ON SUBMANIFOLDS AND HYPERSURFACES OF HYPERBOLIC HERMITIAN MANIFOLDS

Thesis

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By

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Dedicated to my.....

Parents who encouraged me

Teachers who enabled me

Husband who put up with me ....

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CERTIFICATE

This is to certify that the thesis entitled **"On submanifolds and Hypersurfaces of Hyperbolic Hermitian manifolds",** submitted in partial fulfilment of the requirements for the degree of **Doctor of Philosophy** with major in Mathematics and minor in Computer Science of the College of Post-Graduate Studies, G.B. Pant University of Agriculture and Technology, Pantnagar, is a record of *bona fide* research carried out by **Miss Latika Bhatt, Id. No. 23961,** under my supervision, and no part of the thesis has been submitted for any other degree or diploma.

The assistance and help received during the course of this investigation have been duly acknowledged.

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CERTIFICATE

We, the undersigned, members of the Advisory Committee of **Miss Latika Bhatt, Id. No. 23961,** a candidate for the degree of **Doctor of Philosophy** with major in Mathematics and minor in Computer science, agree that the thesis entitled **"On submanifolds and Hypersurfaces of Hyperbolic Hermitian manifolds"**, may be submitted in partial fulfilment of the requirements for the degree.

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INTRODUCTION

Chapter 1 INTRODUCTION

Differential geometry is probably as old as any branch of mathematics and certainly was well launched after Newton and Leibnitz had laid the foundation of calculus. Differential geometry has a long history of change from time to time in its approach due to change in analysis. The introduction of analysis of manifold has given a rigorous approach to it to some extent, its application in various disciplines of mathematics namely relativity, fluid mechanics, etc., have put it on such a height that without its knowledge the study of mathematics is incomplete.

The theory of structure on manifolds is an interesting topic of modern differential geometry and the differential geometric aspects of submanifold of manifolds with certain structures are vast and fruitful fields for Riemannian geometry.

The theory of submanifolds of an almost Hermitian manifold is one of the most interesting topics in differential geometry. In an almost Hermitian manifold, its almost complex structure F transforms a vector into a vector perpendicular to it. Perhaps this was the natural motivation to study submanifolds of an almost Hermitian manifold, according to the

behavior of its tangent bundle under the action of almost complex structure F of the ambient manifold.

There are two well known classes of submanifolds namely invariant submanifold and anti-invariant submanifolds. In the first case tangent space of the submanifold remains invariant under the action of almost complex structure F whereas in the second case, it is mapped into the normal space. Study of differential geometry of CR-submanifolds is a generalization of invariant and anti-invariant submanifold of an almost Hermitian manifold was initiated by Bejancu in 1978 and was followed by several geometers. Chen further generalized the concept of CR-submanifold in 1981 that introduced generic submanifold.

Bejancu and Papaghuic initiated the study of semi-invariant submanifolds in a Sasakian manifolds in 1981. Kobayashi in 1981, Yano-Kon in 1982 and 1983, studied the same concept under the name contact CR-submanifold.

A semi-invariant submanifold is nothing but the extension of the concept of CR-submanifold of a Kaehlerian manifold to submanifold of an almost contact metric manifold.

In 1981 M. Kobayashi studied CR-submanifolds of a Sasakian manifold. Further in 1986, he studied semi-invariant submanifolds of a

certain class of almost contact manifold. In 1985 Oubina introduced a new class of almost contact structure namely the trans Sasakian structure. In 1989, Kalpana gave some results related to almost semi-invariant submanifold of SP-Sasakian manifold. In 1991 Shahid studied CR-submanifolds of trans Sasakian manifold. In 1991 Sinha and Yadav studied some results on semi-invariant submanifolds of a certain class of almost contact manifolds. In 1992 Sinha and Srivastava studied semi-invariant submanifold of Kenmotsu manifold with constant ϕ-holomorphic sectional curvature. In 1993 Shahid studied semi-invariant submanifolds of a nearly Sasakian manifold. In 1996 Tripathi also studied semi-invariant submanifolds of trans Sasakian manifold. Pandey and Verma in 1999 studied para Sasakian manifold. Tripathi in 2000 studied CR-submanifold of a nearly and closely Cosymplectic manifolds.

In 1973 Dube studied almost hyperbolic Hermitian manifolds. In 1978 Dube and Niwas studied almost r-contact hyperbolic structure in a product manifold. In 1980 Pal and Mishra studied Hypersurfaces of almost hyperbolic Hermite manifolds. Dube and Mishra in 1981 studied hypersurface immersed in an almost hyperbolic Hermitian manifold. In 1993 Mishra studied hypersurface of almost Hermite manifold.

The present work, is the study of CR-submanifold of trans para Sasakian manifold and nearly, closely para Cosymplectic manifold, semi-invariant submanifold of nearly r-Sasakian manifold, r-Kenmotsu manifold. We also studied the CR-submanifold of trans hyperbolic Sasakian manifold and semi-invariant submanifold of para Kenmotsu manifold. In the present work, we have also studied hypersurfaces of hyperbolic Hermitian manifolds and obtained the conditions under which the given hypersurfaces will be totally geodesic and hyperbolic Cosymplectic manifolds such as hyperbolic Sasakian, K-contact hyperbolic Riemannian manifold, normal quasi hyperbolic Sasakian manifold. Some other interesting properties of the hypersurfaces of hyperbolic manifolds have also been investigated.

1.1 Structures on Manifolds:

1.1.1 Almost complex manifold:

If in a differentiable manifold $\bar{M}$, there exists a vector valued linear function F, of differentiability class C^{∞} satisfying

(1.1.1) $F^2 = -I$,

then $\bar{M}$ is called an almost complex manifold and the structure F is called an almost complex structure [25].

Nijenhuis tensor with respect to F is a vector valued skew-symmetric bilinear function N given by

(1.1.2) $N(X,Y) = [FX,FY] + F^2[X,Y] - F[FX,Y] - F[X,FY]$,

For all vector fields X,Y of $\bar{M}$.

An almost complex manifold with vanishing Nijenhuis tensor is called a complex manifold.

1.1.2 Almost Hermitian manifold:

An almost Hermitian manifold is an almost complex manifold is an almost complex manifold endowed with a Hermite metric g

(1.1.3) a) $g(FX,FY) = g(X,Y) \Leftrightarrow$ b) $g(FX,Y) = - g(X,FY)$.

The structure {F,g} is called almost Hermitian structure [107].

1.1.3 Almost product manifold:

If in the differentiable manifold $\bar{M}$, there exists a vector valued linear function F, of differentiability class C^∞ satisfying

(1.1.4) $F^2 = I$,

then $\bar{M}$ is said to be an almost product manifold [107, 52].

If in the almost product manifold

(1.1.5) $g(FX,FY) = g(X,Y)$

and

(1.1.6) $g(FX,Y) = g(X,FY)$,

where g is positive definite Riemannian metric, then $\bar{M}$ is called almost product Riemannian manifold. If in the almost product Riemannian manifold $(\nabla_X F)Y = 0$, where ∇ is the Riemannian connection, then $\bar{M}$ is called almost product, almost decomposable manifold.

1.1.4 Almost contact manifold:

If in an odd dimensional differentiable manifold $\bar{M}$, there exists a vector valued linear function F, a vector field U and a 1-form u, satisfying

(1.1.7) $F^2 = -I + u \otimes U, \qquad FU = 0,$

then $\bar{M}$ is called an almost contact manifold and {F,U,u} is called almost contact structure [78].

In an almost contact manifold

(1.1.8) $u(U) = 1, \qquad uoF = 0.$

An almost contact structure can be obtained on a non-invariant hypersurface of an almost Hermitian manifold.

An almost contact manifold in which the metric tensor g satisfies

(1.1.9) $g(FX,FY) = g(X,Y) - u(X)u(Y),$

is called almost contact metric manifold and {F,U,u,g} is called almost contact metric structure.

An almost contact metric manifold for which

(1.1.10) $\nabla_X F = 0$,

is called a Cosymplectic manifold [29].

An almost contact metric manifold for which

(1.1.11) $(\nabla_X F)X = 0$,

is called a nearly Cosymplectic manifold [9].

An almost contact metric manifold is called closely Cosymplectic if F is killing and u is closed, that is [12]

(1.1.12) $\nabla_U F = 0, \qquad \nabla U = 0, \qquad \nabla u = 0.$

An almost contact metric manifold structure (F,U,u,g) is called a Sasakian manifold if [10]

(1.1.13) $(\nabla_X F)(Y) = g(X,Y)U - u(Y)X$

Kenmotsu if [39]

(1.1.14) $(\nabla_X F)(Y) = g(FX,Y)U - u(Y)FX$

and trans Sasakian if [65]

(1.1.15) $(\nabla_X F)(Y) = \alpha(g(X,Y)U - u(Y)X) + \beta(g(FX,Y)U - u(Y)FX)$

for function α, β.

An almost contact metric manifold is called a nearly Sasakian if [10]

(1.1.16) $(\nabla_X F)(Y) + (\nabla_Y F)(X) = u(Y)X + u(X)Y - 2g(X,Y)U.$

1.1.5 Almost r-contact metric manifold:

If in an odd dimensional differentiable manifold $\bar{M}$, there exists a vector valued linear function F, r-vector fields U_p and r 1-form u^p (p=1,2,...,r) such that

(1.1.17) $F^2 = -I + u^p \otimes U_p, \qquad FU_p = 0.$

It gives

(1.1.18) $u^p o F = 0, \qquad u^p(U_q) = \delta_q^p$

and a metric

(1.1.19) $g(FX,FY) = g(X,Y) - u^p(X)u^p(Y),$

then $\bar{M}$ is called an almost r-contact metric manifold and $\{F,U_p,u^p,g\}$ is called almost r-contact metric structure [77, 105].

1.1.6 Almost para contact manifold:

If in an odd dimensional differentiable manifold $\bar{M}$, there exists a vector valued linear function F, a vector field U and a 1-form u satisfying

(1.1.20) $F^2 = I - u \otimes U, \qquad FU = 0.$

It gives

$uoF = 0, \qquad u(U) = 1,$

then $\bar{M}$ is called an almost para contact manifold and {F,U,u} is called almost para contact structure.

An almost para contact manifold in which the metric tensor g satisfies

$$g(FX,FY) = g(X,Y) - u(X)u(Y),$$

is called almost para contact metric manifold and {F,U,u,g} is called almost para contact metric structure [79].

1.1.7 Almost hyperbolic Hermitian manifold:

If in a differentiable manifold $\bar{M}$, there exists a vector valued linear function, F of differentiability class C^∞ satisfying

(1.1.21) a) $F^2 = I$, b) $g(FX,FY) = -g(X,Y)$

then $\bar{M}$ is said to be an almost hyperbolic Hermitian manifold [20].

1.1.8 Almost contact hyperbolic metric manifold:

If in a differentiable manifold $\bar{M}$, there exists a vector valued linear function F, a 1-form u and a vector field U satisfying

(1.1.22) $F^2 = I + u\otimes U$, $FU = 0$.

It gives

(1.1.23) $uoF = 0$ $u(U) = -1$,

then $\bar{M}$ is called almost contact hyperbolic manifold and {F,U,u} is called almost contact hyperbolic structure.

An almost contact hyperbolic manifold in which metric tensor g satisfies

(1.1.24) $g(FX,FY) = -g(X,Y) - u(X)u(Y)$,

is called almost contact hyperbolic metric manifold and $\{F,U,u,g\}$ is called almost contact hyperbolic metric structure [103].

1.2 Submanifolds:

Let M and $\bar{M}$ be two C^{∞} manifolds of dimension n and m (m>n) respectively. A map $i : M \to \bar{M}$ is called an immersion if its differential map i_* is injective for every $x \in M$ and i(M) is called imbedding of M into $\bar{M}$ and the image i(M) is called a submanifold of $\bar{M}$ and it is identified with M. Manifold M is said to be a submanifold of $\bar{M}$ if it is a subset of $\bar{M}$ and if the map $i : M \to \bar{M}$ is immersion and injective both. If M is open, then the submanifold is called open submanifold.

Let M be a submanifold of a Riemannian manifold $\bar{M}$ with a Riemannian metric g, then Gauss and Weingarten formulae are given respectively by

(1.2.1) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$,

(1.2.2) $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

for all $X,Y \in TM$ and $N \in T^{\perp}M$, where $\bar{\nabla}$, ∇, $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in $\bar{M}$, M and the normal bundle of M respectively and h is the second fundamental form related to A by

(1.2.3) $g(h(X,Y),N) = g(A_N X,Y)$.

If F is a (1,1) tensor field on $\bar{M}$ for $X,Y \in TM$ and $N \in T^{\perp}M$, we put

(1.2.4) $FX = PX + QX$, $PX \in TM$, $QX \in T^{\perp}M$,

(1.2.5) $FN = BN + CN$, $BN \in TM$, $CN \in T^{\perp}M$.

Let $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M respectively, then equation of Guass and Coddazzi are given respectively by

(1.2.6) $g(\bar{R}(X,Y)Z,W)=g(R(X,Y)Z,W)-g(h(X,W),h(Y,Z))$
$+g(h(Y,W),h(X,Z))$,

(1.2.7) $(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)$,

where $(\bar{R}(X,Y)Z)^{\perp}$ is the normal component of $\bar{R}(X,Y)Z$, and

(1.2.8) $(\nabla_X h)(Y,Z)=(\nabla_X^{\perp} h)(Y,Z)- h(\nabla_X Y,Z)-h(Y,\nabla_X Z)$, $\forall\ X,Y,Z \in TM$.

The submanifold M is said to be totally geodesic if h = 0 and totally umbilical if

(1.2.9) $h(X,Y) = g(X,Y)H$.

1.3 CR-Submanifolds:

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure tensor F, where F is (1,1) tensor field. We consider a submanifold M of $\bar{M}$ and denote by T_xM and $T_x^{\perp}M$ the tangent space and the normal space of M at $x \in M$ respectively. If T_xM is invariant under the action of F for each $x \in M$, that is if $FT_xM \subset T_xM$ for each $x \in M$, then M is called an invariant (or holomorphic) submanifold of $\bar{M}$. On the other hand, if the transform of T_xM by F is contained in the normal space $T_x^{\perp}M$ for each $x \in M$, that is $FT_xM \subset T_x^{\perp}M$ for each $x \in M$, then M is called an anti-invariant (or totally real) submanifold of $\bar{M}$.

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure tensor F, a submanifold M of $\bar{M}$ is called a CR-submanifold of $\bar{M}$ if there exists a differentiable distribution $D: x \to D_x$ on M satisfying the following conditions:

(i) D is invariant, that is, $FD_x \subset D_x$ for each $x \in M$,

(ii) the complementary orthogonal distribution $D^{\perp}: x \to D_x^{\perp} \subset T_xM$ of D is anti-invariant, that is, $FD_x^{\perp} \subset T_x^{\perp}M$ for each $x \in M$.

1.4 Semi-invariant submanifolds:

A semi-invariant submanifold is nothing but the extension of the concept of the CR-submanifold of Kaehler manifold to submanifold of almost contact metric manifolds.

Let M be submanifold of an almost contact metric manifold $\bar{M}$ with almost contact metric structure (F,U,u,g). Then M is called a semi-invariant submanifold of $\bar{M}$ if there exist two differentiable distributions D and $D^{\perp}$ on M satisfying

(i) $TM = D \oplus D^{\perp} \oplus \{U\}$, where D, $D^{\perp}$ and $\{U\}$ are mutually orthogonal to each other.

(ii) The distribution D is invariant by F, that is $F(D_x) \subset D_x$ for each $x \in M$.

(iii) The distribution $D^{\perp}$ is anti-invariant by F, that is $FD_x^{\perp} \subset T_x^{\perp}M$ for each $x \in M$.

1.5 Connection:

1.5.1 Symmetric Connection:

A linear connection ∇ on a C^{∞} manifold is said to be symmetric if its torsion tensor is zero, that is

(1.5.1) $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

1.5.2 Metric Connection:

A linear connection ∇ on a C^{∞} manifold is said to be metric connection if its tensor g is parallel, that is

(1.5.2) $(\nabla_X g)(Y,Z) = 0.$

1.5.3 Riemannian Connection:

A linear connection ∇ on a C^{∞} manifold is said to be Riemannian connection if it is symmetric as well as metric connection.

1.6 Hypersurfaces:

A submanifold M of $\bar{M}$ is said to be a hypersurface of $\bar{M}$ if dimension of $\bar{M}$ is one greater than dimension of M. In case of a hypersurface there is only one normal vector field to M. Let $\bar{M}$ be an almost contact manifold and M be an orientable hypersurface of $\bar{M}$, and i_* the differential of the immersion i of M into $\bar{M}$. Let X,Y,Z be tangent to M and N a unit normal vector, then we have

(1.6.1) a) $FBX = BfX + u(X)\,N$ b) $FN = -BU$

(1.6.2) $g(BX,BY)ob = h(X,Y),$

where f is a tensor field of type (1,1) and u is a 1-form, U is a vector field and h is a induced metric tensor on M.

If u is identically zero, then M is said to be an invariant hypersurface, that is tangent space of M is invariant by F. If $u \neq 0$, then M is called a non-invariant hypersurface of $\bar{M}$.

Let E be the induced metric connection on the hypersurface M. Then we have

(1.6.3) a) $D_{BX}BY = BE_XY + `H(X,Y)N$, b) $D_{BX}N = -BHX$,

where `H is the second fundamental tensor on M and H is the associate tensor given by

(1.6.4) $`H(X,Y) = h(HX,Y)$.

REVIEW OF LITERATURE

Schouten and Dontzing (1930) first tried to transfer the results in differential geometry to space with Riemannian metric and with affine connection to the case of spaces with complex structures and thus opened a new era of complex manifolds. Later on, Ehresmann (1947) defined an almost complex manifold, which carries a tensor field f whose square is minus unity. Newlander and Nirenberg (1957) studied the case when the complex space is merely differentiable. In solving these problems Nijenhuis (1951) introduced an important tensor, later known as Nijenhuis Tensor. Chern (1946), Bochenner (1947), Sasaki (1960), Kobayashi (1961), Yano (1965) studied differential geometry of complex manifolds and Riemannian manifolds.

Walker (1955) studied the properties of the manifold with an almost product structure, in which there exist a mixed tensor field f whose square is unity. The idea of f structure on a differentiable manifold was first initiated and developed by Yano (1963) and studied some of its properties and integrability condition relating to this structure. He also studied some of the properties on a structure defined by a tensor field f of type (1,1), satisfying $f^3 + f = 0$ and $f^3 - f = 0$.

Tanno (1969), Yano et al. (1969) proved that any invariant submanifold of a normal contact metric manifold is minimal. Mashiro et al. (1973) gave the condition for an invariant submanifold of a K-contact Riemannian manifold to be minimal and also gave the necessary and sufficient conditions for the given manifold to be totally geodesic.

Blair (1971) studied almost contact manifolds with Killing structure tensors (called nearly Cosymplectic) and showed that if this structure is normal, it is Cosymplectic; in particular the (almost) contact distribution is integrable. Blair, Showers and Yano (1976) studied the notion of a nearly Sasakian structure and shown that a normal nearly Sasakian structure is Sasakian structure and hence in particular is contact. They also studied that a hypersurface of a nearly Kaehler manifold is nearly Sasakian if and only if it is quasi Umbilical with respect to the (almost) contact form.

Hyperbolic Kaehlerian space has studied by Pravanovic (1971). Rasevaski (1948) was the first to consider such spaces; later on Sirokov studied it in 1961. Dube (1973) defined and studied almost hyperbolic Hermitian manifold and gave a classification to such spaces as nearly hyperbolic Kaehlerian, quasi hyperbolic Kaehlerian, semi hyperbolic Kaehlerian and hyperbolic Hermitian manifolds. Dube and Niwas (1978) defined r-contact hyperbolic structure in a product manifold, which is

generalization of almost contact hyperbolic structure. Pal and Mishra (1980) shown that a hypersurface of hyperbolic Hermite manifold admits an induced hyperbolic contact structure and also obtained a condition for hypersurface of an almost hyperbolic Hermite manifold with vanishing curvature tensor to be conformally flat. Further Dube and Mishra (1981) studied hypersurfaces of an almost hyperbolic Hermitian manifold.

Mishra (1973) studied a differentiable manifold with f-structure of rank r and also obtained integrability condition of that manifold. M. Kon (1973) studied some properties of invariant submanifold in a K-contact Riemannian manifold. He also studied in 1974 the pinching problem for the length of the second fundamental form of an invariant submanifold of a normal compact metric manifold of constant sectional curvature and gave the condition for an invariant submanifold to be totally geodesic.

K. Matsumoto (1974) have studied a conformal C-killing vector field in a compact Sasakian manifold and shown that a compact (2n+1)-dimensional Sasakian manifold, let u be a conformal C-killing vector field and v be a C-harmonic p-form ($p \leq n$) then we have $L(u)v = 0$, that is the Lie derivative of any C-harmonic p-form ($p \leq n$) with respect to a conformal C-killing vector field always vanishes. H. Endo (1974) studied invariant submanifolds in a K-contact Riemannian manifold and

generalized the result any invariant submanifold of a Sasakian manifold is minimal to K-contact Riemannian manifold. Ram Hit (1974) studied almost contact submanifolds and find the conditions for a metric submanifold of an almost contact metric Cosymplectic manifold to be Cosymplectic. He also obtained the conditions for an almost contact metric submanifold to be nearly Cosymplectic. Stavre, Petre (1975) studied some properties of a regular almost contact metric structure and structure induced on the orbit space.

Upadhyay and Dube (1976) defined and studied almost contact hyperbolic (f,g,η,ξ) structure and obtained some results. Mishra (1976) considers a generalization of an almost contact structure and introduces certain structure in a differentiable manifold and find their properties. Sato (1976) studied a differentiable manifold with structure tensors (ϕ,ξ,η) satisfying $\phi^2 = I - \eta\otimes\xi$ and $\eta(\xi) = 1$, where ϕ is a (1,1) tensor, ξ a vector field, η a 1-form on the manifold and I is the identity such a structure may be considered as analogues to the almost contact structure which is closely related to the almost complex structure. This is closely related to almost product structure. This is known as almost para contact structure and manifold is an almost para contact manifold. Further in

1977 he studied normal para contact Riemannian manifold, which is the most specialized form of almost para contact Riemannian manifolds.

M. Kon (1976) studied an anti-invariant submanifold tangent to the structure vector field of a Sasakian maniold is not totally umbilical and defined the notion of compact totally umbilical submanifolds of a Sasakian manifold which corresponds to that of totally umbilical submanifolds of a Kaehlerian manifold. He proved that any compact totally umbilical, anti-invariant submanifold tangent to the structure vector field of a Sasakian manifold with vanishing compact Bochner curvature tensor is locally a product of a conformally flat Riemannian manifold and 1-dimensional space. Yano and Kon (1976) studied invariant and anti-invariant submanifolds of Sasakian manifolds.

Paljakov (1977) studied detailed account of recent work concerning non-invariant submanifolds of an almost contact manifold as well as certain problems on distribution theory of connection on an almost contact manifold are also discussed.

Bejancu (1978) initiated a new class of submaniolds of a complex manifold, which he called CR-submanifold and obtained some interesting results. Mishra (1978) studied different structures in a differentiable manifold and also some of their important properties. He studied almost

complex structures, almost contact structures, general structures and also studied some of the application of the above structures.

Das, Lovejoy (1979) studied an almost contact manifold satisfying certain natural conditions can be considered as a principal g bundle over an almost complex manifold. The properties of this bundle studied by Staver (1975). The author extends the results of Stavre concerning almost contact manifolds of almost r-contact manifolds. Olszak, Zbigniew (1979) studied a contact metric manifold of dimension (2n+1) and let (ϕ,ξ,η,g) be its contact metric structure where $\phi^2=-I+\eta\otimes\xi$, $\phi\xi=0$, $\eta o\phi=0$, $\eta(\xi) = 1$, $g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$, $g(X,\xi) = \eta(X)$, $\phi(X,Y) = g(X,\phi Y) = d\eta(X,Y)$ and proved some theorems for this structure. He also studied some properties of a nearly Sasakian manifold. Mishra (1979) studied submanifolds of 3-structure metric manifold, 3-structure almost Sasakian manifold, 3-structure quasi Sasakian manifold and Sasakian 3-structure metric manifold.

Adati and A. Kandatu (1980) studied hypersurfaces of P-Sasakian manifolds and manifolds admitting a concircular vector field. T. Miyazawa (1980) have been studied an invariant submanifold V immersed in this almost paracontact Riemannian manifold and showed that the V admits either an almost paracontact Riemannian structure or an

almost product Riemannian structure (ϕ,g) excepting the case where ϕ is trivial.

Zbigniew Olszak (1981) studied the structure of almost Cosymplectic manifolds. He also studied some examples, basic definition and some preliminary properties of an almost Cosymplectic structure. Mishra (1981) defined a conformal K-contact Riemannian manifold as an almost Sasakian manifold with structure tensor (ϕ,ξ,η,g) in which the associated vector field ξ is a conformal killing vector field, that is, $(\nabla_X\eta)Y + (\nabla_Y\eta)X = 2\rho g(X,Y)$, where ∇ is the Riemannian connection, X,Y are arbitrary vector fields and ρ is a scalar. R.S. Mishra (1981) has studied structures in submanifolds of an almost Hermite manifold and proved that the hypersurface of an almost Kaehler manifold is quasi Sasakian manifold. T. Hasanis (1981) has studied necessary and sufficient condition for the hypersurface to be minimal.

M. Kobayashi (1981) has initiated a new class of submanifolds of a Sasakian manifold and introduced the concept of ξ-horizontal and ξ-vertical CR-submanifolds of a Sasakian manifold and gave their basic properties. He also studied the D-parallel normal section and (D,μ) flat normal connection of a CR-submanifold of a Sasakian manifold. Some

kinds of sectional curvatures, the Ricci tensor the scalar curvature of a CR-submanifold of a Sasakian space form were examined.

M. Kobayashi (1982) studied CR-submanifolds of a Sasakian space form with flat normal connection. He also studied the integrability of horizontal and vertical distribution on CR-submanifolds of a nearly Sasakian manifold. Further he studied contact CR-products of Sasakian manifolds.

K. Yano and M. Kon (1983) studied generic submanifolds of a Kaehlerian product manifolds. Bucki (1984) studied invariant submanifolds of an almost r-paracontact manifold. H. Endo (1985) studied invariant submanifolds in an almost Cosymplectic manifold.

S. Deshmukh, A. Sharfuddin and S.J. Husain (1986) studied invariant and non-invariant hypersurfaces of almost para contact metric manifold be an almost product metric manifold and an invariant hypersurface of a para contact metric (para Sasakian) manifold. M. Kobayashi (1986) studied semi-invariant submanifolds of a certain class of almost contact manifold and obtained many important results. Gupta and Prasad (1986) defined and studied almost para r-contact structure manifold. They also defined the Nijenhuis tensor of this structure and almost para r-contact Riemannian manifold and established some results.

M.D. Upadhyay and C.S. Prasad (1987) obtained integrability condition of an almost para contact structure. S.I. Husain and S. Deshmukh (1987) initiated the study of CR-submanifolds of a nearly Kaehler manifold and obtained their fundamental properties. The conditions under which the distributions required by CR-submanifolds are integrable were obtained. Mixed totally geodesic D-totally geodesic and $D^{\perp}$-totally geodesic CR-submanifolds have been studied as well as totally umbilical CR-submanifolds under some condition have been shown to be totally geodesic.

R. Niwas, R. Singh (1988) studied almost r-contact structure manifolds. They considered r-linearly independent C^{∞} vector fields U_x and $r(C^{\infty})$ 1-forms u^x, r some finite integer and studied the notion of almost r-contact structure.

Kalpana (1989) obtained some of the properties of an almost semi-invariant submanifold of an SP-Sasakian manifold. The integrability conditions of the distribution D, $D^{\perp}$, $\bar{D}$, $D\oplus\{\xi\}$, $D^{\perp}\oplus\{\xi\}$ and $\bar{D}\oplus\{\xi\}$ have also been studied. B. Ravi and C S. Bagewadi (1989) have obtained some conditions for the invariant submanifold of a conformal K-contact Riemannian manifold to be minimal and totally geodesic. D.E. Blair

(1990) showed non existence of 4-dimensional almost Kaehler manifold of constant curvature.

R.S. Mishra (1991) studied an almost contact metric manifold and discovered three new manifold namely, generalized quasi-Sasakian manifold of first kind, almost contact metric pseudo-normal manifold, quasi-normal manifold. Hasan Shahid (1991) studied CR-submanifolds of a trans Sasakian manifold that generalizes both α-Sasakian and β-Kenmotsu structure. He obtained the integrability conditions for distributions naturally arise in CR-submanifold. Further in 1994 he studied geometry of leaves on CR-submanifolds of a trans Sasakian manifold and also CR-submanifolds with parallel structures. Sinha and Yadav (1991) studied results on semi-invariant submanifolds of a certain class of almost contact manifold, in which they studied the $(D,D^{\perp})$ geodesic semi-invariant submanifolds of Kenmotsu manifold.

Hasan Shahid (1992) studied CR-submanifolds of Kaehlerian product manifold. Sinha and Srivastava (1992) studied submanifolds of a Kenmotsu manifold to be a semi-invariant submanifolds in particular case when it is a Kenmotsu space form of constant ϕ-holomorphic sectional curvature. Tripathi and Singh (1992) studied some properties of submanifolds of an almost r-para contact Riemannian manifold of s-para contact type. They proved result concerning non-existence of an anti-

invariant distributionon the submanifold of an almost r-para contact Riemannian manifold of P-Sasakian type or SP-Sasakian type.

Hasan Shahid (1993) studied semi-invariant submanifold of a nearly Sasakian manifold and obtained some basic lemmas and properties of parallel horizontal distribution and also obtained the integrability condition of distributions involved in semi-invariant submanifold. Das Lovejoy (1993) studied a differentiable manifold satisfying an algebraic structure $X = a^2X + \sum A_p(X)T^p$, which is generalization of almost r-contact hyperbolic structure. C. S. Prasad (1993) obtained integrability condition of a manifold admitting almost para r-contact structure. Prasad and Ojha (1993) studied CR-submanifolds of trans Sasakian manifolds and obtained integrability conditions of horizontal and vertical distributions. Maeda, Ohnita and Udagawa (1993) studied slant immersions into Kaehler manifolds, in which they studied slant submanifolds in complex projective space and obtained some sufficient conditions for an isometric immersion of a compact Kaehler manifold into a Kaehler manifold to be slant. Mishra (1993) studied the hypersurfaces of almost Hermite manifolds in which some of new manifolds have been obtained geometrically as the hypersurfaces of almost Hermite manifolds. In 1995 he studied normality of the hypersurfaces of almost Hermite manifolds.

Dutta (1994) studied (1,2) symplectic structures, nearly Kaehler structures and S^6 and obtained exact relations between the Hermitian, the symplectic and nearly Kaehler structures. Mishra (1994) studied anti-invariant submanifolds of a Kenmotsu manifold. Sinha and Yadav (1994) studied semi-invariant submanifolds of a Kenmotsu manifold, in which they obtained the integrability conditions of distributions and the geometry of leaves on a semi-invariant submanifolds in a Kenmotsu manifold.

Tripathi (1995) studied almost semi-invariant $\xi^{\perp}$ submanifolds of trans Sasakian manifolds. He proved that an almost semi-invariant $\xi^{\perp}$ submanifold of a normal almost contact metric (and hence trans Sasakian) manifold with proper invariant distribution always possesses a CR-structure. He also obtained integrability conditions for certain natural distributions on almost semi-invariant $\xi^{\perp}$ submanifolds. He showed that the anti-invariant distribution of almost semi-invariant $\xi^{\perp}$ submanifold of a trans Sasakian manifold is always integrable. The author also showed the non existence of proper mixed foliated semi-invariant submanifolds of Sasakian manifolds. Gadea and Oubina (1995) studied homogeneous almost para Hermitian structures. They gave classification of

homogeneous almost para Hermitian structures by using their results for the almost para Hermitian case.

Dube (1996) defined para Kenmotsu manifold and studied different curvature tensors on para Kenmotsu manifold and also the property of parallel π null planes. Tripathi, Singh (1996) studied almost semi-invariant submanifolds of a manifold with a ε-framed metric structure. They established some necessary and sufficient conditions for a submanifold to be an almost semi-invariant submanifold. He also obtained some necessary and sufficient conditions for submanifolds of an s-manifold to be f-invariant, anti-invariant and totally geodesic. Tripathi (1996) studied almost semi-invariant submanifolds of trans Sasakian manifolds, in which he obtained some basic results concerning submanifolds of almost contact metric manifolds and trans Sasakian manifolds and defined almost semi-invariant submanifold of an almost contact metric manifold. He also studied some properties and integrability conditions for certain distribution on almost semi-invariant submanifolds of trans Sasakian manifolds. Totally umbilical, minimal and totally geodesic submanifolds have been studied. Tripathi (1996) studied CR-submanifolds of generalized complex space form. The author obtained some necessary and sufficient conditions for submanifolds of a generalized complex space form to be CR-submanifolds.

Tripathi (1997) studied generic submanifolds of generalized complex space forms, where such submanifolds generalize holomorphic, totally real, slant CR-antiholomorphic, f-generic, generalized CR-submanifolds and skew CR-submanifolds. The author obtained necessary and sufficient conditions for integrability of totally real distribution. He also studied almost semi-invariant submanifold of a parallel ε-framed metric manifold and obtained some necessary and sufficient conditions for certain distributions on the submanifolds to be integrable. Totally umbilical and totally geodesic almost semi-invariant submanifolds are also studied. G. Zhen, J.L. Cabrerizo, L. M. Fernandrz and M. Fernandez (1997) studied the notion of ξ-conformally flat on a contact metric structure and it is proved that any K-contact metric manifold is ξ-conformally flat if and only if it is η-Einstein Sasakian manifold.

Pandey and Verma (1999) have studied some properties of para Sasakian manifold satisfying $R(X,Y)R = 0$ and $R(X,Y)C = 0$, where $R(X,Y)$ is the endomorphism of the Riemannian curvature tensor R and $C(X,Y,Z)$ is conformal curvature tensor. They have proved that if in a P-Sasakian manifold $R(X,Y)C = 0$, then either it is SP-Sasakian manifold or the structure vector ξ normal to $T(M^n)$. Bhatt, Joshi and Dube (1999) have studied non-invariant hypersurface of Kenmotsu manifold and

obtained some interesting results. Prasad and Bagewadi (1999) obtained necessary and sufficient condition for a semi-invariant submanifold of a Kenmotsu manifold to be ξ-horizontal and ξ-vertical in terms of integrability of distributions D and $D^{\perp}$.

Jong Tock Cho (2000) have given a classification of real hypersurfaces of a complex hyperbolic space $\not\subset H^n$ satisfying a point wise nullity condition for the structure vector field ξ, that is $k\{\eta(Y)X - \eta(X)Y\}$, k characterize a horo sphere in $\not\subset H^n$ by the condition $R(X,Y)\xi = \{\eta(Y)X - \eta(X)Y\}$,that is k = 1. Prasad (2000) obtained certain results on P-Sasakian manifolds. Pujar (2000) introduced a P-structure on a differentiable manifold of dimension n that is a tensor field of type (1,1) satisfying $P^3 - P = 0$. Further if there exists a P-structure of rank (n-1), then it is shown that there exists a Riemannian metric g such that $g(PX,PY) = g(X,Y) + \eta^x(X)\eta^y(Y)$ and $g(PX,Y) = g(X,PY)$, where η^x and η^y $(x,y = r+1, r = 2,\ldots,n)$ are the linearly independent covariant vectors (that is 1-form) corresponding to the distribution D_T of dimension n-r with respect to the operator T and r is the dimension of the distribution D_L (orthogonal to D_T) with respect to the operator L. It is also shown that CR-submanifolds of a Lorentzian para Sasakian manifold carry the P-structure. M.M. Tripathi (2000) has studied CR-submanifold of nearly Cosymplectic and Closely Cosymplectic

manifolds. Properties such as integrability autoparallelness of certain distributions on the submanifold were studied. Totally umbilical, totally contact umbilical totally geodesic and totally contact geodesic submanifolds were also studied. Shukla and Tripathi (2000) studied the hypersurfaces of a Riemannian manifold equipped with a Kenmotsu 3-structure. In 2001 they studied generic submanifolds of a Riemannian manifold equppied with a Kenmotsu 3-structure.

I.K. Mishra and R.H.Ojha (2001) have been introduced D-conformal transformation in a SP-Sasakian manifold. Some properties of curvature tensors are also obtained. M.Tarafdar and A. Bhattacharyya (2001) have studied Lorentzian para Sasakian (LP-Sasakian) manifolds with conformally flat and quasi conformally flat curvature tensor. It was shown that in both the cases, the manifold is locally isometric with an unit sphere $S^n(1)$. Further, it was shown that a LP-Sasakian manifold with $R(X,Y)C = 0$ was locally isometric with a unit sphere $S^n(1)$. Joshi and Dube (2001) defined and studied semi-invariant submanifold of an almost r-contact hyperbolic metric manifold and obtained some results. A.K. Sengupta and U.C. De (2001) studied the generalized CR-submanifolds of a trans Sasakian manifold and obtained their basic properties. The conditions of integrability of the distributions of generalized CR-submanifold were also obtained.

MATERIALS AND METHODS

Chapter 3 *MATERIALS AND METHODS*

This chapter is divided into 5 sections. Section 1 is divided in two subsections 1(A) and 1(B). We have studied CR-submanifolds of a trans para Sasakian manifold and CR-submanifold of nearly and closely para Cosymplectic manifold in section 1(A) and 1(B) respectively. Further section 2 is divided in two subsections 2(A) and 2(B). We have studied semi-invariant submanifolds of a nearly r-Sasakian manifold and semi-invariant submanifolds of r-Kenmotsu manifold in section 2(A) and 2(B) respectively. Section 3 is divided in two subsections 3(A) and 3(B). Section 3(A) and 3(B) is devoted to CR-submanifolds of a trans hyperbolic Sasakian manifold I and CR-submanifolds of a trans hyperbolic Sasakian manifold II respectively. Section 4 is divided in two subsections 4(A) and 4(B). We have studied semi-invariant submanifolds of para Kenmotsu manifold I and semi-invariant submanifolds of para Kenmotsu manifold II respectively in section 4(A) and 4(B). Section 5 is devoted to the study of hypersurfaces of almost hyperbolic Hermitian manifolds.

3.1 SECTION 1(A):

ON CR-SUBMANIFOLDS OF A TRANS PARA SASAKIAN MANIFOLD

Let $\bar{M}$ be an n-dimensional almost para contact metric manifold with structure tensors (F,U,u,g).

An almost para contact metric structure (F,U,u,g) on $\bar{M}$ is called trans para Sasakian if

(3.1.1)(A) $(\bar{\nabla}_X F)(Y) = \alpha(g(X,Y)U - u(Y)X) + \beta(g(FX,Y)U - u(Y)FX)$

for α,β non zero constant, then it is called trans para Sasakian structure is of type (α,β).

From the above equation (3.1.1)(A), we have

(3.1.2)(A) $\bar{\nabla}_X U = \alpha FX+\beta(X - u(X)U)$.

A submanifold M of $\bar{M}$ is called a CR-submanifold if U is tangent to M and there exist on M a differentiable distribution D: $x \to D_x \subset T_xM$ satisfying the following conditions:

(i) D_x is invariant under F that is $FD_x \subset D_x$ for each $x \in M$,

(ii) the complimentary orthogonal distribution $D^{\perp}$: $x \to D_x^{\perp} \subset T_xM$ is totally real under F, that is $FD_x^{\perp} \subset T_x^{\perp}M$ for each $x \in M$, where T_xM and $T_x^{\perp}M$ are tangent and normal space of M at x respectively.

M is an invariant (resp. anti-invariant) submanifold of $\bar{M}$ when $\dim D^{\perp} = 0$ (resp. $\dim D = 0$), where D (resp.$D^{\perp}$) is the horizontal (resp. vertical) distribution. The pair $(D,D^{\perp})$ is called U-horizontal (U-vertical) if $U_x \in D_x$ (resp.$U_x \in D_x^{\perp}$) for each $x \in M$.

For a vector field X tangent to M, we put

(3.1.3)(A) $X = PX+QX$,

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. Also for a vector field N normal to M, we put

(3.1.4)(A) $FN = BN+CN$,

where BN (resp. CN) denotes the tangential (resp. normal) component of FN.

The Gauss and Weingarten formulas are given by

(3.1.5)(A) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$, $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

$X,Y \in TM$, $N \in T^{\perp}M$,

where $\nabla^{\perp}$ is the normal connection, h (resp.A) is the second fundamental form (resp. tensor) of M in $\bar{M}$ satisfying

(3.1.6)(A) $g(A_N X,Y) = g(h(X,Y),N)$.

If we denote the orthogonal component of $FD^{\perp}$ in $T^{\perp}M$ by μ, then we have

$T^{\perp}M = FD^{\perp} \oplus \mu$, it is obvious that $F\mu = \mu$.

Lemma 3.1.1(A): Let M be a CR-submanifold of a trans para Sasakian manifold $\bar{M}$. Then we have

(3.1.7)(A) $P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + \alpha g(X,Y)PU + \beta g(FPX,Y)PU$

$$-\alpha u(Y)PX - \beta u(Y)FPX,$$

(3.1.8)(A) $Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) + (\alpha g(X,Y) + \beta g(FQX,Y))QU$

$$-\alpha u(Y)QX,$$

(3.1.9)(A) $h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y) - \beta u(Y)FQX,$

for any $X,Y \in TM$.

Proof: From equations (3.1.1)(A), (3.1.3)(A), (3.1.4)(A) and (3.1.5)(A), we have

$$\bar{\nabla}_X FX - F\bar{\nabla}_X Y = \alpha(g(X,Y)U - u(Y)X) + \beta(g(FX,Y)U - u(Y)FX)$$

or $\nabla_X FPY + h(X,FPY) + \nabla_X^{\perp} FQY - A_{FQY}X - F\nabla_X Y - Fh(X,Y)$

$$= \alpha g(X,Y)PU + \alpha g(X,Y)QU - \alpha u(Y)PX - \alpha u(Y)QX + \beta g(FPX + FQX,Y)U$$

$$-\beta u(Y)FPX - \beta u(Y)FQX,$$

or $P\nabla_X FPY + Q\nabla_X FPY + h(X,FPY) + \nabla_X^{\perp} FQY - PA_{FQY}X - QA_{FQY}X$

$$= FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y) + \alpha g(X,Y)PU + \alpha g(X,Y)QU$$

$$-\alpha u(Y)PX - \alpha u(Y)QX + \beta g(FPX,Y)PU + \beta g(FQX,Y)QU - \beta u(Y)FPX$$

$$-\beta u(Y)FQX.$$

Now equating the horizontal, vertical and normal components we get the required results, which proves our lemma.

The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 3.1.1(A): Let M be a U-horizon~~tal~~ CR-submanifold of a trans para Sasakian manifold $\bar{M}$. Then the distribution D is parallel if and only if

(3.1.10)(A) $h(X,FY) = h(FX,Y) = Fh(X,Y)$, for all $X, Y \in D$.

Proof: parallel distribution is involutive, that is

(3.1.11)(A) $h(X,FY) = h(FX,Y)$, for all $X, Y \in D$.

From (3.1.9)(A) and (3.1.11)(A), we have

(3.1.12)(A) $h(X,FY) = Ch(X,Y)$.

Also $\nabla_X FY \in D$, $\nabla_Y FX \in D$, $\forall$ $X, Y \in D$, so from (3.1.8)(A) and using D-parallelness, we get $Bh(X,Y) = 0$, $\forall$ $X, Y \in D$.

From (3.1.4)(A), we get

$$Fh(X,Y) = Bh(X,Y) + Ch(X,Y).$$

From (3.1.12)(A), $Bh(X,Y) = 0$ and the above equation, we get

$$Fh(X,Y) = Ch(X,Y) = h(X,FY), \forall X, Y \in D,$$

which proves (3.1.10)(A). This completes the proof of the theorem.

A CR-submanifold M of a trans para Sasakian manifold $\bar{M}$ is said to be mixed totally geodesic if $h(X,Y) = 0$, for $X \in D$ and $Y \in D^{\perp}$.

A CR-submanifold M is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$.

A normal vector field $N \neq 0$ is D-parallel normal section on M if $\nabla_X^{\perp} N = 0$, for all $X \in D$.

Proposition 3.1.2(A): Let M be a mixed totally geodesic U-vertical CR-submanifold of a trans para Sasakian manifold $\bar{M}$. Then the normal section $N \in FD^{\perp}$ is a D-parallel if and only if $\nabla_X FN \in D$, for all $X \in D$.

Proof: Let $N \in FD^{\perp}$ and as M be a mixed totally geodesic, we have

$$\nabla_X(FN) = \bar{\nabla}_X(FN)$$

or $$\nabla_X(FN) = (\bar{\nabla}_X F)N + F\bar{\nabla}_X N$$

(3.1.13)(A) $$\nabla_X(FN) = F\nabla_X^{\perp} N - A_N X.$$

Let normal section be D-parallel means $\nabla_X^{\perp} N = 0$. Let we have $A_N X \in D$ and $\nabla_X^{\perp} N = 0$ then from equation (3.1.13)(A), we get $\nabla_X FN \in D$, for all $X \in D$.

Conversely, we have $A_N X \in D$ and $\nabla_X FN \in D$, then from (3.1.13)(A), we get $\nabla_X^{\perp} N = 0$, for all $X \in D$.

This implies that normal section N is D-parallel.

This proves our assertion.

Lemma 3.1.2(A): Let M be a CR-submanifold of a trans para Sasakian manifold $\bar{M}$. Then we have

(3.1.14)(A) $A_{FY}Z-A_{FZ}Y+\alpha(u(Z)Y-u(Y)Z) = FP[Y,Z]$, for any $Y,Z\in D^{\perp}$.

Proof: We have

$$\bar{\nabla}_Y FZ = (\bar{\nabla}_Y F)(Z)+F\bar{\nabla}_Y Z.$$

Using (3.1.1)(A) in the above equation, we get

$$\bar{\nabla}_Y FZ = \alpha(g(Y,Z)U-u(Z)Y)+\beta(g(FY,Z)U-u(Z)FY)+F\nabla_Y Z+Fh(Y,Z)$$

$$= \alpha(g(Y,Z)U-u(Z)Y)+FP\nabla_Y Z+FQ\nabla_Y Z+Bh(Y,Z)+Ch(Y,Z)$$

$$-\beta u(Z)FQY.$$

In view of (3.1.5)(A) and the above equation, we have

(3.1.15)(A) $-A_{FZ}Y+\nabla_Y^{\perp}FZ=\alpha(g(Y,Z)U-u(Z)Y)-\beta u(Z)FQY+FP\nabla_Y Z$

$+FQ\nabla_Y Z+Bh(Y,Z) +Ch(Y,Z)$, for all $Y,Z\in D^{\perp}$.

From (3.1.9)(A), for all $Y,Z\in D^{\perp}$, we obtain

(3.1.16)(A) $\nabla_Y^{\perp}FZ = FQ\nabla_Y Z + Ch(Y,Z) + \beta u(Z)FQY$.

Now from (3.1.15)(A) and (3.1.16)(A), we get

$$FP\nabla_Y Z = -A_{FZ}Y - \alpha g(Y,Z)U+\alpha u(Z)Y - Bh(Y,Z).$$

Similarly we have

$$FP\nabla_Z Y = -A_{FY}Z - \alpha g(Y,Z)U + \alpha u(Y)Z- Bh(Y,Z).$$

Thus from the above two equations, we get

$$FP[Y,Z] = A_{FY}Z - A_{FZ}Y + \alpha(u(Z)Y - u(Y)Z), \text{ for all } Y,Z \in D^{\perp}.$$

This completes the proof of the theorem.

A CR-submanifold of a trans para Sasakian manifold $\bar{M}$ is called D-umbilic (resp. $D^{\perp}$ umbilic) if $h(X,Y) = g(X,Y)H$ holds for all $X,Y \in D$ (resp.$X,Y \in D^{\perp}$), where H is a mean curvature vector field.

3.2 SECTION 1(B):

ON CR–SUBMANIFOLD OF NEARLY AND CLOSELY PARA COSYMPLECTIC MANIFOLDS

Let $\bar{M}$ be an almost para contact metric manifold with structure tensors (F,U,u,g).

An almost para contact metric manifold is called a para Cosymplectic manifold if [30]

(3.1.1)(B) $(\nabla_X F)Y = 0.$

An almost para contact metric manifold is called nearly para Cosymplectic if F is a killing, that is,

(3.1.2)(B) $(\bar{\nabla}_X F)Y + (\bar{\nabla}_Y F)X = 0,$

where $\bar{\nabla}$ is the operator of covariant differentiation with respect to g. On nearly para Cosymplectic manifold, U is a killing vector field. That is,

(3.1.3)(B) $g(\bar{\nabla}_X U,Y) + g(\bar{\nabla}_Y U,X) = 0, \quad \forall, X,Y \in T\bar{V}.$

An almost para contact metric manifold is called closely para Cosymplectic if F is a killing and u is a closed. On closely para Cosymplectic manifold we have

(3.1.4)(B) $\bar{\nabla}_U F = 0, \quad \bar{\nabla} U = 0, \quad \bar{\nabla} u = 0.$

Let M be a submanifold of a Riemannian manifold $\bar{M}$ with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

(3.1.5)(B) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in TM,$

(3.1.6)(B) $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad N \in T^{\perp}M,$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\bar{M}$, M and the normal bundle $T^{\perp}M$ of M respectively and h is the second fundamental form related to A by

(3.1.7)(B) $g(h(X,Y),N) = g(A_N X,Y).$

F is a (1,1) tensor field on M, for $X \in TM$ and $N \in T^{\perp}M$ we have [102]

(3.1.8)(B) $(\bar{\nabla}_X F)Y = ((\nabla_X P)Y - A_{QY}Y - th(X,Y)) + ((\nabla_X Q)Y + h(X,PY) - fh(X,Y)),$

(3.1.9)(B) $(\bar{\nabla}_X F)N = ((\nabla_X t)N - A_{FN}Y - PA_N X) + ((\nabla_X f)N + h(X,tN) + h(X,tN)),$

where,

(3.1.10)(B) $FX \equiv PX + QX, \quad PX \in TM, \quad QX \in T^{\perp}M$

(3.1.11)(B) $FN \equiv tN + fN, \quad tN \in TM, \quad fN \in T^{\perp}M,$

here PX and tN are tangential parts, while QX and fN are normal parts of FX and FN respectively.

Also we have

(3.1.12)(B) $(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y$,

(3.1.13)(B) $(\nabla_X Q)Y \equiv \nabla_X^{\perp} QY - Q\nabla_X Y$,

(3.1.14)(B) $(\nabla_X t)N \equiv \nabla_X tN - t\nabla_X^{\perp} N$,

(3.1.15)(B) $(\nabla_X f)N \equiv \nabla_X^{\perp} fN - f\nabla_X^{\perp} N$.

The submanifolds M is said to be totally geodesic in $\bar{M}$ if h =0 and totally umbilical in $\bar{M}$ if

$h(X,Y) = g(X,Y)K$.

For a distribution D on M, M is said to be D-totally geodesic if $h(X,Y)=0$, $\forall$ $X,Y \in D$. M is said to be D-totally umbilical if we have $h(X,Y) = h(X,Y)K$, where K is a normal vector field $\forall$ $X,Y \in D$. M is said to be (D,E)-mixed totally geodesic if $h(X,Y) = 0$, $\forall$ $X \in D$ and $Y \in E$.

Let D and E be two distributions defined on a manifold M. D is said to be E-parallel if we have $\nabla_X Y \in D$, $\forall$ $X \in E$ and $Y \in D$. If D is D-parallel then it is called autoparallel. D is said to be X-parallel if we have $\nabla_X Y \in D$, $\forall$ $X \in TM$ and $Y \in D$. D is said to be parallel if $\forall$ $X \in TM$ and $Y \in D$, $\nabla_X Y \in D$.

If a distribution D on M is autoparallel then it is integrable, and by Gauss formula D is totally geodesic in M. If D is parallel then the orthogonal complementary distribution $D^{\perp}$ is also parallel.

A submanifold M of an almost para contact metric manifold $\bar{M}$ with $U \in TM$ is called a CR-submanifold of $\bar{M}$ if for each $x \in M$, $T_xM = D_x^1 \oplus D_x^0 \oplus \{U\}_x$, where,

$D_x^1 = Ker(Q|_{\{U\}}^{\perp})_x = \{ X_x \in \{U\}_x^{\perp} : ||X_x|| = ||PX_x|| \} = T_xM \cap F(T_xM)$,

$D_x^0 = Ker(P|_{\{U\}}^{\perp})_x = (X_x \in \{U\}_x^{\perp} : ||X_x|| = ||QX_x|| \} = T_xM \cap F(T_x^{\perp}M)$.

The condition $T_xM = D_x^1 \oplus D_x^0 \oplus \{U\}_x$ implies that $P^3 - P = 0$ [74] on M and hence

$Dim(D_x^1) = Rank(P_x)$ is independent of $x \in M$ and so is that D_x^0.

Now we have $TM = D^1 \oplus D^0 \oplus \{U\}$.

These distributions are also differentiable. We have $T^{\perp}M = \bar{D}^1 \oplus \bar{D}^0$,

where $\bar{D}^1 = Ker(t) = T^{\perp}M \cap F(T^{\perp}M)$, $\bar{D}^0 = Ker(f) = T^{\perp}M \cap F(TM)$,

$QD^0 = \bar{D}^0$ and $t\ \bar{D}^0 = D^0$.

A CR-submanifold of an almost para contact manifold reduces to invariant submanifold (resp. anti-invariant submanifold [4, 111]) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$).

Let M be a submanifold of a nearly para Cosymplectic manifold, tangent to U. By virtue of equation (3.1.2)(B) and equation (3.1.8)(B) we have

(3.1.16)(B) $((\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2th(X,Y)) + ((\nabla_X Q)Y$

$+ (\nabla_Y Q)X + h(X,PY) + h(PX,Y) - 2fh(X,Y)) = 0.$

Proposition 3.1.1(B): Let M be a submanifold of a nearly para Cosymplectic manifold $\bar{M}$. If $U \in TM$ then $\forall$ $X,Y \in TM$ we have

(3.1.17)(B) $(\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2th(X,Y) = 0,$

(3.1.18)(B) $(\nabla_X Q)Y + (\nabla_Y Q)X + h(X,PY) + h(PX,Y) - 2fh(X,Y) = 0.$

Proof: Equating tangential and normal parts of equation (3.1.16)(B), we get the desired results

Proposition 3.1.2(B): Let M be a submanifold of a nearly para Cosymplectic manifold $\bar{M}$. If $U \in TM$ then $\forall$ $X,Y \in TM$ we have

(3.1.19)(B) $\bar{\nabla}_X FY + \bar{\nabla}_Y FX - F[X,Y] = 2((\nabla_X P)Y - A_{QY}Y - th(X,Y))$

$+2((\nabla_X Q)Y + h(X,PY) - fh(X,Y)).$

The proof is obvious and hence omitted.

Proposition 3.1.3(B): Let M be a submanifold of a nearly para Cosymplectic manifold. Then (P,U.u,g) is a nearly para Cosymplectic structure on the distribution $D^1\oplus\{U\}$ if $th(X,Y)=0\ \forall\ X,Y \in D^1\oplus\{U\}$.

Proof: Using $D^1\oplus\{U\} = Ker(Q)$ and $P^2 + tQ = I - u\otimes U$ we obtain $P^2 = I - u\otimes U$ on $D^1\oplus\{U\}$. We also get $PU = 0$, $u(U) = 1$, $uoP= 0$. Using $D^1\oplus\{U\}= Ker(Q)$ and $th(X,Y)=0$ in equation (3.1.17)(B) we have $(\nabla_X P)Y + (\nabla_Y P)X =0$, $\forall\ X,Y \in D^1\oplus\{U\}$.

This proves our assertion.

Let M be a Riemannian manifold with a Riemannian connection ∇. A distribution D on M is said to be nearly autoparallel if $\forall\ X,Y \in D$ we have $(\nabla_X Y + \nabla_Y X) \in D$ or equivalently $\nabla_X X \in D$.

We have

Parallel $\Rightarrow$ Autoparallel $\Rightarrow$ Nearly autoparallel,

Parallel $\Rightarrow$ Integrable,

Autoparallel $\Rightarrow$ Integrable, and

Nearly autoparallel + Integrable $\Rightarrow$ Autoparallel.

Lemma 3.1.1(B): Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$. Then

(3.1.20)(B) $3(A_{QX}Y - A_{QY}X) = P[X,Y], \quad X,Y \in D^0\oplus\{U\}$.

Proof: Let $X,Y \in D^0 \oplus \{U\}$ and $Z \in TM$. We have from equation (3.1.5)(B) and (3.1.6)(B)

$$-A_{FX} Z+\nabla_Z^{\perp} FX= \bar{\nabla}_Z FX=(\bar{\nabla}_Z F)X+F \bar{\nabla}_Z X=- (\bar{\nabla}_X F)Z+ F\nabla_Z X + Fh(Z,X),$$

so that,

$$Fh(Z,X) = -A_{FX} Z + \nabla_Z^{\perp} FX - F\nabla_Z X + (\bar{\nabla}_X F)Z,$$

and hence

$$g(Fh(Z,X),Y) = -g(A_{FX} Z,Y)+g((\bar{\nabla}_X F)Z,Y) = g(A_{FX} Y,Z)+g((\bar{\nabla}_X F)Y,Z).$$

Now we have

$$g(Fh(Z,X),Y) = g(h(Z,X),FY) = g(A_{FY}X,Z).$$

Thus from the above two equations, we have

(3.1.21)(B) $g(A_{FY}X,Z) = g(A_{FX} Y,Z) + g((\bar{\nabla}_X F)Y,Z).$

Now for $X,Y \in D^0 \oplus \{U\}$, we have

$$\bar{\nabla}_X FY- \bar{\nabla}_Y FX = A_{FX}Y - A_{FY}X + \nabla_X^{\perp} FY - \nabla_Y^{\perp} FX$$

and $\bar{\nabla}_X FY - \bar{\nabla}_Y FX = (\bar{\nabla}_X F)Y - (\bar{\nabla}_Y F)X + F[X,Y]$

from above two equations, we get

$$(\bar{\nabla}_X F)Y - (\bar{\nabla}_Y F)X = A_{FX}Y - A_{FY}X + \nabla_X^{\perp} FY - \nabla_Y^{\perp} FX- F[X,Y].$$

Using equation (3.1.2)(B) and above equation, we get

$$(\bar{\nabla}_X F)Y = ½(A_{FX}Y - A_{FY}X + \nabla_X^{\perp} FY - \nabla_Y^{\perp} FX - F[X,Y]).$$

From the above equation and equation (3.1.21)(B) we get the required result.

Lemma 3.1.2(B): Let M be a submanifold of a closely para Cosymplectic manifold $\bar{M}$, tangent to U. Then the integral curve of U in M is geodesic in M, and U is an asymptotic direction.

Proof: Since in a closely para Cosymplectic manifold we have $\bar{\nabla}U = 0$. Now in view of equation (3.1.5)(B), we get $h(U,U) = 0$. This completes the proof.

Proposition 3.1.4(B): Let D be a distribution on a submanifold M of a closely para Cosymplectic manifold such that $U \in TM$. If M is D-totally umbilical then M is D-totally geodesic.

Proof: For D-totally umbilical we have

$$h(X,Y) = g(X,Y)\, K, \ \forall\, X,Y \in D.$$

A direction U at a point of M is an asymptotic direction if normal vector field $K = 0$, which implies that $h(X,Y) = 0$, which shows that M is D-totally geodesic.

Proposition 3.1.5(B): Every totally umbilical submanifold M of a closely para Cosymplectic manifold $\bar{M}$, tangent to U, is totally geodesic.

Proof: The proof follows from proposition (3.1.4)(B).

Let M be a submanifold of an almost para contact metric manifolds, tangent to U.

In this case $TM = \{U\} \oplus \{U\}^{\perp}$, where $\{U\}$ is the distribution spanded by $\{U\}$ and $\{U\}^{\perp}$ is the complementary orthogonal distribution of $\{U\}$ in M. This completes the proof of the theorem.

A submanifold M of an almost para contact metric manifold $\bar{M}$, tangent to U, is called

(1) totally para contact umbilical if it is $\{U\}^{\perp}$-totally umbilical, and

(2) totally para contact geodesic if it is $\{U\}^{\perp}$-totally geodesic.

The condition of totally para contact umbilical and totally para contact geodesic are respectively

(3.1.22)(B) $h(F^2X,F^2Y) = g(F^2X,F^2Y)\,K$, $X,Y \in TM$

(3.1.23)(B) $h(F^2X,F^2Y) = 0$, $X,Y \in TM$

where K is normal vector field.

Using (1.1.20) in (3.1.22)(B) and (3.1.23)(B),we get the following respectively.

(3.1.24)(B) $h(X,Y) = g(FX,FY)\,K + u(X)h(Y,U) + u(Y)h(X,U)$

(3.1.25)(B) $h(X,Y) = u(X)h(Y,U) + u(Y)h(X,U)$

3.3 SECTION 2(A):

ON SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY r-SASAKIAN MANIFOLDS

Let $\bar{M}$ be (2m+1)–dimensional almost r-contact metric manifold with structure tensors (F,U_p,u^p,g).

An almost r-contact metric structure (F,U_p,u^p,g) (where p=1,….,r) on $\bar{M}$ is called nearly r-Sasakian if

(3.2.1)(A) $(\bar{\nabla}_X F)(Y) + (\bar{\nabla}_Y F)(X) = u^p(Y)\,X + u^p(X)Y - 2g(X,Y)U_p,$

where $\bar{\nabla}$ denotes the Riemannian connection with respect to g. If $\bar{M}$ satisfies

(3.2.2)(A) $(\bar{\nabla}_X F)(Y) = u^p(Y)\,X - g(X,Y)U_p, \quad \bar{\nabla}_X U_p = FX,$

then the structure is called r-Sasakian. On a nearly r-Sasakian manifold $\bar{M}$, the r-vector fields U_p (where p = 1,…,r) is killing, that is,

(3.2.3)(A) $g(\bar{\nabla}_X U_p, Y) + g(\bar{\nabla}_Y U_p, X) = 0$, for each $X,Y \in T\bar{M}$.

An n-dimensional Riemannian submanifold M of a nearly r-Sasakian manifold $\bar{M}$ is called a semi-invariant submanifold if U_p is tangent to M and there exist on M a pair of orthogonal distribution $(D,D^{\perp})$ such that

(i) $TM = D \oplus D^{\perp} \oplus \{U_p\},$

(ii) the distribution D is invariant under F, that is $FD_x \subset D_x$, for all $x \in M$,

(iii) the distribution $D^{\perp}$ is anti-invariant under F, that is $F(D_x^{\perp}) \subset T_x^{\perp}M$, for all $x \in M$,

where T_xM and $T_x^{\perp}M$ are tangent space and normal space of M at x.

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp.vertical distribution).A semi-invariant submanifold M is said to be an invariant (resp. anti-invariant) submanifold if we have $D_x^{\perp} = \{0\}$ (resp.$D_x = \{0\}$) for each $x \in M$. We also call M proper if neither D nor $D^{\perp}$ is null.

We denote by g the metric tensor field of $\bar{M}$ as well as that induced on M. Let $\bar{\nabla}$ (resp. ∇) be covariant differentiation with respect to the Levi-Civita connection on $\bar{M}$ (resp. M). The Gauss and Weingarten formulas for M are respectively given by

(3.2.4)(A) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$, $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

for $X,Y \in TM, N \in T^{\perp}M$, where h(resp. A) is the second fundamental form (resp. tensor) of M in $\bar{M}$, and $\nabla^{\perp}$ denotes the operator of the normal connection. We have also

(3.2.5)(A) $g(h(X,Y),N) = g(A_N X,Y)$.

For a vector field X tangent to M, we put

(3.2.6)(A) $X = PX + QX + u^p(X)U_p$,

where P and Q are projection operators and PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any vector field N normal to M, we put

(3.2.7)(A) $FN = BN + CN$,

where B and C are projection operators and BN (resp.CN) denotes the tangential (resp.normal) component of FN.

A semi-invariant submanifold is said to be mixed totally geodesic if $h(X,Z)=0$ for all $X \in D$ and $Z \in D^{\perp}$.

Let us calculate the Nijenhuis tensor on a nearly r-Sasakian manifold $\bar{M}$.

(3.2.8)(A) $N_F(X,Y) = (\bar{\nabla}_{FX}F)(Y) - (\bar{\nabla}_{FY}F)(X) - F(\bar{\nabla}_X F)(Y) + F(\bar{\nabla}_Y F)(X)$,

for any $X,Y \in T\bar{M}$.

From (3.2.1)(A) putting FX in place of X, we have

$$(\bar{\nabla}_{FX}F)(Y) + (\bar{\nabla}_Y F)(FX) = u^p(Y)FX - 2g(FX,Y)U_p,$$

(3.2.9)(A) $(\bar{\nabla}_{FX}F)(Y) = u^p(Y)FX - 2g(FX,Y)Up - (\bar{\nabla}_Y F)(FX)$.

Now, we have

$$(\bar{\nabla}_Y F)(FX) = \bar{\nabla}_Y F^2 X - F\,\bar{\nabla}_Y FX$$

$$(\bar{\nabla}_Y F)(FX) = \bar{\nabla}_Y F^2 X - F(\bar{\nabla}_Y F)(X) - F(F\,\bar{\nabla}_Y X)$$

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$$(\bar{\nabla}_Y F)(FX) = -\bar{\nabla}_Y X + (\bar{\nabla}_Y u^p(X))U_p + u^p(X)\bar{\nabla}_Y U_p - F(\bar{\nabla}_Y F)(X) + \bar{\nabla}_Y X$$
$$-u^p(\bar{\nabla}_Y X)U_p,$$

(3.2.10)(A) $$(\bar{\nabla}_Y F)(FX) = (\bar{\nabla}_Y u^p(X))U_p + u^p(X)\bar{\nabla}_Y U_p - F(\bar{\nabla}_Y F)(X)$$
$$-u^p(\bar{\nabla}_Y X)U_p.$$

Using (3.2.10)(A) in equation (3.2.9)(A), we get

(3.2.11)(A) $$(\bar{\nabla}_{FX} F)(Y) = -2g(FX,Y)U_p + u^p(Y)FX + F(\bar{\nabla}_Y F)(X)$$
$$+u^p(\bar{\nabla}_Y X)U_p - u^p(X)\bar{\nabla}_Y U_p - (\bar{\nabla}_Y u^p(X))U_p.$$

Again using equation (3.2.11)(A) in equation (3.2.8)(A), we get

$$N_F(X,Y) = -2g(FX,Y)U_p + u^p(Y)FX + F(\bar{\nabla}_Y F)(X) + u^p(\bar{\nabla}_Y X)U_p - u^p(X)\bar{\nabla}_Y U_p$$
$$-(\bar{\nabla}_Y u^p(X))U_p + 2g(FY,X)U_p - u^p(X)FY - F(\bar{\nabla}_X F)(Y) - u^p(\bar{\nabla}_X Y)U_p$$
$$+u^p(Y)\bar{\nabla}_X U_p + (\bar{\nabla}_X u^p(Y))U_p - F(\bar{\nabla}_X F)(Y) + F(\bar{\nabla}_Y F)(X)$$

$$N_F(X,Y) = 2F((\bar{\nabla}_Y F)(X) - (\bar{\nabla}_X F)(Y)) + u^p(Y)FX - u^p(X)FY - u^p(X)\bar{\nabla}_Y U_p$$
$$+u^p(Y)\bar{\nabla}_X U_p - 4g(FX,Y)U_p + u^p(\bar{\nabla}_Y X)U_p - u^p(\bar{\nabla}_X Y)U_p$$
$$-\bar{\nabla}_Y u^p(X)U_p + \bar{\nabla}_X u^p(Y)U_p.$$

In view of (3.2.1)(A) and the above equation we get

$$N_F(X,Y) = 2F((\bar{\nabla}_Y F)(X) + (\bar{\nabla}_Y F)(X) - u^p(Y)X - u^p(X)Y + 2g(X,Y)U_p)$$
$$+u^p(Y)FX - u^p(X)FY - u^p(X)\bar{\nabla}_Y U_p + u^p(Y)\bar{\nabla}_X U_p - 4g(FX,Y)U_p$$
$$+2du(X,Y)U_p,$$

$N_F(X,Y)=4F(\bar{\nabla}_Y F)(X)-2u^p(Y)FX-2u^p(X)FY+u^p(Y)FX-u^p(X)FY-u^p(X)FY$

$+ u^p(Y)FX- 2g(FX,Y)U_p,$

(3.2.12)(A) $N_F(X,Y) = 4F(\bar{\nabla}_Y F)(X)-4u^p(X)FY-2g(FX,Y)U_p,$

for any $X,Y \in T\bar{M}$.

Lemma 3.2.1(A): Let M be a semi-invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then

$2(\bar{\nabla}_X F)(Y) = \nabla_X FY-\nabla_Y FX+h(X,FY)-h(Y,FX)-F[X,Y]-2g(X,Y)U_p,$

for any $X,Y \in D$.

Proof: In view of Gauss formula, we get

(3.2.13)(A) $\bar{\nabla}_X FY- \bar{\nabla}_Y FX = \nabla_X FY-\nabla_Y FX+h(X,FY)-h(Y,FX).$

Now on the other hand, we have

(3.2.14)(A) $\bar{\nabla}_X FY- \bar{\nabla}_Y FX = (\bar{\nabla}_X F)(Y)-(\bar{\nabla}_Y F)(X)+F[X,Y].$

From (3.2.13)(A) and (3.2.14)(A), we get

(3.2.15)(A) $(\bar{\nabla}_X F)(Y)-(\bar{\nabla}_Y F)(X)=\nabla_X FY-\nabla_Y FX+h(X,FY)-h(Y,FX)$

$-F[X,Y].$

On the other hand, from (3.2.1)(A) and in view of $u^p(X) = u^p(Y) = 0$ for $X,Y \in D$, we have

(3.2.16)(A) $(\bar{\nabla}_X F)(Y) + (\bar{\nabla}_Y F)(X) = - 2g(X,Y)U_p.$

Hence from (3.2.15)(A) and (3.2.16)(A) we get the result.

Lemma 3.2.2(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then

$$2(\bar{\nabla}_X F)(Y) = -A_{FY}X - \nabla_Y FX + \nabla_X^{\perp} FY - h(Y,FX) - F[X,Y] - 2g(X,Y)U_p,$$

for any $X \in D$ and $Y \in D^{\perp}$.

Proof: From Gauss and Weingarten formula, we get

(3.2.17)(A) $\bar{\nabla}_X FY - \bar{\nabla}_Y FX = -A_{FY}X + \nabla_X^{\perp} FY - \nabla_Y FX - h(Y,FX).$

On the other hand, we have

(3.2.18)(A) $\bar{\nabla}_X FY - \bar{\nabla}_Y FX = (\bar{\nabla}_X F)(Y) - (\bar{\nabla}_Y F)(X) + F[X,Y].$

From (3.2.17)(A) and (3.2.18)(A) we get

(3.2.19)(A) $(\bar{\nabla}_X F)(Y) - (\bar{\nabla}_Y F)(X) = -A_{FY}X + \nabla_X^{\perp} FY - \nabla_Y FX - h(Y,FX)$

$$-F[X,Y].$$

From (3.2.1)(A) and in view of $u^p(X) = u^p(Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$, we have

(3.2.20)(A) $(\bar{\nabla}_X F)(Y) + (\bar{\nabla}_Y F)(X) = -2g(X,Y)U_p.$

Thus from (3.2.19)(A) and (3.2.20)(A) we get the result.

Lemma 3.2.3(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then

(3.2.21)(A) $P\nabla_X FPY + P\nabla_Y FPX - PA_{FQY}X - PA_{FQX}Y$

$$= FP\nabla_X Y + FP\nabla_Y X + u^p(X)PY + u^p(Y)PX,$$

(3.2.22)(A) $Q\nabla_X FPY+Q\nabla_Y FPX-QA_{FQY}X-QA_{FQX}Y$

$$=2Bh(X,Y)+u^p(X)QY+u^p(Y)QX,$$

(3.2.23)(A) $h(X,FPY)+h(Y,FPX)+\nabla_X{}^{\perp}FQY+\nabla_Y{}^{\perp}FQX$

$$=2Ch(X,Y)+FQ\nabla_X Y+FQ\nabla_Y X,$$

(3.2.24)(A) $u^p(\nabla_X FPY+\nabla_Y FPX-A_{FQY}X-A_{FQX}Y) = -2g(FX,FY),$

for $X,Y\in TM$.

Proof: From equation (3.2.1)(A), (3.2.4)(A), (3.2.6)(A) and (3.2.7)(A), we have

$$\bar{\nabla}_X FPY+\bar{\nabla}_X FQY+\bar{\nabla}_Y FPX+\bar{\nabla}_Y FQX=F\bar{\nabla}_X Y+F\bar{\nabla}_Y X+u^p(Y)X+u^p(X)Y$$
$$-2g(X,Y)U_p,$$

$$\nabla_X FPY+h(X,FPY)-A_{FQY}X+\nabla_X{}^{\perp}FQY+\nabla_Y FPX+h(Y,FPX)-A_{FQX}Y+\nabla_Y{}^{\perp}FQX$$
$$=F\bar{\nabla}_X Y+F\bar{\nabla}_Y X+u^p(Y)X+u^p(X)Y-2g(X,Y)U_p, \text{ for all } X,Y\in TM$$

or, $P\nabla_X FPY+Q\nabla_X FPY+u^p(\nabla_X FPY)U_p+h(X,FPY)-PA_{FQY}X-QA_{FQY}X$

$-u^p(A_{FQY}X)U_p+\nabla_X{}^{\perp}FQY+P\nabla_Y FPX+Q\nabla_Y FPX+u^p(\nabla_Y FPX)U_p$

$+h(Y,FPX)-PA_{FQX}Y-QA_{FQX}Y-u^p(A_{FQX}Y)U_p+\nabla_Y{}^{\perp}FQX$

$=FP\nabla_X Y+FQ\nabla_X Y+FP\nabla_Y X+FQ\nabla_Y X+2Bh(X,Y)+2Ch(X,Y)$

$+u^p(X)PY+u^p(X)QY+u^p(Y)PX+u^p(Y)QX+2u^p(X)u^p(Y)U_p$

$-2g(X,Y)U_p.$

Now from the above equation we get (3.2.21)(A), (3.2.22)(A), (3.2.23)(A) and (3.2.24)(A). This completes the proof of the lemma.

The horizontal distribution D is said to be parallel with respect to connection ∇ on M if $\nabla_X Y \in D$ for all $X, Y \in D$.

Lemma 3.2.4(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then $\nabla_X U_p = 0$, for $X \in D^{\perp}$ and $h(X, U_p) = 0$, for $X \in D$.

Proof: From equation (3.2.2)(A), (3.2.4)(A) and (3.2.6)(A) we get

$$FPX + FQX = \nabla_X U_p + h(X, U_p).$$

Now equating tangential and normal parts, we get

$$\nabla_X U_p = FPX \text{ and } h(X, U_p) = FQX.$$

This implies that

$$\nabla_X U_p = 0, \text{ for } X \in D^{\perp} \text{ and } h(X, U_p) = 0, \text{ for } X \in D.$$

This completes the proof of Lemma 3.2.4(A).

Proposition 3.2.1(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. If the horizontal distribution D is parallel then $h(X, FY) = h(Y, FX)$, for all $X, Y \in D$.

Proof: As the horizontal distribution D is parallel, we have $\nabla_X FY \in D$ and $\nabla_Y FX \in D$ for any $X, Y \in D$. Thus by virtue of the above facts and using (3.2.22)(A), we have

$Bh(X, Y) = 0$, for any $X, Y \in D$.

Now $Fh(X, Y) = Bh(X, Y) + Ch(X, Y)$.

We get $Fh(X, Y) = Ch(X, Y)$.

From (3.2.23)(A) and the above facts we have

(3.2.25)(A) $h(X,FY)+h(Y,FX) = 2Ch(X,Y) = 2Fh(X,Y)$, for any $X,Y\in D$.

Taking $X=FX\in D$ into (3.2.25)(A) and using Lemma 3.2.4(A), we get

(3.2.26)(A) $h(FX,FY)-h(X,Y) = 2Fh(FX,Y)$.

Again taking $Y=FY\in D$ into (3.2.25)(A) and using Lemma 3.2.4(A), we get

(3.2.27)(A) $h(FX,FY)-h(X,Y) = 2Fh(X,FY)$.

Hence from (3.2.26)(A) and (3.2.27)(A), we get

$Fh(X,FY) = Fh(FX,Y)$

or $h(X,FY) = h(FX,Y)$, for any $X,Y\in D$.

This completes the proof of proposition (3.2.1)(A).

Lemma 3.2.5(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then M is mixed totally geodesic if and only if $A_N X\in D$ for all $X\in D$.

Proof: M is mixed totally geodesic means $h(X,Y)=0$, for all $X\in D, Y\in D^{\perp}$. Thus in view of equation (3.2.5)(A), we get $g(A_N X,Y)=0$, which proves our assertion.

A normal vector field $N\neq 0$ is D-parallel normal section if $\nabla_X^{\perp} N=0$, for all $X\in D$.

Proposition 3.2.2(A): Let M be a mixed totally geodesic semi-invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then the normal section $N \in FD^{\perp}$ is a D-parallel if and only if $\nabla_X FN \in D$, for all $X \in D$.

Proof: Let $N \in FD^{\perp}$ and as M be a mixed totally geodesic, we have

$$\nabla_X(FN) = \bar{\nabla}_X(FN)$$

or $\nabla_X(FN) = (\bar{\nabla}_X F)N + F\bar{\nabla}_X N$

(3.2.28)(A) $\nabla_X(FN) = F\nabla_X^{\perp}N - A_N X$.

Using Lemma 3.2.5(A) and by (3.2.28)(A) $\nabla_X^{\perp}N = 0$ if and only if $\nabla_X FN \in D$, for all $X \in D$.

Hence this completes the proof of the proposition.

Lemma 3.2.6(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then

$$2(\bar{\nabla}_Y F)Z = A_{FY}Z - A_{FZ}Y - 2g(Y,Z)U_p + \nabla_Y^{\perp}FZ - \nabla_Z^{\perp}FY - F[Y,Z],$$

for any $Y,Z \in D^{\perp}$.

Proof: From Weingarten formula, we get

(3.2.29)(A) $\bar{\nabla}_Y FZ - \bar{\nabla}_Z FY = A_{FY}Z - A_{FZ}Y + \nabla_Y^{\perp}FZ - \nabla_Z^{\perp}FY$.

On the other hand, we have

(3.2.30)(A) $\bar{\nabla}_Y FZ - \bar{\nabla}_Z FY = (\bar{\nabla}_Y F)(Z) - (\bar{\nabla}_Z F)(Y) + F[Y,Z]$.

Now from (3.2.29)(A) and (3.2.30)(A) we get

(3.2.31)(A) $(\bar{\nabla}_Y F)(Z)-(\bar{\nabla}_Z F)(Y)= A_{FY}Z-A_{FZ}Y+\nabla_Y^{\perp}FZ-\nabla_Z^{\perp}FY-F[Y,Z]$.

From (3.2.1)(A) and in view of $u^p(X)=u^p(Y)=0$ for $Y,Z\in D^{\perp}$, we have

(3.2.32)(A) $(\bar{\nabla}_Y F)(Z)+(\bar{\nabla}_Z F)(Y)=-2g(Y,Z)U_p$.

Thus from (3.2.31)(A) and (3.2.32)(A), we get the result.

Proposition 3.2.3(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$.Then

(3.2.33)(A) $A_{FY}Z-A_{FZ}Y=1/3FP[X,Y]$, for $Y,Z\in D^{\perp}$.

Proof: For $Y,Z\in D^{\perp}$ and $X\in TM$. Now from equation (3.2.5)(A) we have

$$2g(A_{FZ}Y,X)=g(h(Y,X),FZ) + g(h(Y,X),FZ)$$

$$=g(\bar{\nabla}_Y X,FZ) + g(\bar{\nabla}_X Y,FZ)$$

$$=-g(F(\bar{\nabla}_Y X + \bar{\nabla}_X Y),Z)$$

$$=-g(\nabla_Y FX,Z) - g(\nabla_X FY,Z)+u^p(X)g(Y,Z)$$

$$=g(\nabla_Y Z,FX) + g(A_{FY}Z,X) +u^p(X)g(Y,Z),$$

which is true for all $X\in TM$ and therefore transvecting X both sides, we have,

$2A_{FZ}Y=A_{FY}Z-F\nabla_Y Z+g(Y,Z)U_p$.

Similarly

$2A_{FY}Z=A_{FZ}Y-F\nabla_Z Y+g(Y,Z)U_p$.

Thus from the above two relations, we have

$A_{FY}Z-A_{FZ}Y=1/3FP[Y,Z]$, for any $Y,Z\in D^{\perp}$. This proves our assertion.

Let M be a semi-invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. We say that M is totally r-contact umbilical if there exist a normal vector field H such that the second fundamental form of M is given by

$h(F^2X,F^2Y)=g(FX,FY)H.$

From the above equation we get

(3.2.34)(A) $h(X,Y)=g(FX,FY)H+u^p(X)h(Y,U_p)+u^p(Y)h(X,U_p),$

for any vector field X,Y tangent to M.

If we have H=0 in (3.2.34)(A), that is the second fundamental form of M is given by

(3.2.35)(A) $h(X,Y)=u^p(X)h(Y,U_p)+u^p(Y)h(X,U_p),$

then we say that M is totally r-contact geodesic.

3.4 SECTION 2(B):

SEMI-INVARIANT SUBMANIFOLDS OF r-KENMOTSU MANIFOLDS

Let $\bar{M}$ be an almost r-contact metric manifold with structure tensors (F,U_p,u^p,g).

An almost r-contact metric manifold structure (F,U_p,u^p,g) is said to be r-Kenmotsu if

(3.2.1)(B) $(\bar{\nabla}_X F)(Y) = g(FX,Y)U_p - u^p(Y)FX,$

for any vector fields $X,Y \in T\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection in $\bar{M}$. From (3.2.1)(B) it follows that

(3.2.2)(B) $\bar{\nabla}_X U_p = X - u^p(X)U_p.$

A submanifold M of an r-Kenmotsu manifold $\bar{M}$ is called a semi-invariant submanifold of $\bar{M}$ if the following conditions are satisfied

(i) $TM = D \oplus D^{\perp}$, where D and $D^{\perp}$ are differentiable orthogonal distributions with respect to the induced metric denoted by g itself on M.

(ii) The distribution D is invariant by F, that is $F(D_x) \subset D_x$, for each $x \in M$.

(iii) The distribution $D^{\perp}$ is anti-invariant by F, that is $F(D_x^{\perp}) \subset T_xM^{\perp}$ for each $x \in M$. where $T_xM^{\perp}$ is the normal space of M at x.

M is an invariant (resp.anti-invariant) submanifold of $\bar{M}$ when $\dim D^{\perp}=0$ (resp.$\dim D=0$).

The equation of Gauss and Weingarten are given by

(3.2.3)(B) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$,

(3.2.4)(B) $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

for $X,Y \in TM$, $N \in T^{\perp}M$, where h(resp. A) is the second fundamental form (resp. tensor) of M in $\bar{M}$, and $\nabla^{\perp}$ denotes the operator of the normal connection. We have

$$g(h(X,Y),N) = g(A_N X,Y).$$

the distribution D (resp. $D^{\perp}$) is defined by projection P(resp. Q) which satisfies the conditions:

(3.2.5)(B) $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, $go(P \times Q) = 0$.

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. Also the pair $(D,D^{\perp})$ is called U_p-horizontal (resp.U_p-vertical) if $(U_p)_x \in D$ (resp. $(U_p)_x \in D^{\perp}$) for all $x \in M$.

For a vector field X tangent to M, let us put

(3.2.6)(B) $X = PX + QX$,

where PX and QX belongs to D and $D^{\perp}$ respectively. Also for a vector field N normal to M, we put

(3.2.7)(B) $FN = BN + CN$,

where BN (resp. CN) denotes the tangential (resp. normal) components of FN.

The horizontal distribution D is said to be parallel on M if $\nabla_X Y \in D$ for any $X,Y \in D$.

A semi-invariant submanifold M of $\bar{M}$ is said to be D-umbilic (resp. $D^{\perp}$-umbilic) if,

(3.2.8)(B) $h(X,Y) = g(X,Y)H$,

holds for all $X,Y \in D$ (resp. $X,Y \in D^{\perp}$), where H being normal vector field. M is said to be D-totally geodesic (resp. $D^{\perp}$ totally geodesic) if $h(X,Y) = 0$ for all $X,Y \in D$ (resp.$X,Y \in D^{\perp}$).

Lemma 3.2.1(B): Let M be a semi-invariant submanifold of a r-Kenmotsu manifold $\bar{M}$. Then we have

(3.2.9)(B) $P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + g(FX,Y)PU_p - u^p(Y)FPX$

(3.2.10)(B) $Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) + g(FX,Y)QU_p - u^p(Y)FQX$

(3.2.11)(B) $h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y)$.

Proof: From equation (3.2.1)(B), we have

(3.2.12)(B) $(\bar{\nabla}_X F)Y = g(FX,Y)PU_p + g(FX,Y)QU_p - u^p(Y)FPX$

$- u^p(Y)FQX$.

We know that

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F\,\bar{\nabla}_X Y,$$

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FPY + \bar{\nabla}_X FQY - F\nabla_X Y - Fh(X,Y),$$

$$(\bar{\nabla}_X F)Y = \nabla_X FPY + h(X,FPY) - A_{FQY}X + \nabla_X^{\perp} FQY - FP\nabla_X Y - FQ\nabla_X Y$$

$$-Bh(X,Y) - Ch(X,Y),$$

(3.2.13)(B) $$(\bar{\nabla}_X F)Y = P\nabla_X FPY + Q\nabla_X FPY + h(X,FPY) - PA_{FQY}X - QA_{FQY}X$$

$$+\nabla_X^{\perp} FQY - FP\nabla_X Y - FQ\nabla_X Y - Bh(X,Y) - Ch(X,Y).$$

From equation (3.2.12)(B) and (3.2.13)(B) equating the horizontal, vertical and normal components, we get the desired results.

A normal vector field $N \neq 0$ is D-parallel normal section on M if $\nabla_X^{\perp} N = 0$, for all $X \in D$.

We say that M is totally r-contact umbilical if there exists a normal vector H on M such that

(3.2.14)(B) $h(X,Y) = g(FX,FY)H + u^p(X)h(Y,U_p) + u^p(Y)h(X,U_p)$,

for all vector fields X,Y tangent to M.

If $H = 0$ in (3.2.14)(B), we have

(3.2.15)(B) $h(X,Y) = u^p(X)h(Y,U_p) + u^p(Y)h(X,U_p)$

Then M is called totally r-contact geodesic.

3.5 SECTION 3(A):

CR-SUBMANIFOLDS OF A TRANS HYPERBOLIC SASAKIAN MANIFOLD I

Let $\bar{M}$ be an n-dimensional almost contact hyperbolic metric manifold with structure tensors (F,U,u,g).

An almost contact hyperbolic metric structure (F,U,u,g) on $\bar{M}$ is called hyperbolic Sasakian if

(3.3.1)(A) $(\bar{\nabla}_X F)(Y) = g(X,Y)U - u(Y)X.$

A hyperbolic Sasakian manifold is called trans hyperbolic Sasakian if

(3.3.2)(A) $(\bar{\nabla}_X F)(Y) = \alpha(g(X,Y)U - u(Y)X) + \beta(g(FX,Y)U - u(Y)FX),$

for function α,β on $\bar{M}$. We say that trans hyperbolic Sasakian structure is of type (α,β).

From the formula (3.3.2)(A) it follows that

(3.3.3)(A) $\bar{\nabla}_X U = -\alpha FX - \beta(X + u(X)U).$

An m-dimensional Riemannian submanifold M of a trans hyperbolic Sasakian manifold $\bar{M}$ is called a CR-submanifold if U is tangent to M and there exist on M a differentiable distribution D: $x \to D_x \subset T_xM$ satisfying the following conditions:

(i) D_x is invariant under F that is $FD_x \subset D_x$ for each $x \in M$,

(ii) the complimentary orthogonal distribution $D^{\perp}$: $x \to D_x^{\perp} \subset T_xM$ is totally real under F, that is $FD_x^{\perp} \subset T_x^{\perp}M$ for each $x \in M$, where T_xM and $T_x^{\perp}M$ are tangent and normal space of M at x respectively.

M is an invariant (resp. anti-invariant) submanifold of $\bar{M}$ when $\dim D^{\perp} = 0$ (resp. $\dim D = 0$), where D (resp.$D^{\perp}$) is the horizontal (resp. vertical) distribution. The pair $(D,D^{\perp})$ is called U-horizontal (U-vertical) if $U_x \in D_x$ (resp.$U_x \in D_x^{\perp}$) for each $x \in M$.

The Gauss and Wiengarten formulas are given by

(3.3.4)(A) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$, $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

$X,Y \in TM$, $N \in T^{\perp}M$,

where $\nabla^{\perp}$ is the normal connection, h (resp.A) is the second fundamental form (resp. tensor) of M in $\bar{M}$.

h is related to A by

(3.3.5)(A) $g(A_N X,Y) = g(h(X,Y),N)$.

For a vector field X tangent to M, we put

(3.3.6)(A) $X = PX + QX$,

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. Also for a vector field N normal to M, we put

(3.3.7)(A) $FN = BN+CN$,

where BN (resp. CN) denotes the tangential (resp. normal) component of FN.

If we denote the orthogonal component of $FD^{\perp}$ in $T^{\perp}M$ by μ, then we have $T^{\perp}M = FD^{\perp} \oplus \mu$, it is obvious that $F\mu = \mu$.

Lemma 3.3.1(A): Let M be a CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. Then we have

(3.3.8)(A) $$P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + \alpha g(X,Y)PU + \beta g(FPX,Y)PU - \alpha u(Y)PX - \beta u(Y)FPX,$$

(3.3.9)(A) $$Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) + (\alpha g(X,Y) + \beta g(FQX,Y))QU - \alpha u(Y)QX,$$

(3.3.10)(A) $$h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y) + \beta u(Y)FQX,$$

for any $X,Y \in TM$.

Proof: From equations (3.3.2)(A),(3.3.4)(A),(3.3.6)(A) and (3.3.7)(A), we have

$$\bar{\nabla}_X FX - F\bar{\nabla}_X Y = \alpha(g(X,Y)U - u(Y)X) + \beta(g(FX,Y)U - u(Y)FX)$$

or $$\nabla_X FPY + h(X,FPY) + \nabla_X^{\perp} FQY - A_{FQY}X - F\nabla_X Y - Fh(X,Y)$$
$$= \alpha g(X,Y)PU + \alpha g(X,Y)QU - \alpha u(Y)PX - \alpha u(Y)QX$$
$$+ \beta g(FPX + FQX,Y)U - \beta u(Y)FPX - \beta u(Y)FQX,$$

or $\quad P\nabla_X FPY+Q\nabla_X FPY+h(X,FPY)+\nabla_X^{\perp}FQY-PA_{FQY}X-QA_{FQY}X$

$=FP\nabla_X Y+FQ\nabla_X Y+Bh(X,Y)+Ch(X,Y)+\alpha g(X,Y)PU+\alpha g(X,Y)QU$

$-\alpha u(Y)PX-\alpha u(Y)QX+\beta g(FPX,Y)PU+\beta g(FQY,Y)QU-\beta u(Y)FPX$

$-\beta u(Y)FQX.$

Now equating the horizontal, vertical and normal components we get the results.

The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X,Y \in D$.

Proposition 3.3.1(A): Let M be a U-horizontal CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. Then the distribution D is parallel if and only if

(3.3.11)(A) $h(X,FY) = h(FX,Y) = Fh(X,Y)$, for all $X,Y \in D$.

Proof: Every parallel distribution is involutive, that is

(3.3.12)(A) $h(X,FY) = h(FX,Y)$, for all $X,Y \in D$.

Now from (3.3.10)(A) and (3.3.12)(A), we have

(3.3.13)(A) $h(X,FY) = Ch(X,Y)$.

Also $\nabla_X FY \in D$, $\nabla_Y FX \in D$, $\forall\ X,Y \in D$, so from (3.3.9)(A) and using D-parallelness, we get $Bh(X,Y) = 0$, $\forall\ X,Y \in D$.

From (3.3.7)(A) we have

$Fh(X,Y) = Bh(X,Y)+Ch(X,Y)$.

Thus in view of (3.3.13)(A), $Bh(X,Y) = 0$ and the above equation, we get

$Fh(X,Y) = Ch(X,Y) = h(X,FY)$, $\forall\ X,Y \in D$,

which proves (3.3.11)(A).

A CR-submanifold M of a trans hyperbolic Sasakian manifold $\bar{M}$ is said to be mixed totally geodesic if $h(X,Y)=0$, for $X \in D$ and $Y \in D^{\perp}$.

A CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$.

A normal vector field $N \neq 0$ is D-parallel normal section if $\nabla_X^{\perp} N=0$, for all $X \in D$.

Proposition 3.3.2(A): Let M be a mixed totally geodesic U-vertical CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. Then the normal section $N \in FD^{\perp}$ is a D-parallel if and only if $\nabla_X FN \in D$, for all $X \in D$.

Proof: Let $N \in FD^{\perp}$ and as M be a mixed totally geodesic, we have

$$\nabla_X(FN) = \bar{\nabla}_X(FN)$$

or $$\nabla_X(FN) = (\bar{\nabla}_X F)N+F\ \bar{\nabla}_X N$$

(3.3.14)(A) $$\nabla_X(FN) = F\nabla_X^{\perp} N-A_N X.$$

But $A_N X \in D$, so by (3.3.14)(A) we see that $\nabla_X^{\perp} N=0$ if and only if $\nabla_X FN \in D$, for all $X \in D$.

This proves our assertion.

Lemma 3.3.2(A): Let M be a CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. Then we have

(3.3.15)(A) $A_{FY}Z-A_{FZ}Y+\alpha(u(Z)Y-u(Y)Z) = FP[Y,Z]$, for any $Y,Z\in D^{\perp}$.

Proof: We have

$$\bar{\nabla}_Y FZ = (\bar{\nabla}_Y F)(Z)+F\bar{\nabla}_Y Z.$$

Using (3.3.2)(A) in the above equation, we get

$$\bar{\nabla}_Y FZ = \alpha(g(Y,Z)U-u(Z)Y)+\beta(g(FY,Z)U-u(Z)FY)+F\nabla_Y Z+Fh(Y,Z)$$

$$= \alpha(g(Y,Z)U-u(Z)Y)+FP\nabla_Y Z+FQ\nabla_Y Z+Bh(Y,Z)+Ch(X,Y)$$

$$-\beta u(Z)FQY.$$

Now using (3.3.4)(A) in the above equation, we obtain

(3.3.16)(A) $-A_{FZ}Y+\nabla_Y^{\perp}FZ=\alpha(g(Y,Z)U-u(Z)Y)-\beta u(Z)FQY+FP\nabla_Y Z$

$$+FQ\nabla_Y Z +Bh(Y,Z)+Ch(Y,Z),$$

for all $Y,Z\in D^{\perp}$.

From (3.3.10)(A), for all $Y,Z\in D^{\perp}$, we have

(3.3.17)(A) $\nabla_Y^{\perp}FZ = FQ\nabla_Y Z + Ch(Y,Z) + \beta u(Z)FQY$.

Now from (3.3.16)(A) and (3.3.17)(A), we have

$$FP\nabla_Y Z = -A_{FZ}Y-\alpha g(Y,Z)U+\alpha u(Z)Y- Bh(Y,Z).$$

Similarly we have

$$FP\nabla_Z Y = -A_{FY}Z - \alpha g(Y,Z)U + \alpha u(Y)Z - Bh(Y,Z).$$

Thus from the above two equations, we get

$$FP[Y,Z] = A_{FY}Z - A_{FZ}Y + \alpha(u(Z)Y - u(Y)Z), \text{ for all } Y,Z \in D^{\perp}.$$

This proves our assertion.

A CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$ is called D-umbilic (resp.$D^{\perp}$ umbilic) if $h(X,Y) = g(X,Y)H$ holds for all $X,Y \in D$ (resp.$X,Y \in D^{\perp}$), where H is a mean curvature vector field.

3.6 SECTION 3(B):

CR-SUBMANIFOLDS OF A TRANS HYPERBOLIC SASAKIAN MANIFOLD II

Let $\bar{M}$ be an almost contact hyperbolic metric manifold with structure tensors (F,U,u,g).

An almost contact hyperbolic metric structure (F,U,u,g) on $\bar{M}$ is called α-hyperbolic Sasakian if

(3.3.1)(B) $(\bar{\nabla}_X F)(Y) = \alpha(g(X,Y)U - u(Y)X), \ \alpha \in R,$

α-hyperbolic Kenmotsu if

(3.3.2)(B) $(\bar{\nabla}_X F)(Y) = \alpha(g(FX,Y)U - u(Y)FX), \ \alpha \in R$

and trans hyperbolic Sasakian if

(3.3.3)(B) $$(\bar{\nabla}_X F)(Y) = 1/2n[(g(X,Y)U - u(Y)X)\delta F(U) + (g(X,FY)U + u(Y)FX)\delta u]$$

where δ is the co-derivative and $\bar{\nabla}$ is the Riemannian connection in $\bar{M}$.

A submanifold M of $\bar{M}$ is called a CR-submanifold if U is tangent to M and there exist on M a differentiable distribution D: $x \to D_x \subset T_xM$ such that:

(i) The distribution D_x is invariant by F, that is $F(D_x) \subset D_x$, for each $x \in M$.

(ii) The distribution $D_x^{\perp}$ is anti-invariant by F, that is $F(D_x^{\perp}) \subset T_xM^{\perp}$ for each $x \in M$. where $T_xM^{\perp}$ is the normal space of M at x.

The equation of Gauss and Weingarten are given by

(3.3.4)(B) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$

where ∇ is the Riemannian connection on M, $\nabla^{\perp}$ is the connection on the normal bundle induced by ∇ and h (resp. A) is the second fundamental form (resp. tensor) of M in $\bar{M}$, satisfying

(3.3.5)(B) $g(h(X,Y),N) = g(A_N X,Y).$

The distribution D (resp. $D^{\perp}$) can be defined by projection P (resp. Q) which satisfies the conditions:

(3.3.6)(B) $P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad go(P \times Q) = 0.$

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. Also the pair $(D,D^{\perp})$ is called U-horizontal (resp.U-vertical) if $U_x \in D$ (resp. $U_x \in D^{\perp}$) for all $x \in M$.

For a vector field X tangent to M, we put

(3.3.7)(B) $X = PX + QX,$

where PX and QX belongs to D and $D^{\perp}$ respectively. Also for a vector field N normal to M, we put

(3.3.8)(B) $FN = BN + CN,$

where BN (resp. CN) denotes the tangential (resp. normal) components of FN.

The horizontal distribution D is said to be parallel if $\nabla_X Y \in D$ for any $X,Y \in D$.

A CR-submanifold M of $\bar{M}$ is said to be D-umbilic (resp. $D^{\perp}$-umbilic) if, $h(X,Y)= g(X,Y)H$ holds for all $X,Y \in D$ (resp.$X,Y \in D^{\perp}$),where H being some normal vector field. M is said to be D-totally geodesic (resp. $D^{\perp}$ totally geodesic) if $h(X,Y) = 0$ for all $X,Y \in D$ (resp. $X,Y \in D^{\perp}$).

Lemma 3.3.1(B): If M be a CR-submanifold of trans hyperbolic Sasakian manifold $\bar{M}$. Then we have

(3.3.9)(B) $P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + 1/2n[(g(X,Y)PU - u(Y)PX)\delta F(U) + (g(X,FY)PU + u(Y)FPX)\delta u]$,

(3.3.10)(B) $Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) + 1/2n[(g(X,Y)QU - u(Y)QX)\delta F(U) + (g(X,FY)QU + u(Y)FQX)\delta u]$,

(3.3.11)(B) $h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y)$.

Proof: From equation (3.3.3)(B), (3.3.4)(B), (3.3.7)(B) and (3.3.8)(B), we get

$$\bar{\nabla}_X FY - F\ \bar{\nabla}_X Y = 1/2n[(g(X,Y)U - u(Y)X)\delta F(U) + (g(X,FY)U + u(Y)FX)\delta u],$$

$$\bar{\nabla}_X FPY + \bar{\nabla}_X FQY - F\nabla_X Y - Fh(X,Y) = 1/2n[(g(X,Y)U - u(Y)X)\delta F(U) + (g(X,FY)U + u(Y)FX)\delta u],$$

$$P\nabla_X FPY + Q\nabla_X FPY + h(X,FPY) - PA_{FQY}X - QA_{FQY}X + \nabla_X^{\perp} FQY - FP\nabla_X Y - FQ\nabla_X Y - Bh(X,Y) - Ch(X,Y) = 1/2n[(g(X,Y)PU + g(X,Y)QU - u(Y)PX - u(Y)QX)\delta F(U) + (g(X,FY)PU + g(X,FY)QU + u(Y)FPX + u(Y)FQX)\delta u].$$

Now equating the horizontal, vertical and normal components we get the results.

Lemma 3.3.2(B): If M be a CR-submanifold of α-hyperbolic Sasakian manifold $\bar{M}$. Then we have

(3.3.12)(B) $P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + \alpha(g(X,Y)PU - u(Y)PX)$,

(3.3.13)(B) $Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) + \alpha(g(X,Y)QU - u(Y)QX)$,

(3.3.14)(B) $h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y)$.

Proof: From equation (3.3.1)(B), (3.3.4)(B), (3.3.7)(B) and (3.3.8)(B), we get

$$\bar{\nabla}_X FY - F\ \bar{\nabla}_X Y = \alpha(g(X,Y)U - u(Y)X),$$

$$\bar{\nabla}_X FPY + \bar{\nabla}_X FQY - F\nabla_X Y - Fh(X,Y) = \alpha(g(X,Y)U - u(Y)X),$$

$$P\nabla_X FPY+Q\nabla_X FPY+h(X,FPY)-PA_{FQY}X-QA_{FQY}X+\nabla_X^{\perp}FQY-FP\nabla_X Y$$

$$-FQ\nabla_X Y-Bh(X,Y)-Ch(X,Y) = \alpha(g(X,Y)PU + g(X,Y)QU - u(Y)PX$$

$$- u(Y)QX).$$

Now equating the horizontal, vertical and normal components we have the results.

Lemma 3.3.3(B): If M be a CR-submanifold of α-hyperbolic Kenmotsu manifold $\bar{M}$. Then we have

(3.3.15)(B) $P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + \alpha(g(X,FY)PU - u(Y)FPX)$,

(3.3.16)(B) $Q\nabla_X FPY - QA_{FQY}X= Bh(X,Y) + \alpha(g(X,FY)QU - u(Y)FQX)$,

(3.3.17)(B) $h(X,FPY) + \nabla_X^{\perp}FQY = FQ\nabla_X Y + Ch(X,Y)$.

Proof: From equation (3.3.2)(B), (3.3.4)(B), (3.3.7)(B) and (3.3.8)(B), we get

$$\bar{\nabla}_X FY - F\ \bar{\nabla}_X Y = \alpha(g(X,FY)U - u(Y)FX),$$

$$\bar{\nabla}_X FPY + \bar{\nabla}_X FQY - F\nabla_X Y - Fh(X,Y) = \alpha(g(X,FY)U - u(Y)FX),$$

$$P\nabla_X FPY+Q\nabla_X FPY+h(X,FPY)-PA_{FQY}X-QA_{FQY}X+\nabla_X^{\perp}FQY-FP\nabla_X Y$$

$$-FQ\nabla_X Y-Bh(X,Y)-Ch(X,Y) = \alpha(g(X,FY)PU + g(X,FY)QU - u(Y)FPX$$

$$- u(Y)FQX).$$

Now equating the horizontal, vertical and normal components we have the results.

Lemma 3.3.4(B): Let M be U-horizontal CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. then horizontal distribution D is parallel if

(3.3.18)(B) $h(X,FY) = h(Y,FX) = Fh(X,Y)$, for all $X,Y \in D$.

Proof: As every parallel distribution is involutive the first equality in (3.3.18)(B) holds that is

(3.3.19)(B) $h(X,FY) = h(FX,Y)$, for all $X,Y \in D$.

Now from (3.3.11)(B) and (3.3.19)(B), we have

(3.3.20)(B) $h(X,FY) = Ch(X,Y)$.

Also $\nabla_X FY \in D$, $\nabla_Y FX \in D$, $\forall$ $X,Y \in D$, so from (3.3.11)(B) and using D-parallelness, we get $Bh(X,Y) = 0$, $\forall$ $X,Y \in D$.

Therefore from $Fh(X,Y)=Bh(X,Y)+Ch(X,Y)$, $Bh(X,Y)=0$ and (3.3.20)(B), we have

$Fh(X,Y) = Ch(X,Y) = h(X,FY)$, $\forall$ $X,Y \in D$,

which proves (3.3.18)(B).

A normal vector field $N \neq 0$ is D-parallel normal section if $\nabla_X^{\perp} N=0$, for all $X \in D$.

Let M be a CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$, then M is said be totally contact hyperbolic umbilical if there exists a normal vector H on M such that

(3.3.21)(B) $h(X,Y) = -g(FX,FY)H - u(X)h(Y,U) - u(Y)h(X,U)$,

for all vector fields X,Y tangent to M.

If H = 0 in (3.3.21)(B), that is, the fundamental form is given by

$$h(X,Y) = -u(X)h(Y,U) - u(Y)h(X,U),$$

then we say that M is totally contact hyperbolic geodesic.

3.7 SECTION 4(A):

SEMI-INVARIANT SUBMANIFOLDS OF PARA KENMOTSU MANIFOLDS I

Let $\bar{M}$ be an almost para contact metric manifold with structure tensors (F,U,u,g).

An almost para contact metric manifold structure (F,U_p,u^p,g) is said to be para Kenmotsu if [21]

(3.4.1)(A) $(\bar{\nabla}_X F)(Y) = -g(X,FY)U - u(Y)FX,$

for any vector fields $X,Y \in T\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection in $\bar{M}$. From (3.4.1)(A) it follows that

(3.4.2)(A) $\bar{\nabla}_X U = X - u(X)U.$

A Riemannian submanifold M of a para Kenmotsu manifold $\bar{M}$ is called a semi-invariant submanifold if U is tangent to M and there exist on M a pair of orthogonal distribution $(D, D^{\perp})$ such that

(i) $TM = D \oplus D^{\perp}$

(ii) the distribution D is invariant under F, that is $FD_x \subset D_x$, for all $x \in M$,

(iii) the distribution $D^{\perp}$ is anti-invariant under F, that is $F(D_x^{\perp}) \subset T_x^{\perp}M$, for all $x \in M$,

where T_xM and $T_x^{\perp}M$ are tangent space and normal space of M at x.

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. A semi-invariant submanifold M is said to be an invariant (resp. anti-invariant) submanifold if we have $D_x^{\perp} = \{0\}$ (resp.$D_x = \{0\}$) for each $x \in M$. We also call M proper if neither D nor $D^{\perp}$ is null.

Let $\bar{\nabla}$ (resp. ∇) be covariant differentiation with respect to the Levi-Civita connection on $\bar{M}$ (resp. M). The Gauss and Weingarten formulas for M are respectively given by

(3.4.3)(A) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$, $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

for $X,Y \in TM$, $N \in T^{\perp}M$, where h (resp. A) is the second fundamental form (resp. tensor) of M in $\bar{M}$, and $\nabla^{\perp}$ denotes the operator of the normal connection. We have

(3.4.4)(A) $g(h(X,Y),N) = g(A_N X,Y)$.

For a vector field X tangent to M, we put

(3.4.5)(A) $X = PX + QX$,

where P and Q are projection operators and PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any vector field N normal to M, we put

(3.4.6)(A) $FN = BN + CN$,

where B and C are projection operators and BN (resp. CN) denotes the tangential (resp. normal) component of FN.

The distribution D (resp. $D^{\perp}$) can be defined by projection P (resp. Q) which satisfies the conditions:

(3.4.7)(A) $P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad go(P\times Q) = 0.$

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. Also the pair $(D, D^{\perp})$ is called U-horizontal (resp.U-vertical) if $U_x \in D$ (resp. $U_x \in D^{\perp}$) for all $x \in M$.

The horizontal distribution D is said to be parallel if $\nabla_X Y \in D$ for any $X,Y \in D$.

A semi-invariant submanifold M of $\bar{M}$ is said to be D-umbilic (resp. $D^{\perp}$-umbilic) if,

(3.4.8)(A) $h(X,Y) = g(X,Y)H.$

holds for all $X,Y \in D$ (resp. $X,Y \in D^{\perp}$), where H is some normal vector field. M is said to be D-totally geodesic (resp. $D^{\perp}$ totally geodesic) if $h(X,Y) = 0$ for all $X,Y \in D$ (resp.$X,Y \in D^{\perp}$).

Lemma 3.4.1(A): Let M be a semi-invariant submanifold of a para Kenmotsu manifold $\bar{M}$. Then we have

(3.4.9)(A) $P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y - g(X,FY)PU - u(Y)FPX,$

(3.4.10)(A) $Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) - g(X,FY)QU - u(Y)FQX,$

(3.4.11)(A) $h(X,FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y).$

Proof: From equation (3.4.1)(A), we have

(3.4.12)(B) $(\bar{\nabla}_X F)Y = -g(X,FY)PU - g(X,FY)QU - u(Y)FPX - u(Y)FQX.$

We know that

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F\,\bar{\nabla}_X Y,$$

$$(\bar{\nabla}_X F)Y = \bar{\nabla}_X FPY + \bar{\nabla}_X FQY - F\nabla_X Y - Fh(X,Y),$$

$$(\bar{\nabla}_X F)Y = \nabla_X FPY + h(X,FPY) - A_{FQY}X + \nabla_X^{\perp} FQY - FP\nabla_X Y - FQ\nabla_X Y$$
$$- Bh(X,Y) - Ch(X,Y),$$

(3.4.13)(A) $(\bar{\nabla}_X F)Y = P\nabla_X FPY + Q\nabla_X FPY + h(X,FPY) - PA_{FQY}X - QA_{FQY}X$
$$+ \nabla_X^{\perp} FQY - FP\nabla_X Y - FQ\nabla_X Y - Bh(X,Y) - Ch(X,Y).$$

From equation (3.4.12)(B) and (3.4.13)(B) equating the horizontal, vertical and normal components we get the desired results.

A normal vector field $N \neq 0$ is D -parallel normal section if $\nabla_X^{\perp} N = 0$, for all $X \in D$.

Submanifold M of para Kenmotsu manifold $\bar{M}$ is called totally para contact umbilical if there exists a normal vector H on M such that

(3.4.14)(A) $h(X,Y) = g(FX,FY)H + u(X)h(Y,U) + u(Y)h(X,U)$

for all vector fields X,Y tangent to M.

If $H = 0$ in (3.4.14)(A), then we have

(3.4.15)(A) $h(X,Y) = u(X)h(Y,U) + u(Y)h(X,U)$

which shows that submanifold M is totally para contact geodesic.

3.8 SECTION 4(B):

ON SEMI-INVARIANT SUBMANIFOLDS OF A PARA KENMOTSU MANIFOLDS II

Let $\bar{M}$ be an differentiable manifold with structure tensors (F,U,u,g).

$\bar{M}$ is a para Kenmotsu manifold if and only if we have [21]

(3.4.1)(B) $(\bar{\nabla}_X F)(Y) = -g(X,FY)U - u(Y)FX,$

from (3.4.1)(B) it follows that

(3.4.2)(B) $\bar{\nabla}_X U = X - u(X)U,$ for any X,Y tangent to $\bar{M}$,

where $\bar{\nabla}$ is the Livi Civita connection on $\bar{M}$.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and suppose that the structure vector field U of $\bar{M}$ is tangent to M. We denote by TM and $T^{\perp}M$ the tangent and normal bundle to M. Then the submanifold M of $\bar{M}$ is called a semi-invariant submanifold if it is endowed with a pair of distributions $(D, D^{\perp})$ satisfying the conditions:

(i) $TM = D \oplus D^{\perp} \oplus \{U\}$, where $\{U\}$ denotes the distribution spanned by U.

(ii) The distribution D is invariant by F that is $F(D_x) \subset D_x$ for each $x \in M$.

(iii) The distribution $D^{\perp}$ is anti-invariant by F that is $F(D_x^{\perp}) \subset T_x^{\perp}M$, for all $x \in M$,

Now Gauss and Weingarten formula are given respectively by

(3.4.3)(B) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$,

(3.4.4)(B) $\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$,

where ∇ is the Riemannian connection determined by the induced metric g on M, $\nabla^{\perp}$ the metric connection in the normal bundle of M and h and A_N are called the second fundamental form and the fundamental tensor with respect to the normal section N, satisfying

(3.4.5)(B) $g(h(X,Y),N) = g(A_N X,Y) = g(A_N Y,X)$.

$\{U\}^{\perp}$ denotes the complimentary orthogonal distribution to $\{U\}$ in TM.

For any $X \in TM$, we have $g(FX,U) = 0$, then we put

(3.4.6)(B) $FX = PX + QX$

where $PX \in \{U\}^{\perp}$ and $QX \in T^{\perp}M$.

Let $\bar{M}$ be a para Kenmotsu space form of constant F-holomorphic sectional curvature tensor C. Then the curvature tensor $\bar{R}$ of $\bar{M}(C)$ is given by

(3.4.7)(B) $$\bar{R}(X,Y)Z = \tfrac{1}{4}(C-3)\{g(Y,Z)X - g(X,Z)Y\} + \tfrac{1}{4}(C+1)\{u(X)u(Z)Y - u(Y)u(Z)X + u(Y)g(X,Z)U - u(X)g(Y,Z)U + g(X,FZ)FY - g(Y,FZ)FX + 2g(X,FY)FZ\}, \ \forall\, X,Y,Z \in T\bar{M}(C)$$

The Gauss equation given by

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W)-g(h(X,W),h(Y,Z))$$
$$+g(h(Y,W),h(X,Z)),$$

$$g(R(X,Y)Z,W) = g(\bar{R}(X,Y)Z,W)+g(h(X,W),h(Y,Z))$$
$$-g(h(Y,W),h(X,Z))$$

Using (3.4.7)(B) in the above equation, we get

(3.4.8)(B) $$g(R(X,Y,Z),W)=\tfrac{1}{4}(C-3)\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}$$
$$+1/4(C+1)\{u(X)u(Z)g(Y,W)$$
$$-u(Y)u(Z)g(X,W)+u(Y)g(X,Z)u(W)$$
$$-u(X)g(Y,Z)u(W)+g(X,FZ)g(FY,W)$$
$$-g(Y,FZ)g(FX,W)+2g(X,FY)g(FZ,W)\}$$
$$+g(h(X,W),h(Y,Z))-g(h(Y,W),h(X,Z)),$$
$$\forall\ X,Y,Z,W\in TM$$

and Coddazzi equation is given by

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)$$

(3.4.9)(B) $$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = \tfrac{1}{4}(C+1)\{g(X,PZ)QY$$
$$- g(Y,PZ)QX + 2g(X,PY)QZ\}$$

where

(3.4.10)(B) $$(\nabla_X h)(Y,Z) = (\nabla_X^{\perp} h)(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z),$$

$\forall\ X,Y,Z\in TM.$

A semi–invariant submanifold of a para Kenmotsu manifold $\bar{M}$ is said to be $(D,D^{\perp})$-geodesic if $h(X,Z) = 0$, $\forall\ X \in D$ and $Z \in D^{\perp}$.

Lemma 3.4.1(B): Let M be a $(D,D^{\perp})$-geodesic semi-invariant submanifold of a para Kenmotsu manifold $\bar{M}$. Then we have

(i) Every $X \in D$ and $Z \in D^{\perp}$ satisfy

(3.4.11)(B) $P\nabla_X Z = -A_{QZ}X + u(Z)PX$,

(3.4.12)(B) $Q\nabla_X Z = \nabla_X^{\perp} QZ$,

(ii) $A_N X$ is a section of the orthogonal compliment of $D^{\perp} \oplus \{U\}$ in TM $\forall\ X \in D$ and $N \in T^{\perp}M$.

Proof: For the first part, since $PZ = 0$, $\forall\ Z \in D^{\perp}$ and $QX = 0$, $\forall\ X \in D$.

Also $g(X,FZ) = 0$, $\forall\ X \in D$ and $Z \in D^{\perp}$, since $FZ \in T^{\perp}M$.

Thus from (3.4.1)(B) we have

(3.4.13)(B) $\bar{\nabla}_X FZ - F\ \bar{\nabla}_X Z = -u(Z)PX$,

or $\bar{\nabla}_X QZ - F(\nabla_X Z + h(X,Z)) = -u(Z)PX$

Since M is $(D,D^{\perp})$-geodesic in $\bar{M}$, we have from above equation

(3.4.14)(B) $-A_{QZ}X + \nabla_X^{\perp} QZ - P\nabla_X Z - Q\nabla_X Z = -u(Z)PX$

Comparing horizontal and normal components of the above equation we get the required results of the first part.

For the second part, since M is $(D, D^{\perp})$-geodesic in $\bar{M}$ and from equation (3.4.5)(B) we get

$g(A_N X,Z) = 0$, $\forall X \in D$ and $Z \in D^{\perp}$, since $N \in T^{\perp}M$.

$\Rightarrow$ $A_N X \in D$ $\Rightarrow$ $A_N X$ is a section of the orthogonal compliment of $D^{\perp} \oplus \{U\}$ in TM.

Lemma 3.4.2(B): Let M be a $(D,D^{\perp})$-geodesic semi-invariant submanifold in a para Kenmotsu space form $\bar{M}(C)$. If D is integrable then we have

(3.4.15)(B) $\frac{1}{2}(C+1)g(X,PY)QZ = h(X,\nabla_Y Z) - h(Y,\nabla_X Z)$,

$\forall X,Y \in D$, $Z \in D^{\perp}$.

Proof: Let $X,Y \in D$ and $Z \in D^{\perp}$, and as M is $(D,D^{\perp})$-geodesic in $\bar{M}$, from (3.4.10)(B) we get

(3.4.16)(B) $(\nabla_X h)(Y,Z) = -h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$,

Similarly, we have

(3.4.17)(B) $(\nabla_Y h)(X,Z) = -h(\nabla_Y X,Z) - h(X,\nabla_Y Z)$.

Using (3.4.16)(B) and (3.4.17)(B) and integrability of D, we get

(3.4.18)(B) $(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = h(X,\nabla_Y Z) - h(Y,\nabla_X Z)$.

Also from (3.4.9)(B), we get

(3.4.19)(B) $(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = \frac{1}{2}(C+1)g(X,PY)QZ$.

From (3.4.18)(B) and (3.4.19)(B), we get the result.

This proves our assertion.

Remark 3.4.1(B): Using (3.4.5)(B) in (3.4.15)(B), we get

$$\tfrac{1}{2}(C+1)g(X,PY)g(QZ,QZ) = g(h(X,\nabla_Y Z),QZ) - g(h(Y,\nabla_X Z),QZ)$$

(3.4.20)(B) $\tfrac{1}{2}(C+1)g(X,PY)\|QZ\|^2 = g(A_{QZ}X,\nabla_Y Z) - g(A_{QZ}Y,\nabla_X Z)$,

$\forall\, X,Y \in D,\ Z \in D^{\perp}$.

Let M be a semi-invariant submanifold of a para Kenmotsu manifold $\bar{M}$. Then $D^{\perp}$ is said to be D-parallel if we have

(3.4.21)(B) $\nabla_X Y \in D^{\perp},\ X \in D,\ Y \in D^{\perp}$.

3.9 SECTION 5:

HYPERSURFACES OF ALMOST HYPERBOLIC HERMITIAN MANIFOLDS

Let $\bar{M}$ be an m dimensional differentiable manifold. An almost hyperbolic Hermitian manifold for which

(3.5.1) $(D_\lambda F)\mu = 0,$

(3.5.2) $(D_\lambda F)\mu + (D_\mu F)\lambda = 0,$

(3.5.3) $d`F = 0 \Leftrightarrow (D_\lambda`F)(\mu,\lambda)+(D_\mu`F)(\nu,\lambda) + (D_\nu`F)(\lambda,\mu)=0,$

(3.5.4) $(D_{F\lambda}F)F\mu = (D_\lambda F)\mu,$

(3.5.5) $(D_{F\lambda}F)F\mu + (D_\lambda F)\mu = 0,$

are satisfied, where D is the Riemannian connection, are called hyperbolic Kaehlerian, nearly hyperbolic Kaehlerian, almost hyperbolic Kaehlerian, hyperbolic Hermitian and quasi hyperbolic Kaehlerian manifold [20].

Let M be a hypersurface of an almost hyperbolic Hermitian manifold $\bar{M}$ with the immersion map b: $M \rightarrow \bar{M}$, such that a point $p \in M$ $\Rightarrow bp \in \bar{M}$. Let T and $\bar{T}$ be the tangent manifolds to M and $\bar{M}$ at p and bp respectively. Let B be the Jacobian map corresponding to the immersion map b.

$B: T \to \bar{T}$, such that $X \in T$ at $p \Rightarrow BX \in \bar{T}$ at bp and $BXh = Xhob$, where h is the C^∞ function on $\bar{T}$.

Let $X,Y,Z \in T$ at p. Then we can write

(3.5.6) a) $FBX = BfX + u(X)N$, b) $FN = -BU$,

(3.5.7) $g(BX,BY)ob = h(X,Y)$,

where N is the unit normal vector to M at p, f is a tensor field of the type (1,1), U is a vector field, u is a 1-form on M and h is the induced metric tensor on M at p.

From (1.1.21)a and (3.5.6) a,b, we get [103]

(3.5.8) $f^2 = I + u \otimes U$, $uof = 0$, $fU = 0$, $u(U) = -1$.

Using (3.5.6)a, and (3.5.7) in (1.1.21)b, we obtain

(3.5.9) a) $h(fX,fY) = -h(X,Y) - u(X)u(Y)$

equivalent to

b) $`f(X,Y) = h(fX,Y)$.

A differentiable manifold M on which there are defined a tensor field f of type (1,1), a vector field U, a 1-form u and a metric tensor field h satisfying (3.5.8), (3.5.9)a for arbitrary vector fields X,Y,Z is called an almost contact hyperbolic metric manifold. Thus the hypersurface of an almost hyperbolic Hermitian manifold is an almost contact hyperbolic metric manifold.

(3.5.15) $\quad d`F = 0 \Leftrightarrow (D_X`F)(Y,Z) + (D_Y`F)(Z,X) + (D_Z`F)(X,Y) = 0$

Thus we call an almost contact hyperbolic metric manifold as a quasi hyperbolic Sasakian manifold if (3.5.15) holds.

An almost contact hyperbolic metric manifold, satisfying

(3.5.16) $\quad (D_X`F)(Y,Z) - u(Y)g(X,Z) + 2u(Z)g(X,Y)$

$$= (D_Y`F)(Z,X) + u(X)g(Y,Z)$$

is called a nearly hyperbolic Sasakian manifold.

HYPERBOLIC COSYMPLECTIC MANIFOLD AS A BASE:

Let us put

(3.5.17) $\quad$ a) $\quad I = I(X,Y,Z) = (D_X`F)(Y,Z)$

b) $\quad \sigma = I(Y,Z,X), \qquad$ c) $\quad \sigma^2 = I(Z,X,Y)$

Now consider a system consisting of all linear combination of I, σ, σ^2, with multiplication as defined by table I

	I	σ	σ^2
I	I	σ	σ^2
σ	σ	σ^2	I
σ^2	σ^2	I	σ

Table I

Then the system is a commutative ring R_1.

From the table, we have $I^2 = I \Leftrightarrow I = 0$,

and $\sigma^3 = I \Leftrightarrow (I-\sigma)(I + \sigma + \sigma^2) = 0$. Now $I = 0 \Leftrightarrow$

(3.5.18) $\quad (D_X`F)(Y,Z) = 0.$

Thus (3.5.18) gives the equation of a manifold and we call it as hyperbolic Cosymplectic manifold.

Let us consider different cases for $\sigma^3 = I$,

Case I: If $I = \sigma$ and $I + \sigma + \sigma^2 \neq 0 \Leftrightarrow$

(3.5.19) $\quad (D_X`F)(Y,Z) = (D_Y`F)(Z,X) \neq 0.$

The above equation we call as nearly hyperbolic Cosymplectic manifold.

Case II: If $I + \sigma + \sigma^2 = 0$ and $I \neq \sigma \Leftrightarrow$

(3.5.20) $\quad (D_X`F)(Y,Z) + (D_Y`F)(Z,X) + (D_Z`F)(X,Y) = 0.$

This gives us the equation of a manifold and we call it as quasi hyperbolic Sasakian manifold.

Case III: If $I = \sigma$ and $I + \sigma + \sigma^2 = 0 \Leftrightarrow I = \sigma = \sigma^2 = 0$.

This shows that the intersection of a nearly hyperbolic Cosymplectic manifold and quasi hyperbolic Sasakian manifold is a hyperbolic Cosymplectic manifold.

Case IV: If $I \neq 0$ and $I + \sigma + \sigma^2 \neq 0$

then the manifold is neither nearly hyperbolic Cosymplectic nor quasi hyperbolic Sasakian manifold.

It can also be proved that

$$(I - \sigma)^r = 0 \text{ and } (I + \sigma + \sigma^2)^r = 0,$$

represents a nearly hyperbolic Cosymplectic and quasi hyperbolic Sasakian manifolds respectively for $r = 1, 2, 3, \ldots$

Let us put

(3.5.21) a) $I_1 = I_1(X,Y,Z) = (D_X`F)(Y,Z) + u(Y)(D_X u)(FZ)$,

b) $\sigma_1 = I_1(Y,Z,X) = (D_Y`F)(Z,X) + u(Z)(D_Y u)(FX)$,

c) $\sigma_1^2 = I_1(Z,X,Y) = (D_Z`F)(X,Y) + u(X)(D_Z u)(FY)$.

(3.5.22) a) $\alpha_1 = I_1(FX,Y,Z)$, b) $\beta_1 = I_1(X,FY,Z)$,

c) $\gamma_1 = I_1(FX,FY,Z)$.

Now consider two systems R_2 and R_3 consisting of all linear combinations of $I_1, \sigma_1, \sigma_1^2$, and $I_1, \alpha_1, \beta_1, \gamma_1$, with multiplication as defined by table II and table III respectively.

	I_1	σ_1	σ_1^2
I_1	I_1	σ_1	σ_1^2
σ_1	σ_1	σ_1^2	I_1
σ_1^2	σ_1^2	I_1	σ_1

Table II

	I_1	α_1	β_1	γ_1
I_1	I_1	α_1	β_1	γ_1
α_1	α_1	I_1	γ_1	β_1
β_1	β_1	γ_1	I_1	α_1
γ_1	γ_1	β_1	α_1	I_1

Table III

Then both the systems R_2 and R_3 are infinite commutative rings.

Table III holds provided

(3.5.23) $(D_U`F) = 0.$

From table II, we have

$$I_1 = 0,\ I_1 = \sigma_1,\ I_1 + \sigma_1 + \sigma_1^2 = 0.$$

Now $I_1 = 0 \Leftrightarrow (D_X`F)(Y,Z) + u(Y)(D_Xu)(FZ) = 0.$

Putting U for Z in the above equation, we get

$$(D_Xu)(FY) = 0.$$

Thus $I_1 = 0$ is equivalent to the equation of a hyperbolic Cosymplectic manifold.

Now $I_1 = \sigma_1 \Leftrightarrow$

$$(D_X`F)(Y,Z) + u(Y)(D_Xu)(FZ) = (D_Y`F)(Z,X) + u(Z)(D_Yu)(FX).$$

Putting U for Z in this equation, we get $(D_Xu)(FY) = 0$. Thus $I_1 = \sigma_1$, represents a nearly hyperbolic Cosymplectic manifold with $(D_Xu)(Y) = 0$. We call this as restricted nearly hyperbolic Cosymplectic manifold.

Also if $I_1 + \sigma_1 + \sigma_1^2 = 0 \Leftrightarrow$

(3.5.24) $$(D_X`F)(Y,Z)+(D_Y`F)(Z,X)+(D_Z`F)(X,Y)+u(Y)(D_Xu)(FZ)$$
$$+u(Z)(D_Yu)(FX)+u(X)(D_Zu)(FY) = 0.$$

Then we call it as generalized quasi hyperbolic Sasakian manifold of the first kind.

From table III, we have $I_1 = 0$, $\gamma_1^2 = I_1$.

Now $\gamma_1^2 = I_1 \Rightarrow \gamma_1 = I_1$ and $\gamma_1 = - I_1$.

Taking $\gamma_1 = I_1 \Leftrightarrow$

(3.5.25) $(D_{FX}`F)(FY,Z) = (D_X`F)(Y,Z) + u(Y)(D_Xu)(FZ)$.

This we call as the equation of an almost contact hyperbolic normal metric manifold.

Putting U for Z in the above equation, we get

(3.5.26) a) $(D_Xu)(FY) = (D_{FX}u)(Y) \Leftrightarrow$

b) $(D_Xu)(Y)= (D_{FX}u)(FY)$

Now taking $\gamma_1 + I_1 = 0 \Leftrightarrow$

(3.5.27) $(D_{FX}`F)(FY,Z) + (D_X`F)(Y,Z) + u(Y)(D_Xu)(FZ) = 0$.

This is called almost contact hyperbolic pseudo normal metric manifold.

Putting U for Z in (3.5.27), we get

(3.5.28) a) $(D_{FX}u)(Y) = -(D_Xu)(FY) \Leftrightarrow$

b) $(D_Xu)(Y) = -(D_{FX}u)(FY)$

Now we have

$(\gamma_1 - I_1)^r = 0$ and $(\gamma_1 + I_1)^r = 0$, $r = 1,2,3,\ldots$.

All represents almost contact hyperbolic normal metric manifold and almost contact hyperbolic pseudo normal metric manifold.

Putting BX for λ and BY for μ in (3.5.4), we get

$$(D_{FBX}F)FBY = (D_{BX}F)BY$$

Using (3.5.6)a in the above equation, we get

$$(D_{BfX}F)FBY - (D_{BX}F)BY + u(X)(D_NF)FBY = 0$$

(3.5.29) $$(D_{BfX}F)BfY-(D_{BX}F)BY+ u(Y)(D_{BfX}F)N + u(X)(D_NF)FBY=0$$

$$-u(X)(D_NF)FBY=D_{BfX}FBfY-FD_{BfX}BfY-D_{BX}FBY+FD_{BX}BY+u(Y)D_{BfX}FN$$
$$-u(Y)FD_{BfX}N.$$

Using (3.5.6)a,b and (3.5.10)a,b in the above equation, we get

$$-u(X)(D_NF)FBY=D_{BfX}Bf^2Y+(fXu(fY))N+u(fY)D_{BfX}N-FBE_{fX}fY$$
$$-F`H(fX,fY)N-D_{BX}BfY-(Xu(Y))N-u(Y)D_{BX}N+FBE_XY$$
$$+F`H(X,Y)N-u(Y)D_{BfX}BU+u(Y)FBHfX$$

$$-u(X)(D_NF)FBY =BE_{fX}f^2Y+`H(fX,f^2Y)N+(fXu(fY))N-BfE_{fX}fY$$
$$-u(E_{fX}fY)N+B`H(fX,fY)U-BE_XfY-`H(X,fY)N-(Xu(Y))N$$
$$+u(Y)BHX+BfE_XY+u(E_XY)N-B`H(X,Y)U-u(Y)BE_{fX}U$$
$$-u(Y)u(HfX)N+u(Y)BfHfX+u(Y)u(HfX)N$$

$$-u(X)(D_NF)FBY=B((E_{fX}f)fY-(E_Xf)Y-u(Y)E_{fX}U+u(Y)(fHf+H)X$$
$$+(`H(fX,fY)-`H(X,Y))U)+((E_{fX}u)(fY)-(E_Xu)(Y)$$
$$-`H(X,fY)+`H(fX,f^2Y))N$$

(3.5.30) $$-u(X)(D_NF)FBY = B(P(X,Y)+Q(X,Y))+(p(X,Y)+q(X,Y))N$$

where

(3.5.31) a) $P(X,Y) = (E_{fX}f)fY - (E_Xf)Y - u(Y)E_{fX}U$

b) $Q(X,Y) = u(Y)(fHf + H)X + (`H(fX,fY) - `H(X,Y))U$

(3.5.32) a) $p(X,Y) = (E_{fX}u)(fY) - (E_Xu)(Y)$

b) $q(X,Y) = -`H(X,fY) + `H(fX,f^2Y)$.

Putting BX for λ and BY for μ in (3.5.5), we get

$$(D_{FBX}F)FBY + (D_{BX}F)BY = 0$$

Using (3.5.6)a in the above equation, we get

$$(D_{BfX}F)FBY + (D_{BX}F)BY + u(X)(D_NF)FBY = 0$$

$$(D_{BfX}F)BfY + (D_{BX}F)BY + u(Y)(D_{BfX}F)N + u(X)(D_NF)FBY = 0$$

$$-u(X)(D_NF)FBY=D_{BfX}FBfY-FD_{BfX}BfY+D_{BX}FBY-FD_{BX}BY+u(Y)D_{BfX}FN$$
$$-u(Y)FD_{BfX}N.$$

Using (3.5.6)a,b and (3.5.10)a,b in the above equation, we get

$$=D_{BfX}Bf^2Y+(fXu(fY))N+u(fY)D_{BfX}N-FBE_{fX}fY-F`H(fX,fY)N+D_{BX}BfY$$
$$+(Xu(Y))N+u(Y)D_{BX}N-FBE_XY-F`H(X,Y)N-u(Y)D_{BfX}BU+u(Y)FBHfX$$
$$=BE_{fX}f^2Y+`H(fX,f^2Y)N+(fXu(fY))N-BfE_{fX}fY-u(E_{fX}fY)N+B`H(fX,fY)U$$
$$+BE_XfY+`H(X,fY)N+(Xu(Y))N-u(Y)BHX-BfE_XY-u(E_XY)N$$
$$+B`H(X,Y)U-u(Y)BE_{fX}U-(Y)u(HfX)N+u(Y)BfHfX+u(Y)u(HfX)N$$
$$-u(X)(D_NF)FBY=B((E_{fX}f)fY+(E_Xf)Y-u(Y)E_{fX}U+u(Y)(fHf-H)X$$
$$+(`H(fX,fY)+`H(X,Y))U)+((E_{fX}u)(fY)+(E_Xu)(Y)$$
$$+`H(X,fY)+`H(fX,f^2Y))N$$

(3.5.33) $-u(Y)(D_N F)FBY = B(R(X,Y) + S(X,Y)) + (r(X,Y) + s(X,Y))N$

where

(3.5.34) a) $R(X,Y) = (E_{fX}f)fY + (E_X f)Y - u(Y)E_{fX}U$

b) $S(X,Y) = u(Y)(fHf-H)X + (`H(fX,fY) + `H(X,Y))U$

(3.5.35) a) $r(X,Y) = (E_{fX}u)(fY) + (E_X u)(Y)$

b) $s(X,Y) = `H(X,fY) + `H(fX,f^2Y)$.

RESULTS AND DISCUSSION

On the basis of analysis carried out in chapter 3 we have been able to deduce following results:

4.1 SECTION 1(A):

CR-SUBMANIFOLDS OF A TRANS PARA SASAKIAN MANIFOLD

Theorem 4.1.1(A): Let M be a CR-submanifold of a trans para Sasakian manifold $\bar{M}$. The distribution $D^{\perp}$ is integrable if and only if

(4.1.1)(A) $A_{FY}Z - A_{FZ}Y = \alpha(u(Y)Z-u(Z)Y)$, for all $Y,Z\in D^{\perp}$.

Proof: Suppose the distribution $D^{\perp}$ is integrable, then $[Y,Z]\in D^{\perp}$ for any $Y,Z\in D^{\perp}$. This gives $P[Y,Z]=0$ and from (3.1.14)(A) we get (4.1.1)(A).

Conversely suppose (4.1.1)A holds. Then by (3.1.14)(A) we have $FP[Y,Z] = 0$, for any $Y, Z\in D^{\perp}$. From this we have $P[Y,Z] = 0$, which is equivalent to $[Y,Z] \in D^{\perp}$ for all $Y,Z\in D^{\perp}$, which implies that $D^{\perp}$ is integrable. This completes the proof of the theorem.

Theorem 4.1.2(A): Let M be a U-horizontal CR-submanifold of a trans para Sasakian manifold $\bar{M}$. The distribution D is integrable if and only if

$h(X,FY) = h(Y,FX)$, for all $X,Y\in D$.

Proof: From (3.1.9)(A) for all $X,Y \in D$, we have

(4.1.2)(A) $h(X,FY) = FQ\nabla_X Y + Ch(X,Y)$.

Similarly, we have

(4.1.3)(A) $h(Y,FX) = FQ\nabla_Y X + Ch(X,Y)$.

From (4.1.2)(A) and (4.1.3)(A), we get

(4.1.4)(A) $h(X,FY) - h(Y,FX) = FQ[X,Y]$.

As the distribution D is integrable, that is, $Q[X,Y] = 0$.

Using $Q[X,Y] = 0$ in equation (4.1.4)(A), we get the result.

Conversely, we have

$$h(X,FY) = h(Y,FX)$$

From equation (4.1.4)(A) and above, we get

$$FQ[X,Y] = 0 \Rightarrow Q[X,Y] = 0$$

$\Rightarrow$ D is integrable. This proves our assertion.

Remark 4.1.1(A): For a trans Sasakian manifold, we have from (3.1.2)(A)

(4.1.5)(A) $\nabla_X U + h(X,U) = \alpha FPX + \alpha FQX + \beta(X - u(X)U)$.

From (4.1.5)(A), we get the following two equations

(4.1.6)(A) $\nabla_X U = \alpha FPX + \beta(X - u(X)U)$,

(4.1.7)(A) $h(X,U) = \alpha FQX$.

Now from (4.1.6)(A) and (4.1.7)(A), we get the following two relations:

(4.1.8)(A) $\nabla_X U = \beta(X - u(X)U)$, for $X \in D^{\perp}$,

(4.1.9)(A) $h(X,U) = 0$, for $X \in D$.

Theorem 4.1.3(A): Let M be a D-umbilic U-horizontal CR-submanifold of a trans para Sasakian manifold $\bar{M}$, then M is D- totally geodesic.

Proof: Let M be a D-umbilic U-horizontal CR-submanifold, that is

$$h(X,Y)=g(X,Y)H, \text{ for all } X,Y \in D.$$

Putting Y=U in the above equation, we have

$$h(X,U) = g(X,U)H$$

Using (4.1.9)(A) in the above equation, we get

$H = 0 \Rightarrow h(X,Y) = 0$.This shows that M is D-totally geodesic.

4.2 SECTION 1(B):

CR-SUBMANIFOLDS OF NEARLY AND CLOSELY PARA COSYMPLECTIC MANIFOLDS

Theorem 4.1.1(B): Let M be a submanifold of a nearly para Cosymplectic manifold $\bar{M}$. If

$U \in TM$ then $\forall$ $X,Y \in TM$ we get

(4.1.1)(B) $P[X,Y] = -\nabla_X PY - \nabla_Y PX + A_{QY}X + A_{QX}Y + 2P\nabla_X Y + 2th(X,Y)$,

(4.1.2)(B) $Q[X,Y] = -\nabla_X^{\perp} QY - \nabla_Y^{\perp} QX - h(X,PY) - h(PX,Y) + 2Q\nabla_X Y$

$+2fh(X,Y)$.

Proof: By virtue of equation (3.1.8)(B) and (3.1.19)(B) we get

$(\nabla_X PY - P\nabla_X Y - \nabla_Y PX + P\nabla_Y X - A_{QY}X + A_{QX}Y - 2\nabla_X PY + 2P\nabla_X Y$

$+ 2A_{QY}X + 2th(X,Y)) + (\nabla_X^{\perp} QY - Q\nabla_X Y - \nabla_Y^{\perp} QX - Q\nabla_Y X + h(X,PY)$

$- h(PX,Y) - 2\nabla_X^{\perp} QY + 2Q\nabla_X Y - 2h(X,PY) + 2fh(X,Y)) = 0.$

Now equating tangential and normal parts of the above equation we get equation (4.1.1)(B) and (4.1.2)(B).

Theorem 4.1.2(B): Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$. We have

(a) if $D^0 \oplus \{U\}$ is autoparallel then

$A_{QY}X + A_{QX}Y + 2th(X,Y) = 0, \forall$ $X,Y \in D^0 \oplus \{U\}$,

(b) if $D^1 \oplus \{U\}$ is autoparallel then

$$h(X,PY) + h(PX,Y) = 2fh(X,Y)\ ,\ \forall\ X,Y \in D^1 \oplus \{U\}.$$

Proof: Using equation (3.1.17)(B) and autoparallelness of $D^0 \oplus \{U\}$ we get (a) and using equation (3.1.18)(B) and autoparallelness of $D^1 \oplus \{U\}$ we get (b). This completes the proof.

Theorem 4.1.3(B): Let M be a submanifold of a nearly para Cosymplectic manifold $\bar{M}$ with $U \in TM$. If M is invariant then M is nearly para Cosymplectic manifold $\bar{M}$. Moreover

(4.1.3)(B) $h(X,PY) + h(PX,Y) - 2fh(X,Y) = 0,\ \forall\ X,Y \in TM.$

Proof: From $D^1 \oplus \{U\} = Ker(Q)$ and equation (3.1.18)(B) we get equation (4.1.3)(B).

Lemma 4.1.1(B): Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$, $\forall\ X,Y \in D^1 \oplus \{U\}$ we get

(4.1.4)(B) $Q[X,Y] = -\ h(X,PY) - h(PX,Y) + 2Q\nabla_X Y + 2fh(X,Y)$

or equivalently

(4.1.5)(B) $-\ h(X,PX) + Q\nabla_X X + fh(X,X) = 0.$

Proof: Using $D^1 \oplus \{U\} = Ker(Q)$ and equation (4.1.2)(B) we get equation (4.1.4)(B) and using X =Y in equation (4.1.4)(B) we get the required result.

Theorem 4.1.4(B): The distribution $D^1 \oplus \{U\}$ on a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$ is integrable if and only if

(4.1.6)(B) $h(X,PY) + h(PX,Y) = 2(Q\nabla_X Y + fh(X,Y))$.

Proof: From $D^1 \oplus \{U\} = Ker(Q)$ and using equation (4.1.4)(B) we get (4.1.5)(B).

Theorem 4.1.5(B): Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$. Then the following statements:

(I) the distribution $D^1 \oplus \{U\}$ is autoparallel,

(II) $h(X,PY) + h(PX,Y) = 2fh(X,Y)$, $X,Y \in D^1 \oplus \{U\}$,

(III) $h(X,PX) = fh(X,X)$, $X \in D^1 \oplus \{U\}$,

(IV) the distribution $D^1 \oplus \{U\}$ is nearly autoparallel,

are related by (I)$\Rightarrow$(II)$\Leftrightarrow$ (III)$\Rightarrow$(IV) .In particular if $D^1 \oplus \{U\}$ is integrable then the above four statements are equivalent.

Proof: (I)$\Rightarrow$(II) follows from theorem (4.1.2)(B)(b). Putting X=Y in (II) we get (II)$\Leftrightarrow$(III). From (4.1.5)(B) we get (III)$\Rightarrow$(IV). This completes the proof of the theorem.

Theorem 4.1.6(B): Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$, such that M is $D^1 \oplus \{U\}$-totally umbilical, then

(I) the distribution $D^1 \oplus \{U\}$ is nearly autoparallel.

Consequently, the following two statements becomes equivalent:

(II) the distribution $D^1 \oplus \{U\}$ is integable,

(III) the distribution $D^1 \oplus \{U\}$ is autoparallel.

Proof: From proposition (3.1.4)(B), M is $D^1 \oplus \{U\}$-totally geodesic that is $h = 0$. Thus from (4.1.5)(B) we get $\nabla_X X = 0$ the statement (I) holds. Hence from definition we get (II)$\Leftrightarrow$(III).

Corollary 4.1.1(B): In a totally umbilical CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$, $D^1 \oplus \{U\}$ is autoparallel.

Proof: Using theorem (4.1.6)(B) we get the result.

Theorem 4.1.7(B): Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$. Then the distribution $D^0 \oplus \{U\}$ is integrable if and only if

$$A_{QX}Y - A_{QY}X = 0, \ \forall\, X,Y \in D^0 \oplus \{U\}.$$

Proof: From $D^0 \oplus \{U\} = \mathrm{Ker}(P)$ and equation (3.1.20)(B), we get the result.

Theorem 4.1.8: Let M be a CR-submanifold of a nearly para Cosymplectic manifold $\bar{M}$. Then the distribution D^0 is integrable if and only if

$$A_{QX}Y - A_{QY}X = 0, \ \forall\, X,Y \in D^0.$$

Proof: By definition of D^0 and equation (3.1.20)(B), we get the result.

Theorem 4.1.9(B): Let M be a CR-submanifold of a para Cosymplectic manifold $\bar{M}$. Then the distribution D^0 and $D^0 \oplus \{U\}$ are integrable.

Proof: The result follows from theorem (4.1.7)(B) and theorem (4.1.8)(B).

Theorem 4.1.10(B): If M is a totally para contact umbilical CR-submanifold of a closely para Cosymplectic manifold $\bar{M}$, then M is (D^1,D^0)-mixed totally geodesic .

Proof: Now we have $h(X,Y) = g(X,Y)\,K$, for $X,Y \in \{U\}^{\perp}$,

$h(U,U) = g(U,U)\,K$,

$0 = g(U,U)\,K$, by using Gauss equation

$\Rightarrow K = 0$.

Therefore M is (D^1,D^0)-mixed totally geodesic .This completes our assertions.

Theorem 4.1.11(B): Let M be a totally para contact umbilical submanifold of a closely para Cosymplectic manifold $\bar{M}$, then either $D^0 = \{0\}$ or $\mathrm{Dim}(D^0) = 1$ or the normal vector field K is orthogonal to FD^0.

Proof: If $\mathrm{Dim}\,(D^0) > 1$, for each $H \in D^0$, $\exists\, X \in D^0$ such that $g(X,H) = 0$ and $||X|| = 0$, then

$g(K,FH) = g(h(X,X),FH) = g(A_{FH}X,X) = g(A_{FX}H,X) = g(h(X,H),FX) = 0$.

This gives desired result.

4.3 SECTION 2(A):

SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY r-SASAKIAN MANIFOLDS

Theorem 4.2.1(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then the distribution $D\oplus\{U_p\}$ is integrable if the following conditions are satisfied:

(4.2.1)(A) $S(X,Y)\in D\oplus\{U_p\}$,

where S(X,Y) is the torsion tensor in equation (4.2.1)(A)

(4.2.2)(A) $h(X,FY)=h(FX,Y)$, for any $X,Y\in D\oplus\{U_p\}$.

Proof: The torsion tensor S(X,Y) of an almost r-contact structure (F,U_p,u^p,g) is given by $S(X,Y)=N(X,Y)+2du^p(X,Y)U_p$,

where N(X,Y) is Nijenhuis tensor of F.

Thus we have

(4.2.3)(A) $S(X,Y)=[FX,FY]-F[FX,Y]-F[X,FY]+F^2[X,Y]+2du^p(X,Y)U_p$,

for any $X,Y\in TM$.

Suppose that the distribution $D\oplus\{U_p\}$ is integrable and for all $X,Y\in D\oplus\{U_p\}$, $Q[X,Y]=0$ and therefore (4.2.3)(A) gives (4.2.1)(A).

Using (3.2.1)(A) we get

$S(X,Y)\in D\oplus\{U_p\}$ if and only if $Q(\nabla_Y F)(X)=0$, for any $X,Y\in D\oplus\{U_p\}$.

Now taking FZ where $Z\in D$ instead of Y in the above equation, we get

$FQ\nabla_{FZ}X + Ch(FX,FZ) + h(X,FZ)=0$, for any $X,Z\in D$.

Interchanging X and Z and subtracting these relations, we get

$FQ[FX,FZ]+h(Z,FX)-h(X,FZ)=0$.

Thus from the above equation we get the result.

Theorem 4.2.2(A): Let M be a semi–invariant submanifold of a nearly r-Sasakian manifold $\bar{M}$. Then the distribution $D^{\perp}$ is integrable if and only if

(4.2.4)(A) $A_{FY}Z-A_{FZ}Y=0$, for any $Y, Z\in D^{\perp}$.

Proof: Suppose the distribution $D^{\perp}$ is integrable, then $[Y,Z]\in D^{\perp}$ for any $Y,Z\in D^{\perp}$. This gives $P[Y,Z]=0$ and from (3.2.33)(A) we get (4.2.4)(A).

Conversely suppose (4.2.4)(A) holds. Then in view of (3.2.33)(A) we have $FP[Y,Z]=0$, for any $Y,Z\in D^{\perp}$. Since rank $F=2n$, therefore we have $P[Y,Z]=0$, which is equivalent to $[Y,Z] \in D^{\perp}$ for all $Y,Z\in D^{\perp}$, which shows that $D^{\perp}$ is integrable.

Theorem 4.2.3(A) Any proper totally r-contact umbilical semi-invariant submanifold of a nearly r-Sasakian manifold is totally r-contact geodesic.

Proof: We know that

$(\bar{\nabla}_X F)(FX)= \bar{\nabla}_X F^2X-F\ \bar{\nabla}_X FY$.

Using (1.1.17) in the above equation, we have

$$(\bar{\nabla}_X F)(FX)=-\ \bar{\nabla}_X X+(\bar{\nabla}_X u^p(X))U_p+u^p(X)\ \bar{\nabla}_X U_p-F\ \bar{\nabla}_X FX,$$

$$=-\ \bar{\nabla}_X X+(\bar{\nabla}_X u^p)(X)U_p+u^p(\bar{\nabla}_X X)U_p+u^p(X)\ \bar{\nabla}_X U_p-F\ \bar{\nabla}_X FX,$$

$$=F^2\ \bar{\nabla}_X X+(\bar{\nabla}_X u^p)(X)U_p+u^p(X)\ \bar{\nabla}_X U_p-F\ \bar{\nabla}_X FX,$$

(4.2.5)(A) $(\bar{\nabla}_X F)(FX) = -F((\bar{\nabla}_X F)(X)) + (\bar{\nabla}_X u^p)(X)U_p + u^p(X)\,\bar{\nabla}_X U_p$,

for any $X \in T\bar{M}$.

From equation (3.2.1)(A) and (3.2.3)(A), we have

$(\bar{\nabla}_X F)(X) = u^p(X)X - g(X,X)U_p$

and

$g(\bar{\nabla}_X U_p, X) = (\bar{\nabla}_X u^p)(X) = 0.$

From equation (4.2.5)(A) and the above equation, we get

(4.2.6)(A) $(\bar{\nabla}_X F)(FX) = -u^p(X)FX + u^p(X)\,\bar{\nabla}_X U_p.$

Now let $X \in D$ and from (4.2.6)(A), we have

$$g((\bar{\nabla}_X F)(FX), H) = 0,$$

$$g(\bar{\nabla}_X F^2 X, H) - g(F(\bar{\nabla}_X FX), H) = 0,$$

$$g(\bar{\nabla}_X FX, FH) - g(\bar{\nabla}_X X, H) = 0,$$

$$g(\bar{\nabla}_X H, X) - g(\bar{\nabla}_X FH, FX) = 0,$$

$$g(A_{FH}X, FX) - g(A_H X, X) = 0.$$

Using (3.2.5)(A) in the above equation, we have

(4.2.7)(A) $g(h(X,FX),FH) - g(h(X,X),H) = 0.$

Using (3.2.34)(A) in (4.2.7)(A), we get

$$g(g(FX,F^2X)H,FH) - g(g(FX,FX)H,H) = 0,$$

$$-g(FX,X)g(H,FH) - g(FX,FX)g(H,H) = 0,$$

$$g(X,X)g(H,H) = 0.$$

Consequently, $H=0$ which completes the proof.

4.4 SECTION 2(B):

SEMI-INVARIANT SUBMANIFOLDS OF r-KENMOTSU MANIFOLDS

Theorem 4.2.1(B): Let M be a semi–invariant submanifold of a r-Kenmotsu manifold $\bar{M}$. If M is U_p-horizontal, then the distribution D is integrable if and only if,

(4.2.1)(B) $h(X,FY)=h(Y,FX)$, for all $X,Y\in D$,

and if M is U_p-vertical, then the distribution $D^{\perp}$ is integrable if and only if,

(4.2.2)(B) $A_{FX}Y - A_{FY}X = u^p(X)FY - u^p(Y)FX + 2g(FX,Y)U_p$.

Proof: Let M be U_P-horizontal then equation (3.2.11)(B) reduces to

$$h(X,FY) = FQ\nabla_X Y + Ch(X,Y),$$

Similarly, we have

$$h(Y,FX) = FQ\nabla_Y X + Ch(X,Y).$$

From these two equations, we have

(4.2.3)(B) $h(X,FY) - h(Y,FX) = FQ[X,Y]$.

Thus if M is U_p-horizontal then $[X,Y]\in D$, that is, $Q[X,Y] = 0$.

Using this in equation (4.2.3)(B), we get the result.

Conversely, using (4.2.1)(B) in (4.2.3)(B), we get

$FQ[X,Y] = 0 \Rightarrow Q[X,Y] = 0 \Rightarrow [X,Y] \in D \Rightarrow D$ is integrable.

This proves the first part.

Let M be U_p-vertical then (3.2.11)(B) reduces to

(4.2.4)(B) $\nabla_X{}^{\perp}FY = FQ\nabla_X Y + Ch(X,Y)$, for all $X,Y \in D^{\perp}$.

We know that

$$\bar{\nabla}_X FY = (\bar{\nabla}_X F)Y + F\bar{\nabla}_X Y$$

$$-A_{FY}X+\nabla_X{}^{\perp}FY=g(FX,Y)U_p-u^p(Y)FX+FP\nabla_X Y+FQ\nabla_X Y+Bh(X,Y)$$
$$+Ch(X,Y)$$

(4.2.5)(B) $\nabla_X{}^{\perp}FY=g(FX,Y)U_p-u^p(Y)FX+FP\nabla_X Y+FQ\nabla_X Y+Bh(X,Y)$
$$+Ch(X,Y)+A_{FY}X$$

From (4.2.4)(B) and (4.2.5)(B), we have

$$FP\nabla_X Y = u^p(Y)FX - g(FX,Y)U_p - Bh(X,Y) - A_{FY}X$$

Similarly we have

$$FP\nabla_Y X = u^p(X)FY - g(FY,X)U_p - Bh(X,Y) - A_{FX}Y$$

From the above two equations we have

(4.2.6)(B) $FP[X,Y] = u^p(Y)FX - u^p(X)FY - 2g(FX,Y)U_p + A_{FX}Y - A_{FY}X$

Thus if M is U_p-vertical then $[X,Y] \in D^{\perp}$ that is $P[X,Y] = 0$.

Using this in (4.2.6)(B), we get (4.2.2)(B).

Conversely, using (4.2.2)(B) in (4.2.6)(B), we get

$FP[X,Y] = 0 \Rightarrow P[X,Y] = 0 \Rightarrow [X,Y] \in D^{\perp} \Rightarrow D^{\perp}$ is integrable.

This proves the second part.

Theorem 4.2.2(B): If M is totally r-contact umbilical semi-invariant submanifold of a r-Kenmotsu manifold with parallel horizontal distributions, then M is totally r-contact geodesic.

Proof: Using (3.2.9)(B) and (3.2.10)(B), we have

$$g(\nabla_X FPY+\nabla_Y FPX-A_{FQY}X-A_{FQX}Y,Z)=g(FP\nabla_X Y+FP\nabla_Y X+2Bh(X,Y),Z)$$
$$+g(FX,Y)u^p(Z)-g(FX,Z)u^p(Y).$$

Since M is totally r-contact umbilical, we have

$$g(\nabla_X FPY,Z)+g(\nabla_Y FPX,Z)-g(h(X,Z),FQY)-g(h(Y,Z),FQX)$$
$$=g(FP\nabla_X Y,Z)+g(FP\nabla_Y X,Z)+2g(X,Y)g(BH,Z)-2u^p(X)u^p(Y)g(BH,Z)$$
$$-2u^p(X)g(h(Y,U_p),FZ)-2u^p(Y)g(h(X,U_p),FZ)+u^p(Z)g(FX,Y)$$
$$-u^p(Y)g(FX,Z)$$

Again using (3.2.14)(B) in the above equation and replacing Y by BH and Z by X we get

(4.2.7)(B) $$g(\nabla_{BH}FPX,X)-g(X,X)g(H,FBH)-g(BH,X)g(H,FQX)$$
$$=g(FP\nabla_X BH,X)+g(FP\nabla_{BH}X,X)+2g(X,BH)g(BH,X)$$
$$-2u^p(X)u^p(BH)g(BH,X)-2u^p(X)g(h(BH,U_p),FX)$$
$$-2u^p(BH)g(h(X,U_p),FX)+u^p(X)g(FX,BH)$$
$$-u^p(BH)g(FX,X)$$

Also for each $X\in D$, we have

$$g(X,BH) = g(FX,BH) = 0.$$

Now differentiating above equation with respect to X, we get

$$g(\nabla_X FX,BH) + g(FX,\nabla_X BH) = 0.$$

Since the horizontal distribution D is parallel, we have

(4.2.8)(B) $\quad g(FX,\nabla_X BH) = 0.$

Using the above facts in (4.2.7)(B) and taking $X \in D$ as unit vector, we get

$$g(\nabla_{BH} FPX,X) - g(H,FBH) = g(FP\nabla_{BH} X,X)$$

(4.2.9)(B) $\quad g((\nabla_{BH} FP)X,X) = -g(FH,BH) = -g(BH,BH).$

Using Gauss formula we have

$$(\bar{\nabla}_{BH} FP)(X) = (\nabla_{BH} FP)(X)$$

Again from Gauss formula, we have

$$g((\nabla_{BH} FP)X,X) = 0.$$

Hence from (4.2.9)(B) and above equation, we get $g(BH,BH)=0 \Rightarrow BH= 0$.

Since $FH \in D^{\perp}$, we have $CH = 0$, hence $FH = 0 \Rightarrow H = 0$. Which gives us that M is totally r-contact geodesic.

This completes the proof of the theorem.

Remark 4.2.1(B): For r-Kenmotsu manifold we have from equation (3.2.2)(B), (3.2.3)(B) and (3.2.6)(B)

(4.2.10)(B) $\quad \bar{\nabla}_X U_p = \nabla_X U_p + h(X,U_p) = PX + QX - u^p(X)PU_p.$

Equating the tangential and normal components of the above equation, we have

(4.2.11)(B) $\nabla_X U_p = PX - u^p(X)PU_p$.

(4.2.12)(B) $h(X,U_p) = QX$

From (4.2.11)(B) and (4.2.12)(B) we have

(4.2.13)(B) $\nabla_X U_p = 0, \forall X \in D^\perp$ and

(4.2.14)(B) $h(X,U_p) = 0, \forall X \in D$.

Note that, for $X \in D$ we have, $g(A_N U_p, X) = g(h(X,U_p),N) = 0$

And so we have $A_N U_p \in D^\perp$.

Theorem 4.2.3(B): Let M be D-umbilic (resp. $D^\perp$-umbilic) semi-invariant submanifold of a r-Kenmotsu manifold. If M is U_p-horizontal (resp.U_p-vertical) then M is D totally geodesic (resp.$D^\perp$-totally geodesic).

Proof: Let M is D-umbilical and U_P-horizontal then $h(X.Y) = g(X,Y)H$.

Putting $Y = U_P$ in the above equation, we obtain

$$h(X,U_p) = g(X,U_p)H$$

Again using (4.2.14)(B) in the above equation, we get

$$H = 0 \Rightarrow h(X,Y) = 0.$$

Hence M is D-totally geodesic.

Similarly if M is $D^\perp$-umbilical, U_p-vertical semi-invariant submanifold then M is $D^\perp$-totally geodesic.

4.5 SECTION 3(A):

CR-SUBMANIFOLDS OF A TRANS HYPERBOLIC SASAKIAN MANIFOLD I

Theorem 4.3.1(A): Let M be a CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. The distribution $D^{\perp}$ is integrable if and only if

(4.3.1)(A) $A_{FY}Z - A_{FZ}Y = \alpha(u(Y)Z - u(Z)Y)$, for all $Y,Z \in D^{\perp}$.

Proof: Suppose the distribution $D^{\perp}$ is integrable, then $[Y,Z] \in D^{\perp}$ for any $Y,Z \in D^{\perp}$. This gives $P[Y,Z]=0$ and from (3.3.15)(A) we get (4.3.1)(A).

Conversely suppose (4.3.1)(A) holds. Then by (3.3.15)(A) we have $FP[Y,Z] = 0$, for any $Y,Z \in D^{\perp}$. From this we have $P[Y,Z]=0$, which is equivalent to $[Y,Z] \in D^{\perp}$ for all $Y,Z \in D^{\perp}$ and $D^{\perp}$ is integrable. This completes the proof of the theorem.

Theorem 4.3.2(A): Let M be a U-horizontal CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. The distribution D is integrable if and only if

$h(X,FY) = h(Y,FX)$, for all $X,Y \in D$.

Proof: Let the disribution D is integrable. Now from (3.3.10)(A) for all $X,Y \in D$, we have

(4.3.2)(A) $h(X,FY) = FQ\nabla_X Y + Ch(X,Y)$.

Similarly, we have

(4.3.3)(A) $h(Y,FX) = FQ\nabla_Y X + Ch(X,Y)$.

From (4.3.2)(A) and (4.3.3)(A), we get

(4.3.4)(A) $h(X,FY) - h(Y,FX) = FQ[X,Y]$.

As the distribution D is integrable $Q[X,Y] = 0$.

Using $Q[X,Y] = 0$ in (4.3.4)(A),we get the result.

Conversely, using $h(X,FY) = h(Y,FX)$ in (4.3.4)(A), we get

$FQ[X,Y] = 0 \Rightarrow Q[X,Y] = 0 \Rightarrow [X,Y] \in D \Rightarrow D$ is integrable.

Remark: Now from (3.3.3)(A), we have

(4.3.5)(A) $\nabla_X U + h(X,U) = -\alpha FPX - \alpha FQX - \beta(X + u(X)U)$.

From (4.3.5)(A), we get the following equations

(4.3.6)(A) $\nabla_X U = -\alpha FPX - \beta(X + u(X)U)$,

(4.3.7)(A) $h(X,U) = -\alpha FQX$.

Now from (4.3.6)(A) and (4.3.7)(A), we get the following relations:

(4.3.8)(A) $\nabla_X U = -\beta(X + u(X)U)$, for $X \in D^{\perp}$,

(4.3.9)(A) $h(X,U) = 0$, for $X \in D$.

Theorem 4.3.3(A): Let M be a D-umbilic U-horizontal CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$, then M is D- totally geodesic.

Proof: Let M be a D-umblic U-horizontal CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$, then we have

$h(X,Y) = g(X,Y)H$, for all $X,Y \in D$.

Putting Y=U in the above equation, we get

$h(X,U) = g(X,U)H$.

Using (4.3.9)(A) in the above equation, we get

$\Rightarrow H = 0 \quad \Rightarrow h(X,Y) = 0.$

This shows that M is D-totally geodesic.

4.6 SECTION 3(B):

CR-SUBMANIFOLDS OF A TRANS HYPERBOLIC SASAKIAN MANIFOLD II

Theorem 4.3.1(B): Let M be a CR-submanifold of a trans hyperbolic Sasakian manifold $\bar{M}$. If M is U-horizontal, then the distribution D is integrable if and only if,

(4.3.1)(B) $h(X,FY)=h(Y,FX)$, for all $X,Y\in D$,

and if M is U-vertical then the distribution $D^{\perp}$ is integrable if and only if,

(4.3.2)(B) $A_{FX}Y-A_{FY}X=1/2n[(u(X)Y-u(Y)X)\delta F(U)$

$$-(u(X)FY-u(Y)FX-2g(X,FY)U)\delta u]$$

Proof: For the first part, let M be U-horizontal then (3.3.11)(B) reduces to

$$h(X,FY) = FQ\nabla_X Y + Ch(X,Y), \text{ for all } X,Y\in D.$$

Similarly, we have

$$h(Y,FX) = FQ\nabla_Y X + Ch(X,Y).$$

From the above two equations, we have

(4.3.3)(B) $h(X,FY) - h(Y,FX) = FQ[X,Y]$.

If M is U-horizontal it means $[X,Y]\in D \Rightarrow Q[X,Y] = 0$. Now from equation (4.3.3)(B), we get

$$h(X,FY) = h(Y,FX).$$

Conversely, using (4.3.1)(B) in (4.3.3)(B), we get

$FQ[X,Y] = 0 \Rightarrow Q[X,Y] = 0 \Rightarrow [X,Y] \in D \Rightarrow D$ is integrable.

This proves the first part.

For the second part, let M be U-vertical, then (3.3.11)(B) reduces to

(4.3.4)(B) $\quad \nabla_X^{\perp} FY = FQ\nabla_X Y + Ch(X,Y)$, for all $X,Y \in D^{\perp}$.

We know that

$$\bar{\nabla}_X FY = (\bar{\nabla}_X F)Y + F\,\bar{\nabla}_X Y$$

$$-A_{FY}X + \nabla_X^{\perp} FY = 1/2n[(g(X,Y)U - u(Y)X)\delta F(U) + (g(X,FY)U + u(Y)FX)\delta u]$$

$$+ FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y)$$

(4.3.5)(B) $\quad \nabla_X^{\perp} FY = 1/2n[(g(X,Y)U - u(Y)X)\delta F(U) + (g(X,FY)U$

$$+u(Y)FX)\delta u] + FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y)$$

$$+A_{FY}X$$

From (4.3.4)(B) and (4.3.5)(B), we have

$$FP\nabla_X Y = -1/2n[(g(X,Y)U - u(Y)X)\delta F(U) + (g(X,FY)U + u(Y)FX)\delta u]$$

$$- Bh(X,Y) - A_{FY}X$$

Similarly we have

$$FP\nabla_Y X = -1/2n[(g(X,Y)U - u(X)Y)\delta F(U) + (g(Y,FX)U + u(X)FY)\delta u]$$

$$- Bh(X,Y) - A_{FX}Y$$

Now from above two equations, we have

(4.3.6)(B) $FP[X,Y]=-1/2n[(u(X)Y-u(Y)X)\delta F(U)+(u(Y)FX-u(X)FY$

$+2g(X,FY)U)\delta u]+ A_{FX}Y - A_{FY}X.$

Thus if M is U-vertical, it means $[X,Y] \in D^{\perp} \Rightarrow P[X,Y] = 0$. Using this in the above equation, we get

$A_{FX}Y - A_{FY}X = 1/2n[(u(X)Y - u(Y)X)\delta F(U) + (u(Y)FX - u(X)FY$

$+ 2g(X,FY)U)\delta u].$

Conversely, using (4.3.2)(B) in (4.3.6)(B), we get

$FP[X,Y] = 0 \Rightarrow P[X,Y] = 0 \Rightarrow [X,Y] \in D^{\perp} \Rightarrow D^{\perp}$ is integrable.

This proves the second part.

This completes the proof of the theorem.

Theorem 4.3.2(B): If M is totally hyperbolic contact umbilical CR-submanifold of a trans hyperbolic Sasakian manifold with parallel horizontal distribution, then M is totally hyperbolic contact geodesic.

Proof: From (3.3.9)(B) and (3.3.10)(B), we have

$g(\nabla_X FPY+\nabla_Y FPX-A_{FQY}X-A_{FQX}Y,Z)=g(FP\nabla_X Y+FP\nabla_Y X+2Bh(X,Y),Z)$

$+1/2n[(g(X,Y)u(Z)-g(X,Z)u(Y))\delta F(U)$

$+(g(X,FY)u(Z)-g(X,FZ)u(Y))\delta u].$

Since M is totally hyperbolic contact umbilical, we have

$g(\nabla_X FPY,Z)+g(\nabla_Y FPX,Z)-g(X,Z)g(FQY,H)-g(Y,Z)g(FQX,H)$

$= g(FP\nabla_X Y,Z)+g(FP\nabla_Y X,Z)+2g(X,Y)g(BH,Z)+2u(X)u(Y)g(BH,Z)$

$-2u(X)g(h(Y,U),FZ)-2u(Y)g(h(X,U),FZ)+1/2n[(g(X,Y)u(Z)$

$-g(X,Z)u(Y))\delta F(U)+(g(X,FY)u(Z)-g(X,FZ)u(Y)\delta u].$

Putting BH for Y and X for Z in the above equation, we get

(4.3.7)(B) $g(\nabla_{BH}FPX,X)-g(X,X)g(H,FBH)-g(BH,X)g(H,FQX)$

$=g(FP\nabla_X BH,X)+g(FP\nabla_{BH}X,X)+2g(X,BH)g(BH,X)$

$+2u(X)u(BH)g(BH,X)-2u(X)g(h(BH,U),FX)$

$-2u(BH)g(h(X,U),FX)+1/2n[(g(X,BH)u(X)$

$-g(X,X)u(BH))\delta F(U)+(g(X,FBH)u(X)$

$-g(X,FX)u(BH)\delta u].$

Suppose $X\in D$ be a unit vector so we have

$g(X,BH) = g(FX,BH) = 0.$

Taking covariant differentiation of the above with respect to X, we have

$g(\nabla_X FX,BH) + g(FX,\nabla_X BH) = 0.$

Since D is parallel, we have

(4.3.8)(B) $g(FX,\nabla_X BH) = 0.$

Using above facts in (4.3.7)(B) and taking $X\in D$ as unit vector, we get

$g(\nabla_{BH}FPX,X) - g(H,FBH) = g(FP\nabla_{BH}X,X)$

(4.3.9)(B) $g((\nabla_{BH}FP)X,X) = -g(FH,BH) = -g(BH,BH).$

Also from Gauss formula, we have

$$(\bar{\nabla}_{BH}FP)(X) = (\nabla_{BH}FP)(X)$$

Again from Gauss formula, we have

$$g((\nabla_{BH}FP)X,X) = 0.$$

Hence from (4.3.9)(B) and above equation, we get $g(BH,BH)=0 \Rightarrow BH=0$.

Since $FH \in D^{\perp}$, therefore $CH = 0$, that is $FH = 0 \Rightarrow H = 0$. Hence M is totally hyperbolic contact geodesic.

Remark 4.3.1(B): For a trans hyperbolic Sasakian manifold, we have

$$\bar{\nabla}_X U = 1/2n[(X + u(X)U)\delta u - FX\delta F(U)]$$

Equating the tangential and normal components of the above equation, we get

$$\nabla_X U = 1/2n[(PX + u(X)PU)\delta u - FPX\delta F(U)]$$

$$h(X,U) = 1/2n[(QX + u(X)QU)\delta u - FQX\delta F(U)]$$

From above two equations, we have

(4.3.10)(B) $\nabla_X U = 0, \forall X \in D^{\perp}$

(4.3.11)(B) $h(X,U) = 0, \forall X \in D.$

Theorem 4.3.3(B): Let M be D-umbilic (resp. $D^{\perp}$-umbilic) CR-submanifold of a trans hyperbolic Sasakian manifold. If M is U-

horizontal (resp.U-vertical) then M is D-totally geodesic (resp.$D^{\perp}$-totally geodesic).

Proof: If M is U-horizontal D-umbilical CR-submanifold of a trans hyperbolic Sasakian manifold then we have $h(X,Y) = g(X,Y)H$.

Putting $Y = U$ in the above equation, we have

$$h(X,U) = g(X,U)H.$$

Using (4.3.11)(B) in the above equation, we get

$$H = 0 \Rightarrow h(X,Y) = 0.$$

Hence M is D-totally geodesic.

Similarly we can easily prove that if M is $D^{\perp}$-umbilical CR-submanifold with U-vertical then M is $D^{\perp}$-totally geodesic.

4.7 SECTION 4(A):

SEMI-INVARIANT SUBMANIFOLDS OF PARA KENMOTSU MANIFOLDS I

Theorem 4.4.1(A): Let M be a semi-invariant submanifold of a para Kenmotsu manifold $\bar{M}$. If M is U-horizontal, then the distribution D is integrable if and only if,

(4.4.1)(A) $h(X,FY)=h(Y,FX)$, for all $X,Y\in D$,

and if M is U-vertical then the distribution $D^{\perp}$ is integrable if and only if,

(4.4.2)(A) $A_{FX}Y - A_{FY}X = u(X)FY - u(Y)FX$, for all $X,Y\in D^{\perp}$.

Proof: For the first part, let M be U-horizontal then equation (3.4.11)(A) reduces to

$$h(X,FY) = FQ\nabla_X Y + Ch(X,Y),$$

Similarly, we have

$$h(Y,FX) = FQ\nabla_Y X + Ch(X,Y).$$

From above two equations, we have

(4.4.3)(A) $h(X,FY) - h(Y,FX) = FQ[X,Y]$.

If $[X,Y]\in D$, then $Q[X,Y] = 0$. Thus from above equation, we get (4.4.1)(A).

Conversely, using (4.4.1)(A) in (4.4.3)(A), we get

$FQ[X,Y] = 0 \Rightarrow Q[X,Y] = 0 \Rightarrow [X,Y] \in D \Rightarrow D$ is integrable.

This proves the first part.

For the second part, let M be U-vertical then (3.4.11)(A) reduces to

(4.4.4)(A) $\nabla_X^{\perp} FY = FQ\nabla_X Y + Ch(X,Y)$, for all $X,Y \in D^{\perp}$.

We know that

$$\bar{\nabla}_X FY = (\bar{\nabla}_X F)Y + F\bar{\nabla}_X Y,$$

$$-A_{FY}X + \nabla_X^{\perp} FY = -g(X,FY)U - u(Y)FX + FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y),$$

(4.4.5)(A) $\nabla_X^{\perp} FY = -g(X,FY)U - u(Y)FX + FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y) + A_{FY}X.$

From (4.4.4)(A) and (4.4.5)(A), we have

$$FP\nabla_X Y = u(Y)FX + g(X,FY)U - Bh(X,Y) - A_{FY}X.$$

Similarly we have

$$FP\nabla_Y X = u(X)FY + g(Y,FX)U - Bh(X,Y) - A_{FX}Y.$$

From above two equations we have

(4.4.6)(A) $FP[X,Y] = u(Y)FX - u(X)FY + A_{FX}Y - A_{FY}X$

If $[X,Y] \in D^{\perp}$ then is $P[X,Y] = 0$. Now from (4.4.6)(A) and $P[X,Y] = 0$, we get (4.4.2)(A).

Conversely, using (4.4.2)(A) in ((4.4.6)(A), we get

$FP[X,Y] = 0 \Rightarrow P[X,Y] = 0 \Rightarrow [X,Y] \in D^{\perp} \Rightarrow D^{\perp}$ is integrable.

This proves the second part.

Theorem 4.4.2(A): If M is totally para contact umbilical semi-invariant submanifold of a para Kenmotsu manifold with parallel horizontal distributions, then M is totally para contact geodesic.

Proof: From (3.4.9)(A) and (3.4.10)(A), we have

$$g(\nabla_X FPY+\nabla_Y FPX-A_{FQY}X-A_{FQX}Y,Z)=g(FP\nabla_X Y+FP\nabla_Y X+2Bh(X,Y),Z)$$
$$-g(X,FY)u(Z)-g(FX,Z)u(Y).$$

Since M is totally para contact umbilical, we have

$$g(\nabla_X FPY,Z)+g(\nabla_Y FPX,Z)-g(h(X,Z),FQY)-g(h(Y,Z),FQX)$$
$$= g(FP\nabla_X Y,Z)+g(FP\nabla_Y X,Z)+2g(X,Y)g(BH,Z)-2u(X)u(Y)g(BH,Z)$$
$$+2u(X)g(h(Y,U),FZ)+2u(Y)g(h(X,U),FZ)-u(Z)g(FX,Y)$$
$$-u(Y)g(FX,Z).$$

Using (3.4.14)(A) in the above equation and replacing Y by BH and Z by X we get

(4.4.7)(A) $$g(\nabla_{BH}FPX,X)-g(X,X)g(H,FBH)-g(BH,X)g(H,FQX)$$
$$=g(FP\nabla_X BH,X)+g(FP\nabla_{BH}X,X)+2g(X,BH)g(BH,X)$$
$$-2u(X)u(BH)g(BH,X)+2u(X)g(h(BH,U),FX)$$
$$+2u(BH)g(h(X,U),FX)-u(X)g(FX,BH)-u(BH)g(FX,X).$$

If $X\in D$ is a unit vector then we have

$$g(X,BH) = g(FX,BH) = 0.$$

Taking differentiation of the above with respect to X, we obtain

$$g(\nabla_X FX,BH) + g(FX,\nabla_X BH) = 0.$$

Since D is parallel, we have

(4.4.8)(A) $g(FX,\nabla_X BH) = 0.$

Using above facts in (4.4.7)(A) and taking $X \in D$ as unit vector, we get

$$g(\nabla_{BH}FPX,X) - g(H,FBH) = g(FP\nabla_{BH}X,X),$$

(4.4.9)(A) $g((\nabla_{BH}FP)X,X) = -g(FH,BH) = -g(BH,BH).$

Now in view of Gauss formula, we have

$$(\bar{\nabla}_{BH}FP)(X) = (\nabla_{BH}FP)(X).$$

Again using Gauss formula, we get

$$g((\nabla_{BH}FP)X,X) = 0.$$

Hence from (4.4.9)(A) and above equation, we get $g(BH,BH)=0 \Rightarrow BH=0$.

Since $FH \in D^{\perp}$, we have $CH = 0$, hence $FH = 0 \Rightarrow H = 0$. Which shows that M is totally para contact geodesic.

Remark 4.4.1(A): For a U-horizontal semi-invariant submanifold of a para Kenmotsu manifold, from equation (3.4.2)(A), (3.4.3)(A) and (3.4.5)(A), we have for any $X \in TM$

(4.4.10)(A) $\bar{\nabla}_X U = \nabla_X U + h(X,U) = PX + QX - u(X)PU.$

Equating the tangential and normal components of the above equation, we get

(4.4.11)(A) $\nabla_X U = PX + QX - u(X)PU$

and

(4.4.12)(A) $h(X,U) = 0$

Note that, for $X \in D$ we have, $g(A_N U,X) = g(h(X,U),N) = 0$ and hence that $A_N U \in D^{\perp}$.

Theorem 4.4.3(A): Let M be D-umbilic (resp. $D^{\perp}$-umbilic) semi-invariant submanifold of a para Kenmotsu manifold. If M is U-horizontal (resp.U-vertical) then M is D totally geodesic (resp.$D^{\perp}$-totally geodesic).

Proof: Since M is D-umbilical semi-invariant submanifold of a para Kenmotsu manifold, we have that $h(X,Y) = g(X,Y)H$ for any $X,Y \in D$. Since M is U-horizontal we may put $Y = U$ in the above equation and, using (4.4.12)(A), we get

$$H = 0 \Rightarrow h(X,Y) = 0.$$

Hence M is D-totally geodesic.

Similarly we can prove that if M is $D^{\perp}$-umbilical semi-invariant submanifold with U-vertical then M is $D^{\perp}$-totally geodesic.

4.8 SECTION 4(B):

SEMI-INVARIANT SUBMANIFOLDS OF PARA KENMOTSU MANIFOLDS II

Theorem 4.4.1(B): Let M be a $(D,D^{\perp})$-geodesic semi-invariant submanifold of a para Kenmotsu space form $\bar{M}(C)$ and D is an integrable distribution then for all $X \in D$, $Z \in D^{\perp}$ with $u(X) = 0$, we have

(4.4.1)(B) $\quad \frac{1}{2}(C+1)\|X\|^2 \|QZ\|^2 = 0, \forall X \in D, Z \in D^{\perp}.$

Proof: From second part of the Lemma 3.4.1(B), we get

(4.4.2)(B) $\quad A_{QZ}X \in D, \forall X \in D, Z \in D^{\perp},$

and using (1.1.20), (3.4.3)(B), (3.4.4)(B), (3.4.6)(B) and (4.4.2)(B), we have

$$g(A_{QZ}Y,\nabla_X Z) = g(A_{QZ}Y, \bar{\nabla}_X Z)$$

$$= g(A_{QZ}Y, F\bar{\nabla}_X FZ)$$

$$= g(FA_{QZ}Y, \bar{\nabla}_X FZ)$$

(4.4.3)(B) $\quad = -g(FA_{QZ}Y, A_{QZ}X), \forall X,Y \in D, Z \in D^{\perp}.$

From (3.4.5)(B) and (4.4.2)(B), we have

(4.4.4)(B) $\quad A_{QZ}FX = FA_{QZ}X, \forall X \in D, Z \in D^{\perp}.$

From (4.4.3)(B) and (4.4.4)(B), we get

(4.4.5)(B) $\quad g(A_{QZ}Y,\nabla_X Z) = -g(A_{QZ}FY, A_{QZ}X), \forall X,Y \in D, Z \in D^{\perp}.$

Similarly, we have

(4.4.6)(B) $g(A_{QZ}X,\nabla_Y Z) = -g(A_{QZ}FX, A_{QZ}Y)$, $\forall\ X,Y \in D, Z \in D^{\perp}$.

Substituting (4.4.5)(B), (4.4.6)(B) in (3.4.20)(B) and then taking into account $Y = FX$ and using (1.1.20), (4.4.4)(B), we get

$$\tfrac{1}{2}(C+1)g(X,X)\,\|QZ\|^2 = -g(A_{QZ}FX,A_{QZ}FX) + g(A_{QZ}F^2X,A_{QZ}X)$$

$$= -\,g(FA_{QZ}X,FA_{QZ}X) + g(A_{QZ}X,A_{QZ}X)$$

$$= -\,g(A_{QZ}X,A_{QZ}X) + g(A_{QZ}X,A_{QZ}X)$$

$$= 0$$

This completes the proof of the theorem.

Theorem 4.4.2(B): Let M be a $(D,D^{\perp})$-geodesic submanifold of a para Kenmotsu manifold $\bar{M}$ and suppose D is an integrable distribution. If $C \neq -1$, then we have either M is an anti-invariant submanifold of $\bar{M}$ or M is an invariant submanifold of $\bar{M}$. If M is proper semi-invariant submanifold, then we have $C = -1$.

Proof: Now suppose that $C \neq -1$, then it follows that M must be non-proper. This fact can be seen as follows:

If $X\neq 0$, then (4.4.1)(B) implies that $\|QZ\|=0$ hence $QZ=0$ and so we have $Z=0$, thus M is an invariant submanifold of $\bar{M}$. If $Z\neq 0$ then $\|QZ\| \neq$

0 and hence from (4.4.1)(B) it follows that $||X|| = 0$ and so $X = 0$. Thus M is an anti-invariant submanifold of $\bar{M}$.

Now if M is a proper semi-invariant submanifold of $\bar{M}$, then there exist non-zero vector $X \in D$ and $Z \in D^{\perp}$. Thus from (4.4.1)(B) it follows that $C = -1$.

This completes the proof of the theorem.

4.9 SECTION 5:

HYPERSURFACES OF ALMOST HYPERBOLIC HERMITIAN MANIFOLDS

Theorem 4.5.1: The hypersurface of a hyperbolic Kaehlerian manifold is an almost contact hyperbolic metric manifold on which

(4.5.1) a) $(E_X f)Y = u(Y)HX - `H(X,Y)U$,

b) $(E_X u)(Y) = -`H(X,fY)$,

(4.5.2) $E_X U = fHX$, are satisfied.

Proof: Putting BX for λ and BY for μ in (3.5.1), we get

$$D_{BX}FBY = FD_{BX}BY$$

Using (3.5.6)a in the above equation, we get

$$D_{BX}BfY + (Xu(Y))N + u(Y)D_{BX}N = FD_{BX}BY.$$

Using (3.5.6)a,b and (3.5.10)a,b in the above equation, we get

$$BE_X fY + `H(X,fY)N + Xu(Y)N - u(Y)BHX$$

$$= FBE_X Y + F`H(X,Y)N$$

$$= BfE_X Y + u(E_X Y)N - BU`H(X,Y).$$

Now equating tangential and normal parts of the above equation, we get (4.5.1)a and (4.5.1)b.

Similarly putting BX for λ and N for μ in (3.5.1), we get

$$D_{BX}FN = FD_{BX}N.$$

Using (3.5.6)a,b and (3.5.10)a,b in the above equation, we get

$$BE_XU + `H(X,U)N = BfHX + u(HX)N$$

Now equating tangential parts of the above equation, we get (4.5.2).

Theorem 4.5.2: The hypersurface of a hyperbolic Kaehlerian manifold is given by

$$f(E_Xf)fY = 0 \text{ or } (E_Xf)fY = (E_Xu)(Y)U,$$

where $E_XU = fHX$.

Proof: Premultiplying equation (4.5.1)a by f and putting fY for Y, we get

$$f(E_Xf)fY = 0.$$

Now putting fY for Y in (4.5.1)a, we get

$$(E_Xf)fY = -`H(X,fY)U.$$

Using (4.5.1)b in the above equation, we get

$$(E_Xf)fY = (E_Xu)(Y)U.$$

This completes the proof of the theorem.

Theorem 4.5.3: The totally geodesic hypersurface or the hypersurface on which

(4.5.3) a) $fH = Hf = 0 \Leftrightarrow$ b) $H = u(HU)u\otimes U$

holds, for a hyperbolic Kaehlerian manifold, is a hyperbolic Cosymplectic manifold.

Proof: In equation (4.5.1)a,b or (4.5.2) the right hand side contains H is not an element of the almost contact hyperbolic metric structure. The right hand side of (4.5.1)a,b and (4.5.2) vanishes when

$$fH = Hf = 0$$

$$\Leftrightarrow \quad f^2HU = 0$$

$$\Leftrightarrow \quad (I + u\otimes U)HU = 0$$

$$\Leftrightarrow \quad Hu(U) + (u\otimes U)u(HU) = 0$$

$$\Leftrightarrow \quad -H + u(HU)\, u\otimes U = 0$$

$$\Leftrightarrow \quad H = u(HU)u\otimes U$$

The above equation is satisfied if, in particular H = 0, that is, the hypersurface is totally geodesic.

Putting the right hand side of the equation (4.5.1)a equal to zero, we get

$$(E_X f)(Y) = 0.$$

This completes the proof of the theorem.

Theorem 4.5.4: The totally umbilic hypersurface of a hyperbolic Kaehlerian manifold is a hyperbolic Sasakian manifold.

Proof: Let the hypersurface of the hyperbolic Kaehlerian manifold be totally umbilic, that is,

(4.5.4) a) $`H = h \Leftrightarrow$ b) $H = I$

Using this in equation (4.5.1)a, we get

$$(E_X f)Y = u(Y)X - h(X,Y)U.$$

This shows that the totally umbilic hypersurface of a hyperbolic Kaehlerian manifold is a hyperbolic Sasakian manifold.

This proves our assertion.

Theorem 4.5.5: The necessary and sufficient condition that the hyperbolic Kaehlerian manifold be K-contact hyperbolic Riemannian manifold is that

(4.5.5) a) $fHX = fX$ or b) $HX = X + u(HX - X)U$.

Proof: Let the hypersuface of the hyperbolic Kaehlerian manifold be K-contact hyperbolic Remannian manifold. Then

$$E_X U = fX.$$

Comparing the above equation with (4.5.2), we get

$$fHX = fX,$$

$$f^2HX = f^2X,$$

$$HX + u(HX)U = X + u(X)U,$$

$$HX = X + u(X - HX)U.$$

Conversely, pre multiplying equation (4.5.5)b by f, we get

$$fHX = fX.$$

Comparing the above equation with (4.5.2), we get

$$E_X U = fX.$$

This shows that the hypersuface of the hyperbolic Kaehlerian manifold be K-contact hyperbolic Riemannian manifold.

Corollary 4.5.1: Let the hypersurface of a hyperbolic Kaehlerian manifold be K-contact hyperbolic Riemannian manifold. Then it is a hyperbolic Sasakian manifold.

Proof: The equation (4.5.5)a, b are satisfied by HX = X, which is the necessary and sufficient condition for the hypersurface to be totally umbilic.

Now from theorem 4.5.4, we get the result.

Theorem 4.5.6: The hypersurface of a nearly hyperbolic Kaehlerian manifold is an almost contact hyperbolic metric manifold satisfying

(4.5.6) a) $(E_X f)Y + (E_Y f)X = u(X)HY + u(Y)HX - 2`H(X,Y)U$

$$=-u(X)f^2HY-u(Y)f^2HX+\{`H(X,f^2Y)$$

$$+`H(f^2X,Y)\}U$$

b) $(E_X u)(Y) + (E_Y u)(X) = -`H(X,fY) - `H(fX,Y)$

(4.5.7) $(D_N F)BX = B(E_X U - fHX)$

(4.5.8) $(E_U `f)(Y,Z)-u(Y)u(HZ)+u(Z)u(HY)= -(E_Y u)(fZ) - `H(Y,f^2Z)$

$$=(E_Z u)(fY)+ `H(f^2Y,Z).$$

Proof: Putting BX for λ and BY for μ in (3.5.2), we get

$$D_{BX}FBY + D_{BY}FBX = FD_{BX}BY + FD_{BY}BX.$$

Using (3.5.6)a in the above equation, we get

$$D_{BX}BfY+(Xu(Y))N+u(Y)D_{BX}N+D_{BY}BfX+(Yu(X))N+u(X)D_{BY}N$$

$$= FD_{BX}BY+FD_{BY}BX$$

Again using (3.5.6)a,b and (3.5.10)a,b in the above equation, we get

$$BE_XfY+`H(X,fY)N+Xu(Y)N-u(Y)BHX+BE_YfX+`H(Y,fX)N+Yu(X)N$$

$$-u(X)BHY = FBE_XY + F`H(X,Y)N+FBE_YX + F`H(Y,X)N,$$

$$=BfE_XY+u(E_XY)N-BU`H(X,Y)+BfE_YX+u(E_YX)N$$

$$-BU`H(Y,X)$$

Now equating tangential and normal parts, we get (4.5.6)a and (4.5.6)b.

Similarly putting BX for λ and N for μ in (3.5.2), we get

$$D_{BX}FN + (D_NF)BX = FD_{BX}N$$

Using (3.5.6)a,b and (3.5.10)a,b in the above equation, we get

$$(D_NF)BX = BE_XU + `H(X,U)N - BfHX - u(HX)N$$

Now equating tangential parts, we get (4.5.7).

Now from (4.5.6)a, we get

$$E_X`f(Y,Z)-u(Y)`H(X,Z)+u(Z)`H(X,Y)$$

$$=(E_Y`f)(Z,X)-u(Z)`H(X,Y)+u(X)`H(Y,Z).$$

Putting U for X in the above equation, we get (4.5.8).

This completes the proof of the theorem.

Theorem 4.5.7: The hypersurface of a nearly hyperbolic Kaehlerian manifold is an almost contact hyperbolic metric manifold satisfying

$$f(E_X f)fY + f(E_Y f)fX = 0.$$

Proof: Pre multiplying equation (4.5.6)a by f, we get the result.

Theorem 4.5.8: The necessary and sufficient condition that for the hypersurface of a nearly hyperbolic Kaehlerian manifold, U be Killing, is

(4.5.9) a) $`H(X,fY) + `H(fX,Y) = 0 \Leftrightarrow$ b) $Hf = fH = 0 \Leftrightarrow$

c) $`H(fX,fY) = -`H(X,Y) + u(X)u(Y)u(HU) \Leftrightarrow$

d) $fHf - H = -u(HU)u\otimes U.$

Proof: Let on the hypersurface of a nearly hyperbolic Kaehlerian manifold U is Killing, that is,

$$(E_X u)(Y) + (E_Y u)(X) = 0.$$

From (4.5.6)b, we have

$$`H(X,fY) + `H(fX,Y) = 0$$

$$\Leftrightarrow \quad h(HfX,Y) - h(fHX,Y) = 0$$

$$\Leftrightarrow \quad Hf = fH$$

$$\Leftrightarrow \quad `H(fX,fY) + `H(f^2X,Y) = 0$$

$$\Leftrightarrow \quad `H(fX,fY)U + h(HX + Hu(X)U,Y) = 0$$

$$\Leftrightarrow \quad `H(fX,fY) + `H(X,Y) - u(X)u(Y)u(HU) = 0$$

$$\Leftrightarrow \quad `H(fX,fY) = -`H(X,Y) + u(X)u(Y)u(HU)$$

Now from the above equation, we get

$\Leftrightarrow \quad -h(fHfX,Y) = -h(HX,Y) + h(u(HU)X,(u\otimes U)Y)$

$\Leftrightarrow \quad fHf - H = -u(HU)u\otimes U.$

Conversely, from (4.5.9)a and (4.5.6)b, we get

$$(E_X u)(Y) + (E_Y u)(X) = 0.$$

This shows that U is Killing.

Hence we get the required result.

Theorem 4.5.9: The totally umbilical hypersurface of a nearly hyperbolic Sasakian manifold is a nearly hyperbolic Sasakian manifold. Also for this hypersurface, we have

(4.5.10) $\quad (D_N F)BX = B(E_X U - fX)$

Proof: Hypersurface is totally umbilical means H = I.

Now from equation (4.5.6)a and the above facts, we get

(4.5.11) $\quad (E_X f)Y + (E_Y f)X = u(X)Y + u(Y)X + 2h(X,Y)U$

This shows that the totally umbilical hypersurface of a nearly hyperbolic Kaehlerian manifold is a nearly hyperbolic Sasakian manifold.

Using H = I in (4.5.7), we get (4.5.10).

Theorem 4.5.10: The hypersurface of an almost hyperbolic Kaehlerian manifold is a quasi hyperbolic Sasakian manifold. The following equation also holds good for this hypersurface

(4.5.12) $(D_N`F)(BX,BY)ob = -du(X,Y) - `H(X,fY) + `H(fX,Y).$

Proof: We know that

$$(D_{BX}F)BY = D_{BX}FBY - FD_{BX}BY$$

Using (3.5.6)a,b and (3.5.10)a,b in the above equation, we obtain

$$(D_{BX}F)BY = D_{BX}BfY+(Xu(Y))N+u(Y)D_{BX}N-FBE_XY-F`H(X,Y)N$$

$$(D_{BX}F)BY=BE_XfY+`H(X,fY)N+(Xu(Y))N-u(Y)BHX-BfE_XY-u(E_XY)N$$

$$+B`H(X,Y)U$$

(4.5.13) $(D_{BX}F)BY=B((E_Xf)Y-u(Y)HX+`H(X,Y)U)$

$$+((E_Xu)(Y)+`H(X,fY))N.$$

Consequently

$$g((D_{BX}F)BY,BZ)ob = (E_X`f)(Y,Z) - u(Y)`H(X,Z) + u(Z)`H(X,Y)$$

(4.5.14) $(D_{BX}`F)(BY,BZ)ob=(E_X`f)(Y,Z)-u(Y)`H(X,Z)+ u(Z)`H(X,Y).$

Similarly, we have

$$(D_{BY}`F)(BZ,BX)ob = (E_Y`f)(Z,X) - u(Z)`H(Y,X) + u(X)`H(Y,Z),$$

$$(D_{BZ}`F)(BX,BY)ob = (E_Z`f)(X,Y) - u(X)`H(Z,Y) + u(Y)`H(Z,X).$$

Adding above three equations and using (3.5.3), we get

(4.5.15) $(E_X`f)(Y,Z) + (E_Y`f)(Z,X) + (E_Z`f)(X,Y) = 0 \Leftrightarrow d`f = 0.$

This shows that the hypersurface of an almost hyperbolic Kaehlerian manifold is a quasi hyperbolic Sasakian manifold.

Now from (4.5.13), we have

(4.5.16) $(D_{BX}`F)(BY,N)ob = (E_X u)(Y) + `H(X,fY)$.

Putting N, BX, BY for λ,μ,ν in (3.5.3), we get

$$(D_N`f)(BX,BY) + (D_{BX}`F)(BY,N) + (D_{BY}`F)(N,BX) = 0$$

Using (4.5.16) in the above equation, we get

$$(D_N`F)(BX,BY)ob + (E_X u)(Y) + `H(X,fY) - (E_Y u)(X) - `H(Y,fX) = 0$$

$$(D_N`F)(BX,BY)ob = -(E_X u)(Y) + (E_Y u)(X) - `H(X,fY) + `H(fX,Y)$$

$$(D_N`F)(BX,BY)ob = -du(X,Y) - `H(X,fY) + `H(fX,Y).$$

This completes the proof of the theorem.

Theorem 4.5.11: The necessary and sufficient condition that the hypersurface of an almost hyperbolic Kaehlerian manifold be an almost hyperbolic Cosymplectic manifold is that

(4.5.17) $(D_N`F)(BX,BY)ob = `H(fX,Y) - `H(X.fY)$

Proof: We know that an almost contact hyperbolic metric manifold satisfying

$$d`f = 0 \Leftrightarrow (E_X`f)(Y,Z) + (E_Y`f)(Z,X) + (E_Z`f)(X,Y) = 0$$

(4.5.18) $du = 0 \Leftrightarrow (E_X u)(Y) = (E_Y u)(X)$

is called an almost hyperbolic Cosymplectic manifold.

Using (4.5.18) in (4.5.12), we get (4.5.17).

Conversely using (4.5.17) in (4.5.12), we get du = 0.

This shows that hypersurface of an almost hyperbolic Kaehlerian manifold be an almost hyperbolic Cosymplectic manifold.

Corollary 4.5.2: Let

(4.5.19) a) $(D_N`F)(BX,BY) = 0$ and

b) $Hf + fH = 0 \Leftrightarrow `H(fX,Y) = `H(X,fY)$.

Then the hypersurface of an almost hyperbolic Kaehlerian manifold is an almost hyperbolic Cosymplectic manifold.

Proof: Using (4.5.19) a,b in (4.5.12), we get

$du(X,Y) = 0$

This shows that the hypersurface of an almost hyperbolic Kaehlerian manifold is an almost hyperbolic Cosymplectic manifold.

Theorem 4.5.12: The hypersurface of a hyperbolic Hermitian manifold is an almost contact hyperbolic metric manifold for which the following relations hold

(4.5.20) a) $(E_{fX}f)Y-(E_{f^2X}f)fY+u(Y)E_{f^2X}U-u(Y)(fHf^2+f^2Hf)X$
$-(`H(f^2X,fY)-`H(fX,f^2Y))U=0$

b) $(E_{fX}u)(Y)-(E_{f^2X}u)(fY)+`H(fX,fY)-`H(f^2X,f^2Y) = 0$

(4.5.21) a) $(E_{fX}f)fY-(E_{f^2X}f)Y-u(Y)E_{fX}U+u(Y)(fHf+f^2Hf^2)X$
$+(`H(fX,fY)-`H(f^2X,f^2Y))U=0$

b) $(E_{fX}u)(fY)-(E_{f^2X}u)(Y)-`H(f^2X,fY)+`H(fX,f^2Y) = 0$

Proof: Premultiplying X by f in equation (3.5.30), we get

$$B(P(fX,Y) + Q(fX,Y)) + (p(fX,Y) + q(fX,Y))N = 0.$$

Now equating tangential and normal parts, we get

(4.5.22) a) $P(fX,Y) + Q(fX,Y) = 0,$

b) $p(fX,Y) + q(fX,Y) = 0.$

From (3.5.31)a,b and (4.5.22)a, we get (4.5.20)a and from (3.5.32)a,b and (4.5.22)b, we get (4.5.20)b.

Now putting U for X in (3.5.30), we get

(4.5.23) $(D_NF)FBY = B(P(U,Y) + Q(U.Y))+(p(U,Y) + q(U,Y))N.$

Using (4.5.23) in (3.5.30) and then equating tangential and normal parts, we obtain

$$-u(X)\{P(U,Y) + Q(U,Y)\} = P(X,Y) + Q(X,Y) \Leftrightarrow$$

(4.5.24) a) $P(f^2X,Y) + Q(f^2X,Y) = 0$, and

$$-u(X)\{p(U,Y) + q(U,Y)\} = p(X,Y) + q(X,Y) \Leftrightarrow$$

b) $p(f^2X,Y) + q(f^2X,Y) = 0.$

From (3.5.31) a,b and (4.5.24)a, we get (4.5.21)a and from (3.5.32)a,b and (4.5.24)b, we get (4.5.21)b.

This completes the proof of the theorem.

Theorem 4.5.13: Let the hypersurface of a hyperbolic Hermitian manifold be an almost contact hyperbolic metric normal manifold. Then we have

(4.5.25) a) $fHf + f^2Hf^2 = 0 \Leftrightarrow$ a`) $`H(fX,fY) = `H(f^2X,f^2Y) \Leftrightarrow$

b) $fHf^2 + f^2Hf = 0 \Leftrightarrow$ b`) $`H(f^2X,fY) = `H(fX,f^2Y)$.

Proof: We know that almost contact hyperbolic metric manifold is said to be normal is that

$$(E_{fX}f)fY - (E_Xf)Y - u(Y)E_{fX}U = 0$$

Using the above equation in (3.5.31)a, we get P(X,Y)=0. Which gives also

(4.5.26) a) $P(U,Y) = 0 \Leftrightarrow$ a`) $E_Uf = 0 \Rightarrow E_UU = 0$.

From (4.5.20)a and (4.5.21)a, we get

$$u(Y)(fHf^2 + f^2Hf)X + \{`H(f^2X,fY) - `H(fX,f^2Y)\}U = 0, \text{ and}$$

$$u(Y)(fHf + f^2Hf^2)X + \{`H(fX,fY) - `H(f^2X,f^2Y)\}U = 0$$

if and only if

$$fHf + f^2Hf^2 = 0 \Leftrightarrow `H(fX,fY) = `H(f^2X,f^2Y)$$

$$fHf^2 + f^2Hf = 0 \Leftrightarrow `H(f^2X,fY) = `H(fX,f^2Y).$$

This proves our assertion.

Theorem 4.5.14: The necessary and sufficient condition that for the hypersurface of a hyperbolic Hermitian manifold (4.5.25) holds is that

(4.5.27) a) $(E_{fX}u)(fY) = (E_{f^2X}u)(Y)$,

b) $(E_{fX}u)(Y) = (E_{f^2X}u)(fY)$.

Proof: Using (4.5.25) in (4.5.20)b and (4.5.21)b, we get (4.5.27)a,b.

Conversely, using (4.5.27)a in (4.5.21)b, we get (4.5.25)b and using (4.5.27)b in (4.5.20) b, we get (4.5.25)a.

This proves our assertion.

Theorem 4.5.15: A K-contact hyperbolic Remannian manifold on which (4.5.26) holds, is a hyperbolic Sasakian manifold.

Proof: Let the hypersurface be K-contact hyperbolic Riemannian manifold. Then (4.5.20)b and (4.5.21)b gives (4.5.25). If in addition (4.5.26) holds, then K-contact hyperbolic Riemannian manifold is called K-contact hyperbolic Riemannian normal manifold. But a K-contact hyperbolic Riemannian normal manifold is a hyperbolic Sasakian manifold.

Hence we get the desired result.

Theorem 4.5.16: The hypersurface of a quasi hyperbolic Kaehlerian manifold is given by

(4.5.28) a) $(E_{fX}f)Y+(E_{f^2X}f)fY-u(Y)E_{f^2X}U+u(Y)(fHf^2-f^2Hf)X$

$+(`H(f^2X,fY)+`H(fX,f^2Y))U=0$

b) $(E_{fX}u)(Y) + (E_{f^2X}u)(fY) + `H(fX,fY) + `H(f^2X,f^2Y) = 0$

(4.5.29) a) $(E_{fX}f)fY+(E_{f^2X}f)Y-u(Y)E_{fX}U+u(Y)(fHf-f^2Hf^2)X$

$+(`H(fX,fY)+`H(f^2X,f^2Y))U=0$

b) $(E_{fX}u)(fY)+(E_{f^2X}u)(Y)+`H(f^2X,fY)+`H(fX,f^2Y)=0$

Proof: Premultiplying X by f in equation (4.5.25), we get

$$B(R(fX,Y)+S(fX,Y))+(r(fX,Y)+s(fX,Y))N=0.$$

Now equating tangential and normal parts, we get

(4.5.30) a) $R(fX,Y)+S(fX,Y)=0,$

b) $r(fX,Y)+s(fX,Y)=0.$

From (3.5.34)a,b and (4.5.30)a, we get (4.5.28)a and from (3.5.35)a,b and (4.5.30)b, we get (4.5.28)b.

Now putting U for X in (3.5.33), we get

(4.5.31) $(D_NF)FBY=B(R(U,Y)+S(U.Y))+(r(U,Y)+s(U,Y))N$

Using (4.5.31) in (3.5.33) and collecting tangential and normal parts, we obtain

$$-u(X)\{R(U,Y)+S(U,Y)\}=R(X,Y)+S(X,Y)\Leftrightarrow$$

(4.5.32) a) $R(f^2X,Y)+S(f^2X,Y)=0$, and

$-u(X)\{r(U,Y)+s(U,Y)\}=r(X,Y)+s(X,Y)\Leftrightarrow$

b) $r(f^2X,Y)+s(f^2X,Y)=0.$

From (3.5.34) a,b and (4.5.32)a, we get (4.5.29)a and from (3.5.35)a,b and (4.5.32)b, we get (4.5.29)b.

This completes the proof of the theorem.

Theorem 4.5.17: Let the hypersurface of a quasi hyperbolic Kaehlerian manifold be an almost contact hyperbolic metric pseudo normal manifold. Then we have

(4.5.33) a) $fHf^2 - f^2Hf = 0 \Leftrightarrow$ a`) $`H(f^2X,fY)+`H(fX,f^2Y)=0$

$\Leftrightarrow$

b) $fHf - f^2Hf^2 = 0 \Leftrightarrow$ b`) $`H(fX,fY) + `H(f^2X,f^2Y)= 0.$

Proof: We know that equation of an almost contact hyperbolic metric pseudo normal manifold is given by

$$(E_{fX}f)fY + (E_Xf)Y - u(Y)E_{fX}U = 0$$

Using this in (3.5.34)a, we get $R(X,Y) = 0$. Which gives also

(4.5.34) a) $R(U.Y) = 0 \Leftrightarrow$ a`) $E_Uf = 0 \Rightarrow E_UU = 0.$

Now from (4.5.28)a and (4.5.29)a, we have

$$u(Y)(fHf^2 - f^2Hf)X + \{`H(f^2X,fY) + `H(fX,f^2Y)\}U = 0, \text{ and}$$

$$u(Y)(fHf - f^2Hf^2)X + \{`H(fX,fY) + `H(f^2X,f^2Y)\}U = 0$$

if and only if

$$fHf^2 - f^2Hf = 0 \Leftrightarrow `H(f^2X,fY) + `H(fX,f^2Y) = 0,$$

$$fHf - f^2Hf^2 = 0 \Leftrightarrow `H(fX,fY) + `H(f^2X,f^2Y) = 0.$$

This completes the proof of the theorem.

Theorem 4.5.18: The necessary and sufficient conditions that for the hypersurface of a quasi hyperbolic Kaehlerian manifold equation (4.4.33) holds is that

(4.5.35) a) $(E_{fX}u)(fY) + (E_{f^2X}u)(Y) = 0,$

b) $(E_{fX}u)(Y) + (E_{f^2X}u)(fY) = 0.$

Proof: Using (4.5.33) in (4.5.28)b and (4.5.29)b, we get (4.5.35)a,b.

Conversely using (4.5.35)a in (4.5.29)b, we get (4.5.33)a and (4.5.35)b in (4.5.28)b, we get (4.5,33)b.

This proves our assertion.

SUMMARY AND CONCLUSION

Chapter 5 SUMMARY AND CONCLUSION

Yano and Kon studied CR-submanifold of an almost contact metric manifold. Semi-invariant submanifold of an almost contact metric manifold defined and studied by Bejancu and Papaghuic. Dube studied almost hyperbolic Hermitian manifolds and Pal and Mishra studied hypersurface of almost hyperbolic Hermit manifolds.

This work consists of 5 sections. Section 1 is divided into two subsections 1(A) and 1(B). In section 1(A) we studied CR-submanifolds of a trans para Sasakian manifold and find some results. In theorem 4.1.3(A) we have shown D-umbilic U-horizontal CR-submanifold of a trans para Sasakian manifold to be D-totally geodesic. We have also studied integrability conditions of distributions D and $D^{\perp}$.

Section 1(B) is devoted to the study of CR-submanifolds of a nearly and closely para Cosymplectic manifolds. We gave integrability, autoparallelness, nearly autoparallelness of the distribution $D^{1}\oplus\{U\}$ on submanifold and integrability of distribution D^{0}, $D^{0}\oplus\{U\}$. Totally umbilical, totally geodesic, totally para contact umbilical and totally para contact geodesic submanifolds are also studied. In Corollary 4.1.1(B) we have shown a totally umbilical CR-submanifold of a nearly para

Cosymplectic manifold $D^1 \oplus \{U\}$ is autoparallel. In theorem 4.1.10(B) we have shown a totally para contact umbilical CR-submanifold of a closely para Cosymplectic manifold is (D^1, D^0)-mixed totally geodesic.

Section 2 is divided into two subsections 2(A) and 2(B). In section 2(A) we have studied semi-invariant submanifold of a nearly r-Sasakian manifold and obtained some important results. In theorem 4.2.3(A) we have shown that any proper totally r-contact umbilical semi-invariant submanifold of a nearly r-Sasakian manifold is totally r-contact geodesic.

In section 2(B) we have studied semi-invariant submanifolds of r-Kenmotsu manifolds. In theorem 4.2.2(B) we have shown that totally r-contact umbilical semi-invariant submanifold of r-Kenmotsu manifold with parallel horizontal distribution is totally r-contact geodesic and in theorem 4.2.3(B) we have shown that D-umbilical U_P-horizontal (resp. $D^{\perp}$-umbilical, U_P-vertical) semi-invariant submanifold of a r-Kenmotsu manifold is D-totally geodesic (resp. $D^{\perp}$-totally geodesic).

Section 3 is divided into two subsections 3(A) and 3(B). In section 3(A) and 3(B) we have studied CR-submanifolds of a trans hyperbolic Sasakian manifold. In theorem 4.3.3(A) we have shown that a D-umbilical U-horizontal CR-submanifold of a trans hyperbolic Sasakian manifold is D-totally geodesic. In theorem 4.3.2(B) we have shown that

totally contact hyperbolic umbilical CR-submanifold of a trans hyperbolic Sasakian manifold with parallel horizontal distribution is totally contact hyperbolic geodesic and in theorem 4.3.3(B) we have shown that D-Umbilical, U-horizontal (resp. $D^{\perp}$-Umbilical, U-vertical) CR-submanifold of a trans hyperbolic Sasakian manifold is D-totally geodesic (resp. $D^{\perp}$-totally geodesic).

Section 4 is divided into two subsections 4(A) and 4(B). Section 4(A) and 4(B) deals with semi-invariant submanifolds of para Kenmotsu manifolds. In theorem 4.4.2(A) we have shown that totally para contact umbilical semi-invariant submanifold of a para Kenmotsu manifold with parallel horizontal distributions is totally para contact geodesic and in theorem 4.4.3(A) we have shown that D-Umbilical, U-horizontal (resp. $D^{\perp}$-Umbilical, U-vertical) semi-invariant submanifold of a para Kenmotsu manifold is D-totally geodesic (resp. $D^{\perp}$-totally geodesic). In theorem 4.4.2(B) we have shown that for $C \neq -1$, a $(D, D^{\perp})$-geodesic submanifold of a para Kenmotsu manifold is either an anti-invariant submanifold or an invariant submanifold of para Kenmotsu manifold and for $C = -1$, it is proper semi-invariant submanifold.

Section 5 is devoted to the study of hypersurfaces of almost hyperbolic Hermitian manifolds. We defined hyperbolic Cosymplectic

manifold, nearly Cosymplectic manifold, quasi hyperbolic Sasakian manifold, almost contact hyperbolic normal metric manifold and almost contact hyperbolic pseudo normal metric manifold. In theorem 4.5.4 we have shown that totally umbilical hypersurface of a hyperbolic Kaehlerian manifold is a hyperbolic Sasakian manifold. In corollary 4.5.1 we have shown that if the hypersurface of a hyperbolic Kaehlerian manifold be K-contact hyperbolic Riemannian manifold then it is a hyperbolic Sasakian manifold.

LITERATURE CITED

LITERATURE CITED

[1] Adati, T. and Kandatu, A.; On hypersurfaces of P-Sasakian manifolds and manifolds admitting a concircular vector field. Tensor N.S., Vol. 34, 97-102, 1980.

[2] Bejancu, A.; CR-submanifold of a Kaehler manifold I, Proc. Amer. Math. Soc., 69, 135-142, 1978.

[3] Bejancu, A.; CR-submanifold of a Kaehler manifold II, Trans. Amer. Math. Soc., 250, 331-345, 1979.

[4] Bejancu, A.; Geometry of CR-submanifolds, D. Redial Publ. Co., 1986.

[5] Bejancu, A. and Papaghuic; Semi-invariant submanifold of a Sasakian manifold, An St. Univ. Allcuza Iasi, 27, 163-170, 1981.

[6] Bejancu, A. and Papaghuic; CR-submanifold of Kenmotsu manifold, Rend. Math., 7, No. 4, 607-622, 1983.

[7] Beltagy, M.; Complete hypersurfaces in hyperbolic space, J. Inst. Math. Comput. Sci. Math. Ser. 4, 343-346, 1991.

[8] Bhatt, L.; Joshi, N.K., Dube, K.K.; Non–invariant hypersurfaces of Kenmotsu manifold, Acta Ciencia Indica, XXVM 2, 183-186, 1999.

[9] Blair, D.E.; Almost contact manifolds with killing structure tensors. Pacific J. of Math., 39, 285-292, 1971.

[10] Blair, D.E.; Contact manifolds in Riemannian geometry. Lecture notes in Math. 509, Springer Varlag, Berlin, 1976.

[11] Blair, D.E.; Non existence of 4-dimentional almost Kaehler manifolds of constant curvature. Proc. Am. Math. Soc. 110. 1033-1039, 1990.

[12] Blair, D.E and Shower, D.K.; Almost contact manifolds with killing structure tensor II, J. Diff. Geometry, 557-582, 1974.

[13] Blair, D.E; Shower, D.K. and Yano, K.; Nearly Sasakian structures, Kodai Math. Sem. Report, 27, 175-180, 1976.

[14] Bucki, A.; Invariant submanifolds of an almost r-paracontact manifold, Tensor N.S., 41, 261-268, 1984.

[15] Bucki, A.; Some properties of almost r-papacontact manifolds, Lobachevskii J. Math. 8, 19-32, 2001.

[16] Chen, B.Y.; Geometry of submanifolds and its applications, Science Univ. Tokyo, 1981.

[17] Das; Lovejoy, S.K.; On a structure satisfying an algebraic equation $X = a^2X + \sum A_p(X)T^p$. Kyungpook Math. J. 33, 61-67, 1993.

[18] Das; Lovejoy, S.K.; On almost r-contact metric manifold. C.R. Acad. Bulgare Sci. 32, No. 6, 711-714, 1979.

[19] Desmukh, S.; Sharfuddin, A. and Husain, S.I.; Invariant and non-invariant hypersurface of almost para contact manifolds, The Aligarh Bull. Math., Vol II, 31-40, 1986.

[20] Dube, K.K.; Almost hyperbolic Hermitian manifolds, Analele Univ. din Timi., Seria Stiinte Math., Vol XI, fase., 1, 47-54, 1973.

[21] Dube, K.K.; Study of curvatures on para Kenmotsu manifold, The Nepali Math. Sci. Report, Vol. 15, 83-88,1996.

[22] Dube, K.K.; Mishra, M.M.; Hypersurfaces immersed in an almost hyperbolic Hermitian manifold, Comptes Rendus, Tome 34, No. 10, 1343-1345, 1981.

[23] Dube, K.K.; Niwas, R.; Almost r-contact hyperbolic structure in a product manifold, Demonstratio Mathematica, Vol XI, No. 4, 887-897, 1978.

[24] Dutta, B.; (1,2) symplectic structures nearly Kaehler structure and S^6. J. of the Indian Math. Soc. 60, 171-190, 1994.

[25] Ehresmann, C.; Surla theorie des espaces fibers, Coll. Inst. C.N.R.S. Topologie algebrique, Paris, 3-35, 1947.

[26] Endo, H.; Invariant submanifolds in a K-contact Riemannian manifold. Tensor N.S., 28, 154-156,1974.

[27] Endo, H.; Remarks on invariant submanifolds in an almost Cosymplectic manifold. Tensor N.S., 42, 165-167, 1985.

[28] Gadea, P.M. and Oubina, J.A.; Homogeneous almost para Hermitian structures. Indian J. Pure Appl. Math., 26 (4), 351-362, 1995.

[29] Goldberg, S.I.; Totally geodesic hypersurfaces of Kaehler manifold, Pacific J. Math., 27, 275-281, 1968.

[30] Gupta, V.C.; Prasad, C.S.; Almost para r-contact structures manifold. Demonstratio Mathematica XIX (4), 1-15, 1986.

[31] Hasanis, N.; Minimal hypersurfaces of a manifold. Tensor N.S., 35, 262-266, 1981.

[32] Hicks, N.J.; Notes on Differential geometry, D. Van Nostrand Company INC. New York, 1964.

[33] Hit, R.; Almost contact submaniolds. Tensor N.S., 28, 167-172, 1774.

[34] Hus, C.J.; On CR-submanifolds of Sasakian manifold I. Math. Research Center Report, Symposium Seminar, 117-140, 1983.

[35] Husain, S.I. and Deshmukh, S.; CR-submanifolds of a nearly Kaehler manifold. Indian J. Pure Appl. Math., 13(11), 979-990, 1987.

[36] Jong, Taekcho; Real hypersurfaces of complex hyperbolic space satisfying a point wise nullity condition. Indian J. Pure Appl. Math., 31 (3), 265-276, 2000.

[37] Joshi, N.K.; Dube, K.K.; Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold, Demonstratio Mathematica, Vol. XXXIV, No.1, 135-143, 2001.

[38] Kalpana; Some results of almost semi-invariant submanifold of an SP-Sasakian manifold. Indian J. Pure Appl. Math. 20(7), 681-689, 1989.

[39] Kenmotsu, K.; A class of almost contact Riemannian manifolds, Tohoku Math. J., 24, 93-103, 1972.

[40] Kirichenko, V.F.; On geometry of Kenmotsu manifolds, Dokl. Akad. Nauk 380, 585-587, 2001.

[41] Kobayashi, M.; CR-submanifold of a Sasakian manifold, Tensor N.S.,35, 297-307, 1981.

[42] Kobayashi, M.; CR-submanifold of a Sasakian space form with flat normal connection, Tensor N.S., 36, 207-213, 1982.

[43] Kobayashi, M.; Contact CR-products of Sasakian manifolds, Tensor N.S., 36, 281-288, 1982.

[44] Kobayashi, M.; Integabilities of horizontal and vertical distributions on CR-submanifolds of a nearly Sasakian manifold. Tensor N.S., 36, 215-221, 1982.

[45] Kobayashi, M.; Semi-invariant submanifolds of a certain class of almost contact manifolds, Tensor N.S., 43, 28-36, 1986.

[46] Kon, M.; A note on invariant submanifolds in a K-contact Riemannian manifold. Tensor N.S., 27, 158-160, 1973.

[47] Kon, M.; On semi-invariant submanifolds of normal contact metric manifolds. Tensor N.S., 28, 167-172, 1974.

[48] Kon, M.; Remarks on anti-invariant submanifolds of a Sasakian manifold. Tensor N.S., 30, 239-246, 1976.

[49] Maeda, S.; Ohnita, Y. and Udagawa, S.; On slant immersion into Kaehler manifolds. Kodai Math. J., 16, 205-219, 1993.

[50] Matsumoto, K.; On conformal C-killing vector field in a compact Sasakian manifold II. Tensor N.S., 28, 91-94, 1974.

[51] Mishra, I.K. and Ojha, R.H.; On para Sasakian manifolds, Indian J. Pure Appl. Math., 32(9), 1309-1316, 2001.

[52] Mishra, R.S.; An almost product almost decomposable manifold. Tensor N.S., 21, 250-260, 1970.

[53] Mishra, R.S.; A differential manifold with f-structure of rank r. Tensor N.S., 27, 369-378, 1973.

[54] Mishra, R.S.; Structures in differentiable manifold. Indian National Science Academy Scientific Report series no. 3, New Delhi, 1978.

[55] Mishra, R.S.; Conformal K-contact Riemannian manifolds. Tensor N.S, 1981.

[56] Mishra, R.S.; Structures in submanifolds of an almost Hermite manifold. Tensor N.S, Vol. 35, 105-110, 1981.

[57] Mishra, R.S.; Almost contact metric manifold, Monograph No. 1, Tensor Society of India, Lucknow, 1991.

[58] Mishra, R.S.; Hypersurface of almost Hermite manifolds, Monograph No. 2, Tensor Society of India, Lucknow, 1993.

[59] Mishra, R.S.; Normality of the hypersurface of almost Hermite manifold. J. of the Indian Math. Soc. Vol. 61, No. 1-2, 71-79, 1995.

[60] Miyazawa, T.; Invariant submanifolds of an almost para contact Riemannian manifold. Tensor N.S., 35, 205-210, 1981.

[61] Nivas, R.; Singh, R.; On almost r-contact structure manifold, Demonstratio Math., 3, 797-803, 1988.

[62] Olaszak, Z.; On contact metric manifold. Tohoku math. J. (2), 31, No. 2, 247-253, 1979.

[63] Olszak, Z.; Nearly Sasakian manifolds. Tensor N.S., 33, 277-286, 1979.

[64] Olaszak, Z.; On almost Cosymplectic manifolds. Kodai Math. J. 1, 239-250, 1981.

[65] Oubina, J.A.; New classes of almost contact metric structures, Publ. Math. Debrecen, 32, 187-93, 1985.

[66] Pal, R.B.; Submanifolds of hyperbolic contact metric manifold, Acta Ciencia Indica Math. 22, 399-404, 1996.

[67] Pal, R.B. and Mishra, R.S.; Hypersurface of almost hyperbolic Hermite manifolds, Indian J. Pure Appl. Math., 11(5), 628-632, 1980.

[68] Pandey, S.N. and Verma, Savita; On para Sasakian manifold, Indian J. Pure Appl. Math. 30 (1), 15-22, 1999.

[69] Prasad, C.S.; Integrability condition of a manifold admitting almost para r-contact structure. J. of the Tensor Society of India, II, 6-14, 1993.

[70] Prasad, C.S.; On P-Sasakian manifolds. Bull. Call. Math. Soc., 92 (2), 127-132, 2000.

[71] Prasad, C.S. and Ojha, R.H.; CR-submanifolds of trans Sasakian manifolds, Indian J. Pure Appl. Math., 24, 427-434, 1993.

[72] Prasad, V.S. and Bagewadi, C.S.; Semi-invariant submanifold of a Kenmotsu manifolds, Ganita, 50 (1), 73-81, 1999.

[73] Provanovic, M.; Holomorphically projective transformation in a locally product space, Mathematica Balkainica, 195-213, 1971.

[74] Pujar, S.S.; On a structure defined by a tensor field of type (1,1) satisfying $P^3-P = 0$. Indian J. Pure Appl. Math., 31 (10), 1229-1234, 2000.

[75] Rasevaski, R.S.; Scalar field in stratified space, works of the seminar on vector and tensor analysis, No 6, Moscow, 225-248, 1948.

[76] Ravi, B. and Bagewadi, C.S.; Invariant submanifolds in a conformal K-contact Riemannian manifold. Indian J. Pure Appl. Math., 20(11), 1119-1125, 1989.

[77] Sasaki, S.; On differentiable manifold with certain structures which are closely related to almost contact structures Tohoku Math. J. 12, 459-476, 1960.

[78] Sasaki, S. and Hatakeyama, Y.; On differentiable manifolds with certain structures which are closely related to almost contact structures, I, II, Tohoku Math. J. 13, 44-100, 281-94, 1961.

[79] Sato, I.; On a structure similar to the almost contact structure. Tensor N.S., 30, 219-224, 1976.

[80] Sato, I.; On a structure similar to the almost contact structure II. Tensor N.S., 31, 119-205, 1977.

[81] Sengupta, A.K. and De,U.C.; Generalised CR-submanifolds of a trans-Sasakian manifold. Indian J. Pure Appl. Math., 32(4), 573-580, 2001.

[82] Shahid, M.H.; CR-submanifolds of a trans Sasakian manifold, Indian J. Pure Appl. Math. 22 (12), 1007-1012, 1991.

[83] Shahid, M.H.; CR-submanifolds of a Kaehlerian product manifold. Indian J. Pure Appl. Math., 23 (12), 873-879, 1992.

[84] Shahid, M.H.; On semi-invariant submanifold of a nearly Sasakian manifold, Indian J. Pure Appl. Math. 24, 571-580, 1993.

[85] Shahid, M.H.; CR-submanifold of trans Sasakian manifold II, Indian J. Pure Appl. Math., 25(3), 299-307, 1994.

[86] Shukla, S.S. and Tripathi, M.M.; Hypersurfaces of manifolds with a Kenmotsu 3-structure. Bull. Cal. Math. Soc., 92 (5), 365-374, 2000.

[87] Shukla, S.S. and Tripathi, M.M.; Generic submanifolds of manifolds equipped with a Kenmotsu 3-structure. Bull.Cal. Math. Soc., 93 (1), 47-58, 2001.

[88] Sinha, B.B. and Srivastava, A.K.; Semi-invariant submanifold of a Kenmotsu manifold with constant ϕ-holomorphic sectional curvature, Indian J. Pure Appl. Math., 23 (11), 783-789, 1992.

[89] Sinha, B.B. and Yadav, R.N.; Some results on semi-invariant submanifolds of a certain class of almost contact manifolds. Indian J. Pure Appl. Math., 22(6), 455-460, 1991.

[90] Sinha, B.B. and Yadav, R.N.; Semi-invariant submanifolds of a Kenmotsu manifold, Bull Cal. Math. Soc., 86, 405-412, 1994.

[91] Sirokov, A.P.; About one class of space over the algebra. Publication of higher education institution (Mathematics) 1, 163-170, 1961.

[92] Tanno, S.; Isometric immersion of Sasakian manifolds in spheres. Kodai Math. Sem.Rep., 21, 448-458, 1969.

[93] Tarafdar, M. and Bhattacharyya, A.; On Lorentzian Para Sasakian manifold. Ganita, 52(2), 149-155, 2001.

[94] Tripathi, M.M.; Almost semi-invariant submanifold of a trans-Sasakian manifolds, An. St. Univ. Al. cuza Iasi, Tomul XLI, S.I.a. Mathematica, f2, 243-268, 1995.

[95] Tripathi, M.M.; Non existence of proper mixed foliated semi-invariant submanifolds of Sasakian manifold. Riv. Mat. Univ. Parma (5) 4, 101-102, 1995.

[96] Tripathi, M.M.; Almost semi-invariant submanifolds of an ε-framed metric manifold. Demonstratio Mathematica XXIX (2), 413-426, 1996.

[97] Tripathi, M.M. and Singh, K.D.; On submanifolds of s-manifolds. Ganita, 47 (2), 51-54, 1996.

[98] Tripathi, M.M.; Almost semi-invariant submanifolds of trans-Sasakian manifolds. J. of Indian Math. Soc. 62, 1-4, 225-245, 1996.

[99] Tripathi, M.M.; Some characterization of CR-submanifold of generalized complex space forms, Kuwait J. of Science and Engg., 23,133-138, 1996.

[100] Tripathi, M.M.; Almost semi-invariant submanifolds of a parallel ε-framed metric manifold. Ganita, 48 (1), 57-50, 1997.

[101] Tripathi, M.M.; On CR-submanifolds of nearly and closely Cosymplectic manifolds. Ganita, 51, No. 1, 45-56, 2000.

[102] Tripathi, M.M.; Singh, K.D.; Submanifolds of an almost r-para contact Riemannian manifold. Ganita, 43 (1-2), 137-145, 1992.

[103] Upadhyay, M.D. and Dube, K.K.; Almost contact hyperbolic (f,g,η,ξ)-structure, Acta. Math. Acad. Scient. Hung., Tomus, XXVIII, 13-15, 1976.

[104] Upadhyay, M.D. and Prasad, C.S.; Integrability conditions on almost para contact manifold, J. of the Tensor Society of India, 5, 21-31, 1987.

[105] Vanzura, J.; Almost r-contact structure. Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat., 26, 97-115, 1972.

[106] Yano, K.; On a structure defined by a tensor field f of type (1,1) satisfying $f^3 + f = 0$. Tensor N.S. 99-109, 1963.

[107] Yano, K.; Differential geometry of complex and almost complex spaces. Pergamon press, 1965.

[108] Yano, K. and Kon, M.; Anti-invariant submanifolds, Marcel Dekker Inc., New York, 1776.

[109] Yano, K. and Kon, M.; Contact CR-submanifolds, Kodai Math. J. 5. 238-252, 1982.

[110] Yano, K. and Kon, M.; CR-submanifolds of Kaehlerian and Sasakian manifolds, Progress in Maths, Birkhauser, Boston, 1983.

[111] Yano, K. and Kon, M.; On contact CR-submanifold, J. Korean math. Soc., 26,pp. 231-262, 1989 .

[112] Zhen, G.; Cabrerizo, J.L.; Fernandrz, L. M. and Fernandrz, M.; On ξ-conformally flat contact metric manifolds. Indian J. Pure Appl. Math., 28 (6), 725-734, 1997.

APPENDIX

RESEARCH PAPERS

1. On CR-submanifold of nearly and closely para Cosymplectic manifolds. Demonstratio Mathematica Vol. XXXV, No. 2, 405-415, 2002.

2. Semi-invariant submanifold of a nearly r-Sasakian manifolds. Bull. Cal. Math. Soc. Vol. 94, No 2, 137-146, 2002.

3. Semi-invariant submanifolds of r-Kenmotsu manifolds (Communicated).

4. On CR-submanifolds of a trans hyperbolic Sasakain manifold (Communicated).

5. On CR-submanifolds of a trans para Sasakian manifold (Communicated).

6. On semi-invariant submanifold of para Kenmotsu manifold (Communicated).

7. Hypersurfaces of almost hyperbolic Hermitian manifolds I (communicated).

8. Hypersurfaces of almost hyperbolic Hermitian manifolds II (communicated).

9. Hypersurfaces of almost hyperbolic Hermitian manifolds III (communicated).

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ABSTRACT

CR-submanifold of an almost contact metric manifold studied by Yano and Kon. Semi-invariant submanifold defined and studied by Bejancu and Papaghuic. Dube studied almost hyperbolic Hermitian manifold.

The present work is also a step forward in the same direction. Present work consists of 5 sections. Section 1 is divided into two subsections 1(A) and 1(B) in which we studied CR-submanifolds of a trans para Sasakian manifold and CR-submanifolds of a nearly and closely para cosymplectic manifolds respectively and obtained some interesting results. Section 2 is divided into two subsections 2(A) and 2(B). In section 2(A) and 2(B) we studied semi-invariant submanifolds of a nearly r-Sasakian manifold and semi-invariant submanifolds of r-Kenmotsu manifold respectively and find some important results. Section 3 is divided into two subsections 3(A) and 3(B) in which we studied CR-submanifolds of a trans hyperbolic Sasakian manifold and find some results. Section 4 is divided into two subsections 4(A) and 4(B). In section 4(A) and 4(B) we studied semi-invariant submanifolds of para Kenmotsu manifolds and find some conditions for integrability and parallelness of the distributions, Section 5 is devoted to the study of hypersurfaces of almost hyperbolic Hermitian manifolds and defined hyperbolic Cosymplectic, nearly hyperbolic Cosymplectic manifold etc.

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