_i^2\right)d\theta} \qquad \text{....(3.26)}$$

The integration in the denominator is not in the closed form.

Bayes estimator of θ is obtained by solving

$$\tilde{\theta} = E(\theta) = \int_{\theta} \theta . g_1(\theta \mid \tilde{x}) d\theta$$

$$\tilde{\theta} = \frac{\int_0^{\infty} \frac{\theta^{2n}}{(4\theta+1)^n} \exp(-2\theta \sum_{i=1}^{n} x_i) \prod_{i=1}^{n} \left(1 + \frac{\theta}{2} x_i^2\right) d\theta}{\int_0^{\infty} \frac{\theta^{2n-1}}{(4\theta+1)^n} \exp(-2\theta \sum_{i=1}^{n} x_i) \prod_{i=1}^{n} \left(1 + \frac{\theta}{2} x_i^2\right) d\theta} \qquad ....(3.27)$$

The integral so formed is the ratio of two integrals which cannot be solved directly.

Thus we have applied Tierney and Kadane method to solve it.

The method consists of the following steps:

$$L(\theta \mid \tilde{x}) = \log l(\theta \mid \tilde{x}) = n \log 8\theta^2 - n \log(4\theta + 1) - 2\theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left(1 + \frac{\theta}{2} x_i^2\right)$$

$$u(\theta) = \log g_1(\theta) = -\log\theta \qquad ....(3.28)$$

$$\rho_1(\theta) = \theta$$

$$\log \rho_1(\theta) = \log\theta \qquad ....(3.29)$$

$$\delta(\theta) = \frac{1}{n}\left[L(\theta \mid \tilde{x}) + u(\theta)\right]$$

$$= \log 8\theta^2 - \log(4\theta + 1) - 2\theta \frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \frac{\theta}{2} x_i^2\right) - \frac{1}{n} \log\theta \qquad ....(3.30)$$

$$\delta_{\theta}^{*}(\theta) = \delta(\theta) + \frac{1}{n} \log \rho_1(\theta)$$

$$= \delta(\theta) + \frac{1}{n} \log\theta$$

$$= \log 8\theta^2 - \log(4\theta+1) - 2\theta\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{\theta}{2}x_i^2\right) \quad \text{....(3.31)}$$

$$\delta_{\theta^2}^{*}(\theta) = \delta(\theta) + \frac{2}{n}\log\rho_1(\theta)$$

$$= \delta(\theta) + \frac{2}{n}\log\theta$$

$$= \log 8\theta^2 - \log(4\theta+1) - 2\theta\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{\theta}{2}x_i^2\right) + \frac{1}{n}\log\theta \quad \text{....(3.32)}$$

Now,

$$\frac{\partial}{\partial\theta}\delta(\theta) = \frac{2}{\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} - \frac{1}{n\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\delta(\theta) = -\frac{2}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} + \frac{1}{n\theta^2} \quad \text{...(3.33)}$$

$$\frac{\partial}{\partial\theta}\delta_{\theta}^{*}(\theta) = \frac{2}{\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_{\theta}^{*}(\theta) = -\frac{2}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} \quad \text{....(3.34)}$$

$$\frac{\partial}{\partial\theta}\delta_{\theta^2}^{*}(\theta) = \frac{2}{\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} + \frac{1}{n\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_{\theta^2}^{*}(\theta) = -\frac{2}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} - \frac{1}{n\theta^2} \quad \text{....(3.35)}$$

Then we find,

$$\delta(\hat{\theta}), \delta_{\theta}^{*}(\hat{\theta}), \delta_{\theta^2}^{*}\left(\hat{\theta}\right)$$

and $\sum, \sum_{\theta}^{*}, \sum_{\theta^2}^{*}$ as:

$$\Sigma = \left| \frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)} \right|_{\theta=\hat{\theta}} \qquad \Sigma_{\theta}^{*} = \left| \frac{1}{\frac{\partial^2}{\partial\theta^2}\delta_{\theta}^{*}(\theta)} \right|_{\theta=\hat{\theta}} \qquad \Sigma_{\theta^2}^{*} = \left| \frac{1}{\frac{\partial^2}{\partial\theta^2}\delta_{\theta^2}^{*}(\theta)} \right|_{\theta=\hat{\theta}}$$

Substituting the above said values in the following formula, we get the Bayes estimator of θ as

$$\tilde{\theta} = E(\theta) = \sqrt{\frac{\left|\Sigma_{\theta}^{*}\right|}{|\Sigma|}} \exp\left[n\left\{\delta_{\theta}^{*}\left(\hat{\theta}\right) - \delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \hat{\theta}\sqrt{\frac{\left|\Sigma_{\theta}^{*}\right|}{|\Sigma|}} \qquad \text{....(3.36)}$$

Mean square error of $\tilde{\theta}$ is given by

$$MSE\left(\tilde{\theta}\right) = E\left(\tilde{\theta} - \theta\right)^2 \qquad \text{....(3.37)}$$

Now Bayes estimator of θ^2 is obtained by using

$$E(\theta^2) = \sqrt{\frac{\left|\Sigma_{\theta^2}^{*}\right|}{|\Sigma|}} \exp\left[n\left\{\delta_{\theta^2}^{*}\left(\hat{\theta}\right) - \delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \hat{\theta}^2\sqrt{\frac{\left|\Sigma_{\theta^2}^{*}\right|}{|\Sigma|}} \qquad \text{....(3.38)}$$

Risk of $\tilde{\theta}$ is given by

$$Risk\left(\tilde{\theta}\right) = \tilde{\theta}^2 - (\tilde{\theta})^2 \qquad \text{....(3.39)}$$

Bayes estimator of reliability function is obtained as

$$\tilde{R}(t)=E(R(t))=\frac{\int_0^\infty \exp(-2\theta t)\left(1+2\theta t\left(\frac{\theta t+1}{4\theta+1}\right)\right)\frac{\theta^{2n-1}}{(4\theta+1)^n}\exp\left(-2\theta\sum_{i=1}^n x_i\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)d\theta}{\int_0^\infty \frac{\theta^{2n-1}}{(4\theta+1)^n}\exp\left(-2\theta\sum_{i=1}^n x_i\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)d\theta} \quad ....(3.40)$$

The integral so formed is the ratio of two integrals which cannot be solved directly. Thus we have applied Tierney and Kadane method to solve it.

Now,

$$\rho_2(\theta)=\exp(-2\theta t)\left[1+2\theta t\,\frac{(\theta t+1)}{(4\theta+1)}\right]$$

$$\log\rho_2(\theta)=-2\theta t+\log\left[1+2\theta t\,\frac{(\theta t+1)}{(4\theta+1)}\right] \quad ....(3.41)$$

$$\delta^*_{R(t)}(\theta)=\log 8\theta^2-\log(4\theta+1)-\frac{2\theta\sum_{i=1}^n x_i}{n}+\frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right)-\frac{\log\theta}{n}-\frac{2\theta t}{n}$$
$$+\frac{1}{n}\log\left[1+2\theta t\,\frac{(\theta t+1)}{(4\theta+1)}\right] \quad ....(3.42)$$

$$\delta^*_{R^2(t)}(\theta)=\log 8\theta^2-\log(4\theta+1)-\frac{2\theta\sum_{i=1}^n x_i}{n}+\frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right)-\frac{\log\theta}{n}-\frac{4\theta t}{n}$$
$$+\frac{2}{n}\log\left[1+2\theta t\,\frac{(\theta t+1)}{(4\theta+1)}\right] \quad ....(3.43)$$

Now,

$$\frac{\partial}{\partial\theta}\delta^*_{R(t)}(\theta)=\frac{2}{\theta}-\frac{4(n+1)}{n(4\theta+1)}-\frac{2\sum_{i=1}^n x_i}{n}+\frac{1}{n}\sum_{i=1}^n\frac{x_i^2}{(2+\theta x_i^2)}-\frac{1}{n\theta}-\frac{2t}{n}+\frac{2(2\theta t^2+t+2)}{n(2\theta^2t^2+2\theta t+4\theta+1)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{R(t)}(\theta)=-\frac{2}{\theta^2}+\frac{16(n+1)}{n(4\theta+1)^2}-\frac{1}{n}\sum_{i=1}^n\frac{x_i^4}{(2+\theta x_i^2)^2}+\frac{1}{n\theta^2}-\frac{8(\theta^2t^4+\theta t^3+2\theta t^2+2t+2)}{n(2\theta^2t^2+2\theta t+4\theta+1)^2} \quad ...(3.44)$$

$$\frac{\partial}{\partial\theta}\delta^*_{R^2(t)}(\theta)=\frac{2}{\theta}-\frac{4(n+2)}{n(4\theta+1)}-\frac{2\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)}-\frac{1}{n\theta}-\frac{4t}{n}+\frac{4(2\theta t^2+t+2)}{n(2\theta^2t^2+2\theta t+4\theta+1)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{R^2(t)}(\theta)=-\frac{2}{\theta^2}+\frac{16(n+2)}{n(4\theta+1)^2}-\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2}+\frac{1}{n\theta^2}-\frac{16(\theta^2t^4+\theta t^3+2\theta t^2+2t+2)}{n(2\theta^2t^2+2\theta t+4\theta+1)^2} \quad ...(3.45)$$

Calculate,

$$\delta(\hat{\theta}),\delta^*_{R(t)}(\hat{\theta}),\delta^*_{R^2(t)}(\hat{\theta})$$

And then we obtain $\sum,\sum^*_{R(t)},\sum^*_{R^2(t)}$ as:

$$\Sigma=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)}\right|_{\theta=\hat{\theta}} \qquad \Sigma^*_{R(t)}=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{R(t)}(\theta)}\right|_{\theta=\hat{\theta}} \qquad \Sigma^*_{R(t)^2}=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{R^2(t)}(\theta)}\right|_{\theta=\hat{\theta}}$$

Substituting the above said values in the following formula, we get the Bayes estimator of R(t) as

$$\tilde{R}(t)=E(R(t))=\sqrt{\frac{\left|\Sigma^*_{R(t)}\right|}{|\Sigma|}}\exp\left[n\left\{\delta^*_{R(t)}(\hat{\theta})-\delta(\hat{\theta})\right\}\right]$$

$$=\sqrt{\frac{\left|\Sigma^*_{R(t)}\right|}{|\Sigma|}}\left[1+2\hat{\theta}t\,\frac{(\hat{\theta}t+1)}{(4\hat{\theta}+1)}\right]\exp(-2\hat{\theta}t) \quad ...(3.46)$$

Mean square error of $\tilde{R}(t)$ is given by

$$MSE(\tilde{R}(t))=E(\tilde{R}(t)-R(t))^2 \quad ...(3.47)$$

Now Bayes estimator of $R^2(t)$ is obtained by using

$$\tilde{R}^2(t)=E(R^2(t))=\sqrt{\frac{\left|\Sigma^*_{R^2(t)}\right|}{|\Sigma|}}\exp\left[n\left\{\delta^*_{R^2(t)}(\hat{\theta})-\delta(\hat{\theta})\right\}\right]$$

$$= \sqrt{\frac{\left|\Sigma^{*}_{R^2(t)}\right|}{|\Sigma|}} \left[1 + 2\hat{\theta}t \frac{\left(\hat{\theta}t + 1\right)}{\left(4\hat{\theta} + 1\right)}\right]^2 \exp\left(-4\hat{\theta}t\right) \qquad ....(3.48)$$

Risk of $\tilde{R}(t)$ is given by

$$Risk\,\tilde{R}(t) = \tilde{R}^2(t) - \left(\tilde{R}(t)\right)^2 \qquad ....(3.49)$$

Bayes estimator of hazard rate function is calculated as below

$$\tilde{h}(t) = E(h(t)) = \frac{\int_0^\infty \frac{8\theta^{2n+1}}{(4\theta+1)^n}\left(1+\frac{\theta}{2}t^2\right)\exp\left(-2\theta\sum_{i=1}^n x_i\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)\frac{1}{\left(2\theta^2t^2+2\theta t+4\theta+1\right)}d\theta}{\int_0^\infty \frac{\theta^{2n-1}}{(4\theta+1)^n}\exp\left(-2\theta\sum_{i=1}^n x_i\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)d\theta} \quad ...(3.50)$$

$\tilde{h}(t)$ is again the ratio of two integrals which cannot be solved directly.

Thus we have applied Tierney and Kadane method to solve it.

Now,

$$\rho_3(\theta) = \frac{8\theta^2\left(1+\frac{\theta}{2}t^2\right)}{\left(2\theta^2t^2 + 2\theta t + 4\theta + 1\right)}$$

$$\log\rho_3(\theta) = \log 8\theta^2 + \log\left(1+\frac{\theta}{2}t^2\right) - \log\left(2\theta^2t^2 + 2\theta t + 4\theta + 1\right) \qquad ....(3.51)$$

$$\delta^*_{h(t)}(\theta) = \frac{(n+1)}{n}\log 8\theta^2 - \log(4\theta+1) - \frac{2\theta\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{\log\theta}{n} + \frac{1}{n}\log\left(1+\frac{\theta}{2}t^2\right)$$
$$-\frac{1}{n}\log\left(2\theta^2t^2 + 2\theta t + 4\theta + 1\right) \qquad ....(3.52)$$

$$\delta^*_{h^2(t)}(\theta) = \frac{(n+2)}{n}\log 8\theta^2 - \log(4\theta+1) - \frac{2\theta\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{\log\theta}{n} + \frac{2}{n}\log\left(1+\frac{\theta}{2}t^2\right)$$
$$-\frac{2}{n}\log\left(2\theta^2t^2 + 2\theta t + 4\theta + 1\right) \qquad ....(3.53)$$

Now,

$$\frac{\partial}{\partial\theta}\delta^*_{h(t)}(\theta)=\frac{2(n+1)}{n\theta}-\frac{4}{(4\theta+1)}-\frac{2\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)}-\frac{1}{n\theta}+\frac{t^2}{n\left(2+\theta t^2\right)}$$
$$-\frac{2\left(2\theta t^2+t+2\right)}{n\left(2\theta^2t^2+2\theta t+4\theta+1\right)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h(t)}(\theta)=-\frac{2(n+1)}{n\theta^2}+\frac{16}{(4\theta+1)^2}-\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2}+\frac{1}{n\theta^2}-\frac{t^4}{n\left(2+\theta t^2\right)^2}$$
$$+\frac{8\left(\theta^2t^4+\theta t^3+2\theta t^2+2t+2\right)}{n\left(2\theta^2t^2+2\theta t+4\theta+1\right)^2} \qquad ....(3.54)$$

$$\frac{\partial}{\partial\theta}\delta^*_{h^2(t)}(\theta)=\frac{2(n+2)}{n\theta}-\frac{4}{(4\theta+1)}-\frac{2\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)}-\frac{1}{n\theta}+\frac{2t^2}{n\left(2+\theta t^2\right)}$$
$$-\frac{4\left(2\theta t^2+t+2\right)}{n\left(2\theta^2t^2+2\theta t+4\theta+1\right)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h^2(t)}(\theta)=-\frac{2(n+2)}{n\theta^2}+\frac{16}{(4\theta+1)^2}-\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2}+\frac{1}{n\theta^2}-\frac{2t^4}{n\left(2+\theta t^2\right)^2}$$
$$+\frac{16\left(\theta^2t^4+\theta t^3+2\theta t^2+2t+2\right)}{n\left(2\theta^2t^2+2\theta t+4\theta+1\right)^2} \qquad ....(3.55)$$

Then we find,

$$\delta(\hat{\theta}),\delta^*_{h(t)}(\hat{\theta}),\delta^*_{h^2(t)}\left(\hat{\theta}\right)$$

and $\sum,\sum^*_{h(t)},\sum^*_{h^2(t)}$ as:

$$\Sigma=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)}\right|_{\theta=\hat{\theta}} \qquad \Sigma^*_{h(t)}=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{h(t)}(\theta)}\right|_{\theta=\hat{\theta}} \qquad \Sigma^*_{h(t)^2}=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{h^2(t)}(\theta)}\right|_{\theta=\hat{\theta}}$$

Substituting the above said values in the following formula, we get the Bayes estimator of h(t) as

$$\tilde{h}(t)=E(h(t))=\sqrt{\frac{\left|\Sigma^*_{h(t)}\right|}{|\Sigma|}}\exp\left[n\left\{\delta^*_{h(t)}\left(\hat{\theta}\right)-\delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \sqrt{\frac{\left|\Sigma^*_{h(t)}\right|}{|\Sigma|}} \frac{8\hat{\theta}^2\left(1+\frac{\hat{\theta}}{2}t^2\right)}{\left(2\hat{\theta}^2t^2+2\hat{\theta}t+4\hat{\theta}+1\right)} \quad ....(3.56)$$

Mean square error of $\tilde{h}(t)$ is given by

$$MSE\left(\tilde{h}(t)\right) = E\left(\tilde{h}(t)-h(t)\right)^2 \quad ....(3.57)$$

Now Bayes estimator of $h^2(t)$ is obtained by using

$$\tilde{h}^2(t) = E\left(h^2(t)\right) = \sqrt{\frac{\left|\Sigma^*_{h^2(t)}\right|}{|\Sigma|}} \exp\left[n\left\{\delta^*_{h^2(t)}\left(\hat{\theta}\right)-\delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \sqrt{\frac{\left|\Sigma^*_{h^2(t)}\right|}{|\Sigma|}} \frac{64\hat{\theta}^4\left(1+\frac{\hat{\theta}}{2}t^2\right)^2}{\left(2\hat{\theta}^2t^2+2\hat{\theta}t+4\hat{\theta}+1\right)^2} \quad ....(3.58)$$

Risk of $\tilde{h}(t)$ is given by

$$Risk\,\tilde{h}(t) = \tilde{h}^2(t)-\left(\tilde{h}(t)\right)^2 \quad ....(3.59)$$

3.4 Bayes Estimation of θ under Prior 2

The prior PDF of θ is

$$g_2(\theta) = \frac{1}{\Gamma b}\theta^{b-1}\exp(-\theta)$$

Then posterior distribution of θ is given by

$$g_2(\theta|\tilde{x}) = \frac{l(\theta|\tilde{x}).g_2(\theta)}{\int_0^\infty l(\theta|\tilde{x}).g_2(\theta)d\theta}$$

$$= \frac{\frac{\theta^{2n+b-1}}{(4\theta+1)^n}\exp\left(-2\theta\sum_{i-1}^{n}x_i\right)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right).\exp(-\theta)}{\int_0^\infty \frac{\theta^{2n+b-1}}{(4\theta+1)^n}\exp\left(-2\theta\sum_{i-1}^{n}x_i\right)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right).\exp(-\theta)d\theta}$$

$$= \frac{\frac{\theta^{2n+b-1}}{(4\theta+1)^n} \exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right)}{\int_0^\infty \frac{\theta^{2n+b-1}}{(4\theta+1)^n} \exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right)d\theta} \qquad ...(3.60)$$

To solve (3.60) we have used Tierney Kadane method of approximation again.

$$\tilde{\theta} = E(\theta) = \frac{\int_0^\infty \frac{\theta^{2n+b}}{(4\theta+1)^n} \exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right)d\theta}{\int_0^\infty \frac{\theta^{2n+b-1}}{(4\theta+1)^n} \exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}\left(1+\frac{\theta}{2}x_i^2\right)d\theta} \qquad ...(3.61)$$

$$L(\theta \mid \tilde{x}) = \log l(\theta \mid \tilde{x}) = n\log 8\theta^2 - n\log(4\theta+1) - 2\theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log\left(1+\frac{\theta}{2}x_i^2\right)$$

$$u(\theta) = \log\left(\frac{1}{\Gamma b}\theta^{b-1}\exp(-\theta)\right)$$

$$= -\theta - \log(\Gamma b) + (b-1)\log\theta \qquad ...(3.62)$$

$$\rho_1(\theta) = \theta$$

$$\log\rho_1(\theta) = \log\theta \qquad ...(3.63)$$

$$\delta(\theta) = \frac{1}{n}\left[L(\theta \mid \tilde{x}) + u(\theta)\right]$$

$$= \log 8\theta^2 - \log(4\theta+1) - 2\theta\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{\theta}{n} - \frac{1}{n}\log\Gamma b$$

$$+ \frac{(b-1)}{n}\log\theta \qquad ...(3.64)$$

$$\delta_\theta^*(\theta) = \delta(\theta) + \frac{1}{n}\log\theta$$

$$= \log 8\theta^2 - \log(4\theta+1) - 2\theta \frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n} \log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{\theta}{n} - \frac{1}{n}\log\Gamma b$$

$$+\frac{b}{n}\log\theta \qquad \text{...(3.65)}$$

$$\delta^*_{\theta^2}(\theta) = \delta(\theta) + \frac{2}{n}\log\rho_1(\theta)$$

$$= \delta(\theta) + \frac{2}{n}\log\theta$$

$$= \log 8\theta^2 - \log(4\theta+1) - 2\theta \frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n} \log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{\theta}{n} - \frac{1}{n}\log\Gamma b$$

$$+\frac{(b+1)}{n}\log\theta \qquad \text{...(3.66)}$$

$$\frac{\partial}{\partial\theta}\delta(\theta) = \frac{2}{\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} + \frac{(b-1)}{n\theta} - \frac{1}{n}$$

$$\frac{\partial^2}{\partial\theta^2}\delta(\theta) = -\frac{2}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} + \frac{(1-b)}{n\theta^2} \qquad \text{...(3.67)}$$

$$\frac{\partial}{\partial\theta}\delta^*_{\theta}(\theta) = \frac{2}{\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} - \frac{1}{n} + \frac{b}{n\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{\theta}(\theta) = -\frac{2}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} - \frac{b}{n\theta^2} \qquad \text{...(3.68)}$$

$$\frac{\partial}{\partial\theta}\delta^*_{\theta^2}(\theta) = \frac{2}{\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} - \frac{1}{n} + \frac{(b+1)}{n\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{\theta^2}(\theta) = -\frac{2}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} - \frac{(b+1)}{n\theta^2} \qquad \text{....(3.69)}$$

Then we find,

$$\delta(\hat\theta), \delta^*_{R(t)}(\hat\theta), \delta^*_{R^2(t)}(\hat\theta)$$

and $\sum, \sum^*_{R(t)}, \sum^*_{R^2(t)}$ as:

$$\Sigma = \left| \frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)} \right|_{\theta=\hat{\theta}} \qquad \Sigma_\theta^* = \left| \frac{1}{\frac{\partial^2}{\partial\theta^2}\delta_\theta^*(\theta)} \right|_{\theta=\hat{\theta}} \qquad \Sigma_{\theta^2}^* = \left| \frac{1}{\frac{\partial^2}{\partial\theta^2}\delta_{\theta^2}^*(\theta)} \right|_{\theta=\hat{\theta}}$$

Substituting the above said values in the following formula, we get the Bayes estimator of θ as

$$\tilde{\theta} = E(\theta) = \sqrt{\frac{|\Sigma_\theta^*|}{|\Sigma|}} \exp\left[n\left\{\delta_\theta^*\left(\hat{\theta}\right) - \delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \hat{\theta}\sqrt{\frac{|\Sigma_\theta^*|}{|\Sigma|}} \qquad \text{....(3.70)}$$

Mean square error of $\tilde{\theta}$ is given by

$$MSE\left(\tilde{\theta}\right) = E\left(\tilde{\theta} - \theta\right)^2 \qquad \text{....(3.71)}$$

Now Bayes estimator of θ^2 is obtained by using

$$E(\theta^2) = \sqrt{\frac{|\Sigma_{\theta^2}^*|}{|\Sigma|}} \exp\left[n\left\{\delta_{\theta^2}^*\left(\hat{\theta}\right) - \delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \hat{\theta}^2\sqrt{\frac{|\Sigma_{\theta^2}^*|}{|\Sigma|}} \qquad \text{....(3.72)}$$

Risk of $\tilde{\theta}$ is given by

$$Risk\left(\tilde{\theta}\right) = \tilde{\theta}^2 - \left(E\left(\tilde{\theta}\right)\right)^2 \qquad \text{...(3.73)}$$

Bayes estimator of reliability function is calculated as below

$$R(t) = \exp(-2\theta t)\left[1 + 2\theta t\,\frac{(\theta t + 1)}{(4\theta + 1)}\right]$$

$$\tilde{R}(t)=E(R(t))=\frac{\int_0^\infty \exp(-2\theta t)\left(1+2\theta t\left(\frac{\theta t+1}{4\theta+1}\right)\right)\frac{\theta^{2n+b-1}}{(4\theta+1)^n}\exp\left(-\theta\left(2\sum_{i=1}^n x_i+1\right)\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)d\theta}{\int_0^\infty \frac{\theta^{2n+b-1}}{(4\theta+1)^n}\exp\left(-\theta\left(2\sum_{i=1}^n x_i+1\right)\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)d\theta} \quad ....(3.74)$$

To solve (3.74) we have used Tierney Kadane method.

Now,

$$\rho_2(\theta)=\exp(-2\theta t)\left[1+2\theta t\frac{(\theta t+1)}{(4\theta+1)}\right]$$

$$\log\rho_2(\theta)=-2\theta t+\log\left[1+2\theta t\frac{(\theta t+1)}{(4\theta+1)}\right] \quad ....(3.75)$$

$$\delta^*_{R(t)}(\theta)=\log 8\theta^2-\log(4\theta+1)-\frac{2\theta\sum_{i=1}^n x_i}{n}+\frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right)-\frac{\theta}{n}-\frac{\log\Gamma b}{n}$$
$$+\frac{(b-1)}{n}\log\theta-\frac{2\theta t}{n}+\frac{1}{n}\log\left[1+2\theta t\frac{(\theta t+1)}{(4\theta+1)}\right] \quad ....(3.76)$$

$$\delta^*_{R^2(t)}(\theta)=\log 8\theta^2-\log(4\theta+1)-\frac{2\theta\sum_{i=1}^n x_i}{n}+\frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right)-\frac{\theta}{n}-\frac{\log\Gamma b}{n}$$
$$+\frac{(b-1)}{n}\log\theta-\frac{4\theta t}{n}+\frac{2}{n}\log\left[1+2\theta t\frac{(\theta t+1)}{(4\theta+1)}\right] \quad ....(3.77)$$

Now,

$$\frac{\partial}{\partial\theta}\delta^*_{R(t)}(\theta)=\frac{2}{\theta}-\frac{4(n+1)}{n(4\theta+1)}-\frac{2\sum_{i=1}^n x_i}{n}+\frac{1}{n}\sum_{i=1}^n\frac{x_i^2}{(2+\theta x_i^2)}-\frac{1}{n}+\frac{(b-1)}{n\theta}-\frac{2t}{n}$$
$$+\frac{2(2\theta t^2+t+2)}{n(2\theta^2t^2+2\theta t+4\theta+1)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{R(t)}(\theta) = -\frac{2}{\theta^2} + \frac{16(n+1)}{n(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} + \frac{(1-b)}{n\theta^2}$$
$$-\frac{8(\theta^2 t^4 + \theta t^3 + 2\theta t^2 + 2t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)^2} \qquad ...(3.78)$$

$$\frac{\partial}{\partial\theta}\delta^*_{R^2(t)}(\theta) = \frac{2}{\theta} - \frac{4(n+2)}{n(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} - \frac{1}{n} + \frac{(b-1)}{n\theta}$$
$$-\frac{4t}{n} + \frac{4(2\theta t^2 + t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)}$$
$$\frac{\partial^2}{\partial\theta^2}\delta^*_{R(t)}(\theta) = -\frac{2}{\theta^2} + \frac{16(n+2)}{n(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} + \frac{(1-b)}{n\theta^2}$$
$$-\frac{16(\theta^2 t^4 + \theta t^3 + 2\theta t^2 + 2t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)^2} \qquad ....(3.79)$$

Then we find,

$$\delta(\hat{\theta}), \delta^*_{R(t)}(\hat{\theta}), \delta^*_{R^2(t)}(\hat{\theta})$$

and $\Sigma, \Sigma^*_{R(t)}, \Sigma^*_{R^2(t)}$ as:

$$\Sigma = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)}\right|_{\theta=\hat{\theta}} \qquad \Sigma^*_{R(t)} = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{R(t)}(\theta)}\right|_{\theta=\hat{\theta}} \qquad \Sigma^*_{R^2(t)} = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{R^2(t)}(\theta)}\right|_{\theta=\hat{\theta}}$$

Substituting the above said values in the following formula, we get the Bayes estimator of R(t) as

$$\tilde{R}(t) = E(R(t)) = \sqrt{\frac{|\Sigma^*_{R(t)}|}{|\Sigma|}}\exp\left[n\left\{\delta^*_{R(t)}(\hat{\theta}) - \delta(\hat{\theta})\right\}\right]$$

$$= \sqrt{\frac{|\Sigma^*_{R(t)}|}{|\Sigma|}}\left[1 + 2\hat{\theta}t\frac{(\hat{\theta}t+1)}{(4\hat{\theta}+1)}\right]\exp(-2\hat{\theta}t) \qquad ...(3.80)$$

Mean square error of $\tilde{R}(t)$ is given by

$$MSE(\tilde{R}(t)) = E(\tilde{R}(t) - \tilde{R}^2(t))^2 \qquad ....(3.81)$$

Now Bayes estimator of $R^2(t)$ is obtained by using

$$\tilde{R}^2(t) = E\left(R^2(t)\right) = \sqrt{\frac{\left|\Sigma^*_{R^2(t)}\right|}{|\Sigma|}}\exp\left[n\left\{\delta^*_{R^2(t)}\left(\hat{\theta}\right) - \delta\left(\hat{\theta}\right)\right\}\right]$$

$$= \sqrt{\frac{\left|\Sigma^*_{R^2(t)}\right|}{|\Sigma|}}\left[1 + 2\hat{\theta}t\,\frac{\left(\hat{\theta}t+1\right)}{\left(4\hat{\theta}+1\right)}\right]^2 \exp\left(-4\hat{\theta}t\right) \qquad \text{....(3.82)}$$

Risk of $\tilde{R}(t)$ is given by

$$Risk\,\tilde{R}(t) = \tilde{R}^2(t) - \left(\tilde{R}(t)\right)^2 \qquad \text{....(3.83)}$$

Bayes estimator of hazard rate function is calculated as below

$$\tilde{h}(t) = E\left(h(t)\right) = \frac{\int_0^\infty \frac{8\theta^{2n+b+1}}{(4\theta+1)^n}\left(1+\frac{\theta}{2}t^2\right)\exp\left(-\theta\left(2\sum_{i=1}^n x_i + 1\right)\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)\frac{1}{\left(2\theta^2t^2+2\theta t+4\theta+1\right)}d\theta}{\int_0^\infty \frac{\theta^{2n+b-1}}{(4\theta+1)^n}\exp\left(-\theta\left(2\sum_{i=1}^n x_i+1\right)\right)\prod_{i=1}^n\left(1+\frac{\theta}{2}x_i^2\right)d\theta} \qquad \text{....(3.84)}$$

Equation (3.84) has been solved by using Tierney Kadane method

where

$$\rho_3(\theta) = \frac{8\theta^2\left(1+\frac{\theta}{2}t^2\right)}{\left(2\theta^2t^2+2\theta t+4\theta+1\right)}$$

$$\log\rho_3(\theta) = -\log 8\theta^2 + \log\left(1+\frac{\theta}{2}t^2\right) - \log\left(2\theta^2t^2+2\theta t+4\theta+1\right) \qquad \text{....(3.85)}$$

$$\delta^*_{h(t)}(\theta) = \frac{(n+1)}{n}\log 8\theta^2 - \log(4\theta+1) - \frac{2\theta\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{b\theta}{n} - \frac{\log b}{n}$$
$$+\frac{1}{n}\log\left(1+\frac{\theta}{2}t^2\right) - \frac{1}{n}\log\left(2\theta^2t^2+2\theta t+4\theta+1\right) \qquad \text{....(3.86)}$$

$$\delta^*_{h^2(t)}(\theta) = \frac{(n+2)}{n}\log 8\theta^2 - \log(4\theta+1) - \frac{2\theta\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n\log\left(1+\frac{\theta}{2}x_i^2\right) - \frac{b\theta}{n} - \frac{\log b}{n}$$
$$+\frac{2}{n}\log\left(1+\frac{\theta}{2}t^2\right) - \frac{2}{n}\log\left(2\theta^2t^2+2\theta t+4\theta+1\right) \qquad \text{.....(3.87)}$$

Now,

$$\frac{\partial}{\partial\theta}\delta^*_{h(t)}(\theta) = \frac{2(n+1)}{n\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} - \frac{b}{n} + \frac{t^2}{n(2+\theta t^2)} - \frac{2(2\theta t^2 + t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h(t)}(\theta) = -\frac{2(n+1)}{\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} - \frac{t^4}{n(2+\theta t^2)^2} + \frac{8(\theta^2 t^4 + \theta t^3 + 2\theta t^2 + 2t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)^2} \qquad ....(3.88)$$

$$\frac{\partial}{\partial\theta}\delta^*_{h^2(t)}(\theta) = \frac{2(n+2)}{n\theta} - \frac{4}{(4\theta+1)} - \frac{2\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^2}{(2+\theta x_i^2)} - \frac{b}{n} + \frac{2t^2}{n(2+\theta t^2)} - \frac{4(2\theta t^2 + t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h^2(t)}(\theta) = -\frac{2(n+2)}{n\theta^2} + \frac{16}{(4\theta+1)^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i^4}{(2+\theta x_i^2)^2} - \frac{2t^4}{n(2+\theta t^2)^2} + \frac{16(\theta^2 t^4 + \theta t^3 + 2\theta t^2 + 2t + 2)}{n(2\theta^2 t^2 + 2\theta t + 4\theta + 1)^2} \qquad ....(3.89)$$

Then we find,

$$\delta(\hat{\theta}), \delta^*_{h(t)}(\hat{\theta}), \delta^*_{h^2(t)}(\hat{\theta})$$

and $\sum, \sum^*_{h(t)}, \sum^*_{h^2(t)}$ as:

$$\sum = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)}\right|_{\theta=\hat{\theta}} \qquad \sum^*_{h(t)} = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{h(t)}(\theta)}\right|_{\theta=\hat{\theta}} \qquad \sum^*_{h(t)^2} = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{h^2(t)}(\theta)}\right|_{\theta=\hat{\theta}}$$

Substituting the above said values in the following formula, we get the Bayes estimator of h(t) as

$$\tilde{h}(t) = E(h(t)) = \sqrt{\frac{\left|\Sigma^*_{h(t)}\right|}{|\Sigma|}} \exp\left[n\left\{\delta^*_{h(t)}(\hat{\theta}) - \delta(\hat{\theta})\right\}\right]$$

$$= \sqrt{\frac{\left|\Sigma^*_{h(t)}\right|}{|\Sigma|}} \frac{8\hat{\theta}^2\left(1 + \frac{\hat{\theta}}{2}t^2\right)}{\left(2\hat{\theta}^2 t^2 + 2\hat{\theta}t + 4\hat{\theta} + 1\right)} \qquad ...(3.90)$$

Mean square error of $\tilde{h}(t)$ is given by

$$MSE\left(\tilde{h}(t)\right) = E\left(\tilde{h}(t) - \tilde{h}^2(t)\right)^2 \qquad ....(3.91)$$

Now Bayes estimator of $h^2(t)$ is obtained by using

$$\tilde{h}^2(t) = E\left(h^2(t)\right) = \sqrt{\frac{\left|\Sigma^*_{h^2(t)}\right|}{|\Sigma|}} \exp\left[n\left\{\delta^*_{h^2(t)}(\hat{\theta}) - \delta(\hat{\theta})\right\}\right]$$

$$= \sqrt{\frac{\left|\Sigma^*_{h^2(t)}\right|}{|\Sigma|}} \frac{64\hat{\theta}^4\left(1 + \frac{\hat{\theta}}{2}t^2\right)^2}{\left(2\hat{\theta}^2 t^2 + 2\hat{\theta}t + 4\hat{\theta} + 1\right)^2} \qquad ...(3.92)$$

Risk of $\tilde{h}(t)$ is given by

$$Risk\,\tilde{h}(t) = \tilde{h}^2(t) - \left(\tilde{h}(t)\right)^2 \qquad ...(3.93)$$

3.5 Simulation Study

Simulation is a technique applied to a probability model to generate random samples of required sizes. Whenever we estimate parameters for a given model under hypothetical or randomly generated values, the question over its application for different sets is raised. For validating our approximated values for a large population, so large that it can be treated as real, we apply simulation technique.

In present work, the estimated values of parameter, reliability function and all other related measures have been computed using a simulated data replicated 10,000 times using R software.

3.6 Real life application

The following real life data set of size 23 fatigue life for deep-groove ball bearings, compiled by American Standards Association and reported in Lieblein and Zelen (1956) has been analyzed to illustrate the applicability of the proposed model.

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

To see whether the proposed weighted xgamma distribution is a good fit to the above said data, we have classified the values into 8 class intervals as given below:

Class Interval	Observed frequency (O)	F(y)	$\Delta F(y)$	Expected Frequency (E) $N.\Delta F(y)$	$\frac{(O-E)^2}{E}$
0-20	1	0.082764621	0.082764621	1.903586	0.428911
20-40	2	0.254538153	0.171773532	3.950791	0.963247
40-60	8	0.458446915	0.203908762	4.689902	2.336244
60-80	4	0.637605614	0.179158699	4.12065	0.003533
80-100	3	0.771564572	0.133958958	3.081056	0.002132
100-120	2	0.862296048	0.090731476	2.086824	0.003612
120-140	2	0.919795393	0.057499345	1.322485	0.347094
140-160	1	0.974789832	0.054994439	1.844702	0.386795
Total	23			23	4.471567

Table 3.1: chi-square goodness of fit

Here, $\chi^2_{cal} = 4.471567$

which is insignificant as the chi-square value at 7 d.f and at 1% level of significance is 18.47531. Thus the proposed model is a good fit to the above said data set.

Results and Discussion

This chapter documents the Bayes estimates of the parameter θ of the weighted xgamma distribution along with its reliability function R(t), hazard rate function h(t) and other associated measures computed for random samples of various sizes generated through R software. Each random sample is replicated 10,000 times to increase the sample size considerably. The values of Bayes estimates of R(t) and h(t) are obtained at initial time t=1.2 everywhere.

The random samples of sizes 20, 40, 60 and 80 were drawn from the weighted xgamma distribution with values 1.0, 1.5 and 2.0 of θ using R software. Each random sample was replicated 10,000 times. The maximum likelihood estimates of θ, reliability function R(t) and hazard rate function h(t) were evaluated along with their Mean Square Errors for different values of the sample size n. Besides, Bayes estimates of θ, reliability function R(t) and hazard rate function h(t) were evaluated along with their Mean Square Errors and Risks under Squared Error Loss Function(SELF) for two priors mentioned in Chapter 3. The values of Bayes estimates of θ, R(t) and h(t) corresponding to θ=1.0, 1.5 and 2.0 are given in Tables 4.1, 4.2, and 4.3 respectively.

Besides, the values of the maximum likelihood estimates (MLEs) of R(t) and h(t) are calculated for different values of t ranging from 0 to 3.2 and are given in Table 4.4. The figures for reliability function R(t) and hazard rate function h(t) for different values of t are shown in Fig. 4.1 and 4.2 respectively.

Θ=1.0	MLE Estimates			Bayes Estimates					
				Prior 1			Prior 2		
	Θ	R(t)	h(t)	Θ	R(t)	h(t)	Θ	R(t)	H(t)
n=20	1.041736	0.18443	1.412612	1.02338	0.183794	1.379791	1.025139	0.183854	1.382915
MSE	0.04361	0.004493	0.12361	0.040574	0.004476	0.113737	0.040867	0.004477	0.114678
Risk(SELF)				0.000678	8.48E-07	0.00216	0.000556	6.96E-07	0.001776
n=40	1.042759	0.18417	1.414359	1.02438	0.183534	1.381491	1.026141	0.183595	1.38462
MSE	0.044246	0.004501	0.125569	0.041146	0.004485	0.115484	0.041445	0.004486	0.116447
Risk(SELF)				0.000681	8.47E-07	0.002169	0.000558	6.94E-07	0.001782
n=60	1.04254	0.184049	1.413913	1.024163	0.183413	1.381052	1.025924	0.183474	1.384181
MSE	0.04339	0.004463	0.123011	0.040335	0.004446	0.113086	0.04063	0.004448	0.114031
Risk(SELF)				0.00068	8.46E-07	0.002164	0.000557	6.94E-07	0.001779
n=80	1.04561	0.183284	1.419157	1.027168	0.18265	1.386163	1.028936	0.18271	1.389306
MSE	0.045192	0.004506	0.128487	0.041951	0.004491	0.117955	0.042265	0.004492	0.118963
Risk(SELF)				0.000686	8.42E-07	0.002187	0.000562	6.90E-07	0.001797

Table 4.1: Estimated values of θ, R(t) and h(t) along with their MSE and risks

It is revealed from Table 4.1 that Bayes risks under SELF are smaller for prior 2 compared to prior 1 for all sample sizes. The value of Bayes estimate of θ is 1.025139 with minimum risk 0.000556 at n=20 and similarly Bayes estimate of R(t) is 0.18271 with minimum risk 6.90E-07 at n=80 and Bayes estimate of h(t) is 1.382915 with minimum risks 0.001776 at n=20. Finally it is concluded that prior 2 is superior to prior 1 for θ=1.

Θ=1.5	MLE Estimates			Bayes Estimates					
				Prior 1			Prior 2		
	θ	R(t)	h(t)	Θ	R(t)	h(t)	Θ	R(t)	h(t)
n=20	1.571361	0.068902	2.32408	1.54091	0.068581	2.266965	1.544086	0.068614	2.272886
MSE	0.107267	0.001503	0.339341	0.099084	0.001492	0.310183	0.099925	0.001493	0.313147
Risk(SELF)				0.001857	2.38E-07	0.00651	0.001496	1.92E-07	0.005257
n=40	1.570029	0.069676	2.322101	1.539603	0.069354	2.26503	1.542777	0.069387	2.270948
MSE	0.113104	0.001591	0.358195	0.104723	0.001578	0.32823	0.10559	0.001579	0.331294
Risk(SELF)				0.001861	2.42E-07	0.006527	0.001498	1.95E-07	0.00527
n=60	1.563832	0.07003	2.310884	1.533554	0.069706	2.254121	1.53671	0.069739	2.260001
MSE	0.106609	0.001581	0.336925	0.098884	0.001568	0.309286	0.099678	0.001569	0.312095
Risk(SELF)				0.001838	2.43E-07	0.006436	0.00148	1.96E-07	0.005198
n=80	1.571169	0.069401	2.324022	1.540719	0.06908	2.266904	1.543896	0.069113	2.272826
MSE	0.111834	0.001583	0.354084	0.103451	0.00157	0.324149	0.104316	0.001572	0.327205
Risk(SELF)				0.001862	2.41E-07	0.00653	0.001499	1.94E-07	0.005273

Table 4.2: Estimated values of θ, R(t) and h(t) along with their MSE and risks

It is revealed from Table 4.2 that Bayes risks under SELF are smaller for prior 2 compared to prior 1 for all sample sizes. The value of Bayes estimate of θ is 1.53671with minimum risk 0.00148 at n=60 and similarly Bayes estimate of R(t) is 0.068614 with minimum risk 1.92E-07 at n=20 and Bayes estimate of h(t) is 2.260001 with minimum risks 0.005198 at n=60. Finally it is concluded that prior 2 is superior to prior 1 for θ=1.5.

Θ=2.0	MLE Estimates			Bayes Estimates					
				Prior 1			Prior 2		
	Θ	R(t)	h(t)	θ	R(t)	h(t)	Θ	R(t)	h(t)
n=20	2.093607	0.026843	3.262102	2.05071	0.02669	3.179228	2.055412	0.026707	3.188257
MSE	0.210423	0.000446	0.709117	0.194579	0.000441	0.649436	0.19627	0.000442	0.65574
Risk(SELF)				0.003684	6.40E-08	0.013699	0.002934	5.11E-08	0.010942
n=40	2.095749	0.026681	3.265863	2.052802	0.026528	3.182886	2.05751	0.026545	3.191927
MSE	0.205735	0.000446	0.692203	0.189953	0.000441	0.632879	0.191632	0.000441	0.639126
Risk(SELF)				0.003686	6.35E-08	0.013705	0.002936	5.07E-08	0.010947
n=60	2.093173	0.026859	3.261208	2.050287	0.026705	3.178356	2.054987	0.026722	3.187383
MSE	0.205577	0.000454	0.691447	0.190007	0.000449	0.63287	0.191663	0.00045	0.639038
Risk(SELF)				0.003677	6.42E-08	0.013668	0.002929	5.13E-08	0.010918
n=80	2.092435	0.026746	3.259771	2.049567	0.026593	3.176959	2.054265	0.026609	3.18598
MSE	0.204393	0.000439	0.688173	0.18893	0.000434	0.629977	0.190576	0.000435	0.636106
Risk(SELF)				0.003673	6.35E-08	0.013652	0.002926	5.07E-08	0.010905

Table 4.3: Estimated values of θ, R(t) and h(t) along with their MSE and risks

It is revealed from Table 4.3 that Bayes risks under SELF are smaller for prior 2 compared to prior 1 for all sample sizes. The value of Bayes estimate of θ is 2.054265 with minimum risk 0.002926 at n=80 and similarly Bayes estimate of R(t) are 0.026545 and 0.026609 with minimum risk 5.07E-08 at n=40 and n=60. Further, Bayes estimate of h(t) is 3.18598 with minimum risks 0.010905 at n=80. Finally it is concluded that prior 2 is superior to prior 1 for θ=2.0.

Finally, it is concluded from Tables 4.1, 4.2 and 4.3 that Prior 2 is superior to Prior 1 for all values of θ and for all sample sizes for finding Bayes estimates of θ, R(t) and h(t). Further, it is observed that R(t) decreases with increasing value of θ and h(t) increases with increasing value of θ.

Time (t)	$\hat{R}(t)$	$\hat{h}(t)$
0	1	1.68221
0.2	0.72319	1.564755
0.4	0.533486	1.483416
0.6	0.398688	1.433888
0.8	0.300127	1.409433
1	0.226601	1.403222
1.2	0.171073	1.409476
1.4	0.128878	1.423772
1.6	0.096762	1.442926
1.8	0.072351	1.464741
2	0.053854	1.487747
2.2	0.039901	1.510988
2.4	0.029427	1.533859
2.6	0.021605	1.555995
2.8	0.015793	1.577187
3	0.011497	1.597332
3.2	0.008337	1.616392

Table 4.4 MLEs of R(t) and h(t) for different values of t

It is concluded from Table 4.4 that MLE of R(t) decreases with increasing value of time (t). It takes value 1 at initial time 0 and 0.008337 at time 3.2. Moreover the MLE of hazard rate function h(t) takes value 1.68221 at initial time 0 and then decreases up to 1.403222 at time 1 and after this goes on increasing continuously. This is shown graphically in Figures 4.1 and 4.2 respectively.

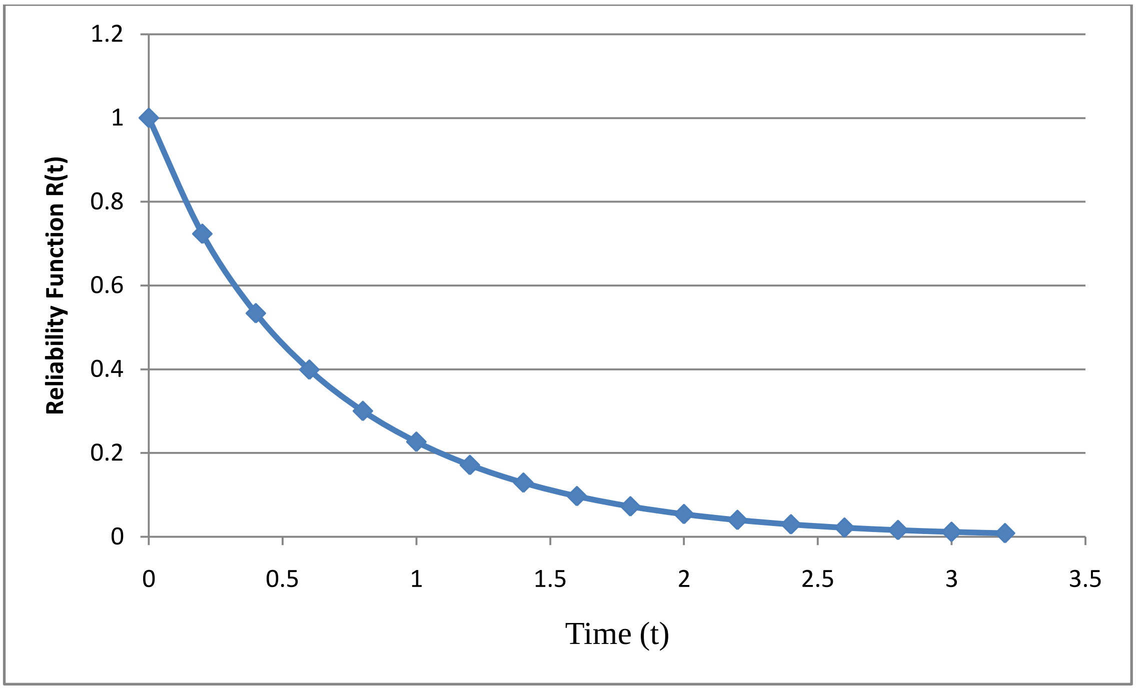

Fig. 4.1: Reliability Curve

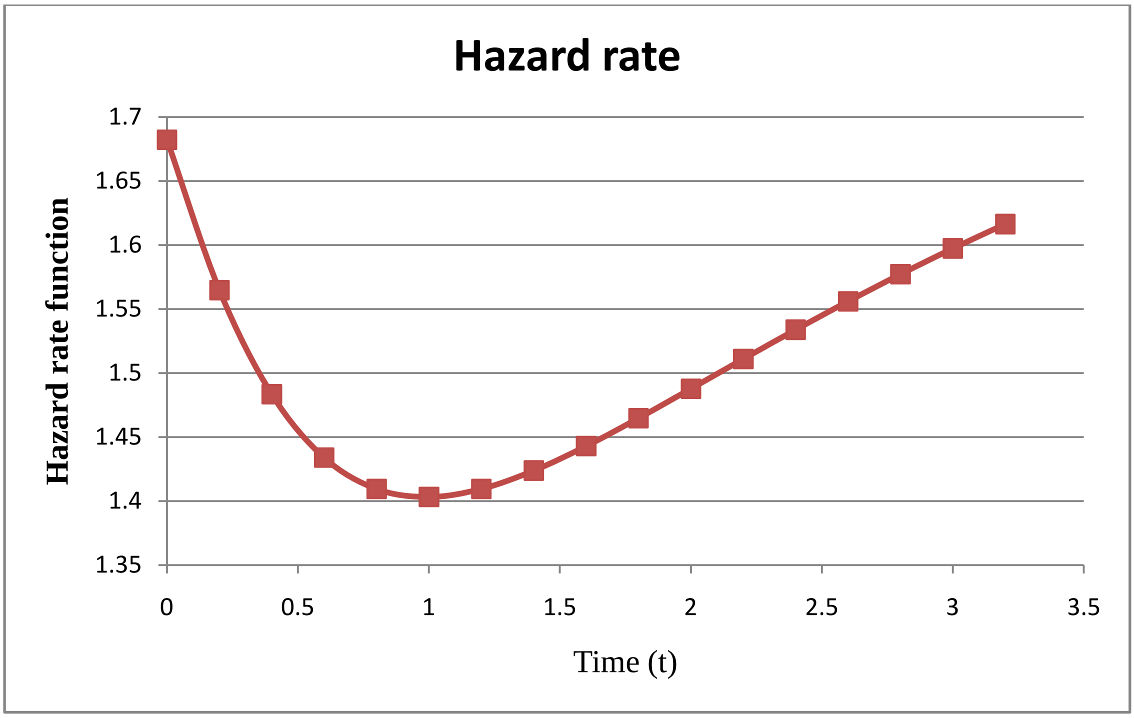

Fig. 4.2: Hazard Rate Curve

(b) Real data set

A real life data set of size 23 fatigue life for deep-groove ball bearings, compiled by American Standards Association and reported in Lieblein and Zelen (1956) has been analyzed to illustrate the applicability of the proposed model. It has already been mentioned in Chapter 3 that the proposed model is a good fit to the above said data. The MLE of θ for this data set is found to be 0.01998142. Using this value of θ we have calculated Bayes estimates of θ, R(t) and h(t) under two different priors which are given in Table 4.5.

	Prior 1			Prior 2		
	θ	R(t)	h(t)	θ	R(t)	h(t)
Estimates	0.019832	0.996475	0.002828	0.019838	0.996476	0.002829
Risk	4.36331E-08	3.79151E-09	3.40913E-09	4.00057E-08	3.46966E-09	3.13142E-09

Table 4.5: Estimated values of θ, R(t) and h(t) along with their MSE and risks for real data set

It is revealed from Table 4.5 that Prior 2 is superior to Prior 1 for obtaining Bayes estimates of θ, R(t) and h(t). The estimates of θ, R(t) and h(t) are 0.019838, 0.996476 and 0.002829 respectively with Risks 4.00057E-08, 3.46966E-09 and 3.13142E-09. This confirms the conclusions drawn from the simulated data generated from the proposed model with different values of θ and establishes the utility of the proposed model for describing real life situations.

Summary and Conclusion

SUMMARY AND CONCLUSIONS

The present study, deals with the synthesis of a new continuous probability distribution, named weighted xgamma distribution and its parameter estimation along with the estimation of its reliability function and hazard rate function. The new distribution comes out to be lifetime distribution which may be used to fit several sets of real life data. The expressions for various distributional properties of weighted xgamma distribution including its moment generating function (mgf), cumulant generating function (cgf), characteristics function, moments etc. have been derived. The classical estimators of parameter θ are obtained using method of moments and method of maximum likelihood estimation. Besides, Bayes estimators of θ, reliability function R(t) and hazard rate function h(t) are obtained under two different priors by means of Tierney and Kadane method of approximation. Maximum Likelihood estimates of θ are obtained by using Newton-Raphson method through R software for the random samples of various sizes generated through simulation technique. Moreover the expressions for Bayes risk of θ, reliability function(R(t)) and hazard rate function(h(t)) under both the priors have also been obtained using Squared Error Loss Function(SELF). Simulation study has also been carried out for getting reliable estimates with varying sample sizes through R software. Each random sample has been replicated 10,000 times to increase the sample size considerably. The present study was undertaken with following objectives:

1. To obtain Bayes estimators of the parameters of the weighted xgamma distribution under different priors.
2. To obtain Bayes estimators of the reliability and hazard rate functions of the weighted xgamma distribution under different priors.
3. To compare the above said estimators by using a suitable loss function.
4. To illustrate the above methodology by means of numerical examples.

Chapter 1 includes the introduction of Bayesian approach of parameter estimation and its importance over classical estimation, weighted distribution, form of its pdf and explains the weighted xgamma distribution in detail along with the objectives of the study. Chapter 2 presents, a brief review of literature relevant to the study. The materials and methods used for synthesis of this new pdf along with its distributional properties, utility and classical and Bayes estimation of its parameter, reliability function R(t) and hazard rate function h(t) have been discussed in Chapter 3. The utility of the proposed model has been illustrated by means of simulated random samples of various sizes generated through R software in Chapter 4. Bayes estimates of θ, R(t) and h(t) for the random samples of different sizes and one real life data have also been obtained under two priors (uniform and gamma priors) through simulation and are given in Chapter 4. It is concluded that gamma prior is superior to uniform prior for all random samples and real data set. The estimated values of θ, R(t) and h(t) have also been calculated using the two priors and it is concluded that reliability function R(t) decreases with increasing value of time t and takes nearly 0 value at t=3.2. Moreover hazard rate function h(t) decreases with increasing value of t up to t=1 and then again goes on increasing continuously. The details of the work done so far in this field mentioned in Chapter 2 have been recorded under the heading "Literature Cited". The coding of R programming for various calculations have been given in Appendix in the end of the thesis.

The major findings of the study from the simulated data are as follows:

Case 1: For randomly generated data

The Bayes estimates of θ, R(t) and h(t) for random samples generated from weighted xgamma distribution with θ=1.0 are (1.025139, 0.18271 and 1.382915) with minimum risks (0.000556, 6.90E-07 and 0.001776) respectively. These values for random samples from weighted xgamma distribution with θ=1.5 and θ=2.0 respectively are (1.53671, 0.068614 and 2.260001) and (2.054265, 0.026545, and 3.18598) with minimum risks (0.00148, 1.92E-07 and 0.005198) and (0.002926, 5.07E-08 and 0.010905). Thus Bayes estimate of θ, R(t) and h(t) are more precise for gamma prior (prior 2) compared to uniform prior (prior 1) for all assumed values of θ.

Case 2: For real data set

It has been observed that for real data set, gamma prior (prior 2) is superior to uniform prior (prior 1) for obtaining Bayes estimates (0.019838, 0.996476 and 0.002829) of parameter θ, reliability function R(t) and hazard rate function h(t) of weighted xgamma distribution with risks (4.0005E-08, 3.46966E-09 and 3.13142E-09) which is consistent with the conclusions drawn from the random samples generated from weighted xgamma distribution through R software.

Therefore, finally it is concluded that the proposed model is useful for describing many real life situations and for Bayes estimation of its parameter θ, reliability function R(t) and hazard rate function h(t), gamma prior (prior 2) should be preferred over uniform prior.

Literature Cited

LITERATURE CITED

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Appendices

```
rm(list=ls(all='TRUE'))
priya=function(n){
 Fx<- function(x,th,u){
   return(1-exp(-2*th*x)*(1+2*th*x*(th*x+1)/(4*th+1))-u)
 }
 dirFx<- function(x,th){
   return(4*th^2*exp(-2*th*x)*(2+th*x^2)/(4*th+1))
 }
 th <- 1
 t <- 1.2
 b <- 3
 n <- 20
 itr <- 100
 # esp <- 0.001
 x <- matrix(c(rep(0,n),rep(0,n),runif(n)), ncol=3,nrow=n)
 j=1
 itr.count <- 0
 for(i in 1:n){
   while(j)
   {
     itr.count <- itr.count+1
     if(itr.count>=itr){
       break
     }
```

```
    else{
      x[i,1] <- x[i,2]
      x[i,2] <- x[i,1] - Fx(x = x[i,1], th = th, u = x[i,3])/dirFx(x = x[i,1],th = th)
    }
  }
  itr.count<-0
 }
 x=x[,2]
 Fmth<- function(n,x,mth){
  return((2*n/mth)-(4*n/(4*mth+1))-(2*sum(x))+sum(x^2/(2+mth*x^2)))
 }
 derFmth<- function(n,mth,x){
  return((-2*n/mth^2)+(16*n/(4*mth+1)^2)-sum(x^4/(2+mth*x^2)^2))
 }
 itr2 <- 100
 mth <- matrix(c(0.1,0.1), ncol=2,nrow=1)
 for (i in 1:itr2) {
  mth[1,1] <- mth[1,2]
  mth[1,2] <- mth[1,1] - Fmth(n=n, x = x, mth = mth[1,1])/derFmth(n=n, x = x, mth =
mth[1,1])
 }
 mth<-mth[1,2]
 mth
 rt0<- function(t,th){
  return(exp(-2*th*t)*(1+2*th*t*(th*t+1)/(4*th+1)))
 }
 rt0_v<-rt0(th = th, t = t)
```

```
rt0_v
ht0<- function(t,th){
  return(8*(th^2)*(1+(th*t^2)/2)/((2*(th^2)*(t^2))+(2*th*t)+(4*th)+1))
}
ht0_v<-ht0(th = th, t = t)
ht0_v
mrt0<- function(t,mth){
  return(exp(-2*mth*t)*(1+2*mth*t*(mth*t+1)/(4*mth+1)))
}
mrt0_v<-mrt0(mth = mth, t = t)
mrt0_v
mht0<- function(t,mth){
  return(8*(mth^2)*(1+(mth*t^2)/2)/((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1))
}
mht0_v<-mht0(mth = mth, t = t)
mht0_v
mseth<- function(th,mth){
  return((mth-th)^2)
}
mseth_v <- mseth(mth = mth, th = th)
mseth_v
msert0<- function(rt0,mrt0){
  return((mrt0-rt0)^2)
}
msert0_v<-msert0(mrt0 = mrt0_v, rt0 = rt0_v)
msert0_v
```

```
mseht0<- function(ht0,mht0){
  return((mht0-ht0)^2)
}
mseht0_v<-mseht0(mht0 = mht0_v, ht0 = ht0_v)
mseht0_v
sigma11=function(n,x,mth){
  return(1/(-(2/mth^2)+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)+(1/(n*mth
^2))))
}
 sigma11_v<-sigma11(mth = mth, n = n, x = x)
sigma11_v
mth
sigma12=function(n,x,mth){
  return(1/(-(2/mth^2)+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)))
}
sigma12_v<-sigma12(mth = mth, n = n, x = x)
sigma12_v
mth
sigma13=function(n,x,mth){
  return(1/(-(2/mth^2)+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)(1/(n*mth
^2))))
}
sigma13_v<-sigma13(mth = mth, n = n, x = x)
sigma13_v
mth
sigma14=function(n,x,mth,t){
  return(1/(-(2/mth^2)+(16*(n+1)/(n*(4*mth+1)^2))-((sum(x^4/(2+mth*x^2)^2))/n)
```

```
+(1/(n*mth^2))-8*(((mth^2)*(t^4))+(mth*(t^3))+(2*mth*(t^2))+(2*t)+2)/(n*((2*
(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
  }
  sigma14_v<-sigma14(mth = mth, n = n, x = x, t = t)
  sigma14_v
  mth
  sigma15=function(n,x,mth,t){
    return(1/(-(2/mth^2)+(16*(n+2)/(n*(4*mth+1)^2))-((sum(x^4/(2+mth*x^2)^2))/n)
+(1/(n*mth^2))-16*(((mth^2)*(t^4))+(mth*(t^3))+(2*mth*(t^2))+(2*t)+2)/(n*
((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
  }
  sigma15_v<-sigma15(mth = mth, n = n, x = x, t = t)
  sigma15_v
  mth
  sigma16=function(n,x,mth,t){
    return(1/(-(2*(n+1)/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)
+(1/(n*mth^2))-((t^4)/(n*(2+mth*t^2)^2))+8*(((mth^2)*(t^4))+(mth*(t^3))+
(2*mth*(t^2))+(2*t)+2)/(n*((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
  }
  sigma16_v<-sigma16(mth = mth, n = n, x = x, t = t)
  sigma16_v
  mth
  sigma17=function(n,x,mth,t){
    return(1/(-(2*(n+2)/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)
+(1/(n*mth^2))-(2*(t^4)/(n*(2+mth*t^2)^2))+16*(((mth^2)*(t^4))+(mth*(t^3))
+(2*mth*(t^2))+(2*t)+2)/(n*((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
  }
  sigma17_v<-sigma17(mth = mth, n = n, x = x, t = t)
```

```
sigma17_v
mth
theta11=function(sigma11,sigma12,mth){
  return((sqrt(sigma12/sigma11))*mth)
}
theta11_v<-theta11(sigma11 = sigma11_v, sigma12 = sigma12_v, mth = mth)
theta11_v
theta12=function(sigma11,sigma13,mth){
  return((sqrt(sigma13/sigma11))*mth^2)
}
theta12_v<-theta12(sigma11 = sigma11_v, sigma13 = sigma13_v, mth = mth)
theta12_v
msetheta1<- function(theta11,th){
  return((theta11-th)^2)
}
msetheta1_v <- msetheta1(theta11 = theta11_v, th = th)
msetheta1_v
risktheta1<- function(theta11,theta12){
  return((theta12-(theta11^2)))
}
risktheta1_v <- risktheta1(theta11 = theta11_v, theta12 = theta12_v)
risktheta1_v
rel11=function(sigma11,sigma14,mrt0){
  return((sqrt(sigma14/sigma11))*mrt0)
}
rel11_v<-rel11(sigma11 = sigma11_v, sigma14 = sigma14_v, mrt0 = mrt0_v)
rel11_v
```

```
rel12=function(sigma11,sigma15,mrt0){
  return((sqrt(sigma15/sigma11))*mrt0^2)
}
rel12_v<-rel12(sigma11 = sigma11_v, sigma15 = sigma15_v, mrt0 = mrt0_v)
rel12_v
mserel1<- function(rel11,rt0){
  return((rel11-rt0)^2)
}
mserel1_v <- mserel1(rel11 = rel11_v, rt0 = rt0_v)
mserel1_v
riskrel1<- function(rel11,rel12){
  return((rel12-(rel11^2)))
}
riskrel1_v <- riskrel1(rel11 = rel11_v, rel12 = rel12_v)
riskrel1_v
haz11=function(sigma11,sigma16,mht0){
  return((sqrt(sigma16/sigma11))*mht0)
}
haz11_v<-haz11(sigma11 = sigma11_v, sigma16 = sigma16_v, mht0 = mht0_v)
haz11_v
haz12=function(sigma11,sigma17,mht0){
  return((sqrt(sigma17/sigma11))*mht0^2)
}
haz12_v<-haz12(sigma11 = sigma11_v, sigma17 = sigma17_v, mht0 = mht0_v)
haz12_v
msehaz1<- function(haz11,ht0){
  return((haz11-ht0)^2)
```

```
  }
  msehaz1_v <- msehaz1(haz11 = haz11_v, ht0 = ht0_v)
  msehaz1_v
  riskhaz1<- function(haz11,haz12){
    return((haz12-(haz11^2)))
  }
  riskhaz1_v <- riskhaz1(haz11 = haz11_v, haz12 = haz12_v)
  riskhaz1_v
  sigma21=function(n,x,b,mth){
    return(1/(-(2/mth^2)+((1-b)/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+
mth*x^2)^2))/n)))
  }
  sigma21_v<-sigma21(mth = mth, n = n, b = b, x = x)
  sigma21_v
  mth
  sigma22=function(n,x,mth,b){
    return(1/(-(2/mth^2)-(b/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2
)^2))/n)))
  }
  sigma22_v<-sigma22(mth = mth, b = b, n = n, x = x)
  sigma22_v
  mth
  sigma23=function(n,x,b,mth){
    return(1/(-(2/mth^2)-((1+b)/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+
mth*x^2)^2))/n)))
  }
  sigma23_v<-sigma23(mth = mth, n = n, b = b, x = x)
  sigma23_v
```

```
mth
sigma24=function(n,x,mth,b,t){
  return(1/(-(2/mth^2)+((1-b)/(n*(mth^2)))+(16*(n+1)/(n*(4*mth+1)^2))-((sum(x^4/(2+mth*x^2)^2))/n)-8*(((mth^2)*(t^4))+(mth*(t^3))+(2*mth*(t^2))+(2*t)+2)/(n*((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
}
sigma24_v<-sigma24(mth = mth, n = n, b = b, x = x, t = t)
sigma24_v
mth
sigma25=function(n,x,mth,b,t){
  return(1/(-(2/mth^2)+((1-b)/(n*(mth^2)))+(16*(n+2)/(n*(4*mth+1)^2))-((sum(x^4/(2+mth*x^2)^2))/n)-16*(((mth^2)*(t^4))+(mth*(t^3))+(2*mth*(t^2))+(2*t)+2)/(n*((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
}
sigma25_v<-sigma25(mth = mth, n = n, b = b, x = x, t = t)
sigma25_v
mth
sigma26=function(n,x,mth,t, b = b){
  return(1/(-(2*(n+1)/(n*(mth^2)))+((1-b)/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)-((t^4)/(n*(2+mth*t^2)^2))+8*(((mth^2)*(t^4))+(mth*(t^3))+(2*mth*(t^2))+(2*t)+2)/(n*((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^2)))
}
sigma26_v<-sigma26(mth = mth, n = n, b = b, x = x, t = t)
sigma26_v
mth
sigma27=function(n,x,mth,b,t){
  return(1/(-(2*(n+2)/(n*(mth^2)))+((1-b)/(n*(mth^2)))+(16/(4*mth+1)^2)-((sum(x^4/(2+mth*x^2)^2))/n)-(2*(t^4)/(n*(2+mth*t^2)^2))+16*(((mth^2)*(t^4))
```

```
+(mth*(t^3))+(2*mth*(t^2))+(2*t)+2)/(n*((2*(mth^2)*(t^2))+(2*mth*t)+(4*mth)+1)^
2)))
 }
 sigma27_v<-sigma27(mth = mth, n = n, b = b, x = x, t = t)
 sigma27_v
 mth
 theta21=function(sigma21,sigma22,mth){
   return((sqrt(sigma22/sigma21))*mth)
 }
 theta21_v<-theta21(sigma21 = sigma21_v, sigma22 = sigma22_v, mth = mth)
 theta21_v
 theta22=function(sigma21,sigma23,mth){
   return((sqrt(sigma23/sigma21))*mth^2)
 }
 theta22_v<-theta22(sigma21 = sigma21_v, sigma23 = sigma23_v, mth = mth)
 theta22_v
 msetheta2<- function(theta21,th){
   return((theta21-th)^2)
 }
 msetheta2_v <- msetheta2(theta21 = theta21_v, th = th)
 msetheta2_v
 risktheta2<- function(theta21,theta22){
   return((theta22-(theta21^2)))
 }
 risktheta2_v <- risktheta2(theta21 = theta21_v, theta22 = theta22_v)
 risktheta2_v
 rel21=function(sigma21,sigma24,mrt0){
```

```
  return((sqrt(sigma24/sigma21))*mrt0)
}
rel21_v<-rel21(sigma21 = sigma21_v, sigma24 = sigma24_v, mrt0 = mrt0_v)
rel21_v
rel22=function(sigma21,sigma25,mrt0){
  return((sqrt(sigma25/sigma21))*mrt0^2)
}
rel22_v<-rel22(sigma21 = sigma21_v, sigma25 = sigma25_v, mrt0 = mrt0_v)
rel22_v
mserel2<- function(rel21,rt0){
  return((rel21-rt0)^2)
}
mserel2_v <- mserel2(rel21 = rel21_v, rt0 = rt0_v)
mserel2_v
riskrel2<- function(rel21,rel22){
  return((rel22-(rel21^2)))
}
riskrel2_v <- riskrel2(rel21 = rel21_v, rel22 = rel22_v)
riskrel2_v



haz21=function(sigma21,sigma26,mht0){
  return((sqrt(sigma26/sigma21))*mht0)
}
haz21_v<-haz21(sigma21 = sigma21_v, sigma26 = sigma26_v, mht0 = mht0_v)
haz21_v
haz22=function(sigma21,sigma27,mht0){
```

```
  return((sqrt(sigma27/sigma21))*mht0^2)
 }
 haz22_v<-haz22(sigma21 = sigma21_v, sigma27 = sigma27_v, mht0 = mht0_v)
 haz22_v
 msehaz2<- function(haz21,ht0){
  return((haz21-ht0)^2)
 }
 msehaz2_v <- msehaz2(haz21 = haz21_v, ht0 = ht0_v)
 msehaz2_v
 riskhaz2<- function(haz21,haz22){
  return((haz22-(haz21^2)))
 }
 riskhaz2_v <- riskhaz2(haz21 = haz21_v, haz22 = haz22_v)
 riskhaz2_v
c(mth,rt0_v,ht0_v,mseth_v,mrt0_v,mht0_v,msert0_v,mseht0_v,theta11_v,msetheta1_v,
risktheta1_v,rel11_v,mserel1_v,riskrel1_v,haz11_v,msehaz1_v,riskhaz1_v,theta21_v,m
setheta2_v,risktheta2_v,rel21_v,mserel2_v,riskrel2_v,haz21_v,msehaz2_v,riskhaz2_v)
}
th=1
valu=replicate(10000,priya(20));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(40));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(60));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(80));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
#########################
```

```
th=1.5
valu=replicate(10000,priya(20));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(40));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(60));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(80));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
##########################
th=2
valu=replicate(10000,priya(20));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(40));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(60));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
valu=replicate(10000,priya(80));asd=rowMeans(valu);
write.table(t(asd),"est2.csv",sep=",",append=T,row.names=F,col.names=F)
```

Vita

The authoress was born on 13 July, 1994 in Khatima. She successfully completed her High School in 2010 and Intermediate in 2012 from Central Board of Secondary Education with first division. The authoress passed her B.Sc. (Physical Sciences) degree in 2015, from D.S.B. Campus, Kumaun University, Nainital. Thereafter the authoress joined the College of Post Graduate Studies in 2015 in Govind Ballabh Pant University of Agriculture and Technology, Pantnagar, Uttarakhand, India, for the degree of M.Sc. with Major in Statistics and Minor in Mathematics.

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Thesis Title : **BAYESIAN ESTIMATION IN WEIGHTED XGAMMA DISTRIBUTION**

Advisor : Dr. Vinod Kumar

ABSTRACT

In the present study, a new lifetime distribution, named weighted xgamma distribution has been proposed and its distributional properties are investigated. The maximum likelihood estimates of the parameter θ have been obtained by means of Newton-Raphson method. The expressions for various distributional properties of weighted xgamma distribution including its moment generating function (mgf), cumulant generating function (cgf), characteristic function, moments etc. have been derived. The Bayes estimators of its parameter (θ), reliability function R(t) and hazard rate function h(t) are obtained using Tierney and Kadane method of approximation under two priors namely uniform and gamma. The results obtained have been illustrated by means of several randomly generated data sets from the proposed model, each sample replicated 10,000 times The Bayes Risks have been evaluated by using Squared Error Loss Function (SELF). A real life data set has also been used to establish its utility. It is concluded that gamma prior is superior to uniform prior for finding Bayes estimates of the parameter θ, reliability function R(t) and hazard rate function h(t) of the proposed weighted xgamma distribution.

(Vinod Kumar)
Advisor

(Priya Agrawal)
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नाम	: प्रिया अग्रवाल	परिचयांक	: ४९३९२
सत्र एवं प्रवेश वर्ष	: प्रथम, २०१५-२०१६	उपाधि	: स्नातकोत्तर
मुख्य विषय	: सांख्यिकी	विभाग	: गणित, सांख्यिकी एवं संगणक विज्ञान
गौण विषय	: गणित		
शोध शीर्षक	: भारित एक्सगामा बंटन में बेज आकलन		
सलाहकार	: डॉ॰ विनोद कुमार		

सारांश

वर्तमान अध्ययन में एक नया भारित एक्सगामा नामक जीवनकाल बंटन प्रस्तावित किया गया है व इस बंटन की विशेषताओं का अन्वेषण किया गया है। न्यूटन-राफ्सन विधि के द्वारा प्राचल (θ) के अधिकतम सम्भावित आकलक ज्ञात किये गये हैं। भारित एक्सगामा बंटन की विभिन्न विशेषताओं जैसे आघूर्ण जनक फलन, क्यूमलेंट जनक फलन, अभिलक्षण फलन, आघूर्ण इत्यादि की व्युत्पत्ति की गयी है। इसके प्राचल (θ), विश्वसनीयता फलन ($R(t)$) व हैजार्ड दर फलन ($h(t)$) के बेज आकलक दो प्रायरों के तहत टियर्नी व कडाने विधि द्वारा ज्ञात किये गये हैं। प्राप्त परिणामों को प्रस्तावित प्रारूप से जनित अनेक यादृच्छिक प्रतिदर्शों जिनमें से प्रत्येक को १०००० बार दोहराया गया है, के उदाहरण द्वारा समझाया गया है। वर्ग त्रुटि हानि फलन (SELF) का उपयोग कर बेज जोखिम ज्ञात किये गये हैं। इस प्रारूप की उपयोगिता स्थापित करने हेतु एक वास्तविक आंकडा समुच्चय का उपयोग किया गया है। अन्ततः निष्कर्ष निकाला गया है कि प्रस्तावित एक्सगामा बंटन के प्राचल (θ), विश्वसनीयता फलन ($R(t)$) व हैजार्ड दर फलन ($h(t)$) के बेज आकलन हेतु गामा प्रायर यूनीफार्म प्रायर की तुलना में श्रेष्ठ है।

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लेखिका